Fakultät für Mathematik
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Mathematical Foundations

04.04.2013

(1) **(5 pt.)** Show that $y_1(x) = e^x$, $y_2(x) = x \cdot e^x$ and $y_3(x) = e^{-x}$ are solutions of the differential equation

$$y''' - y'' - y' + y = 0.$$

(2) **(5 pt.)** Determine all eigenvalues and eigenvectors of

$$M = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(3) **(5 pt.)** Solve the differential equation

$$3 \cdot y(x)^2 \cdot y'(x) = x$$

with $y(0) = 1$.

(4) **(5 pt.)** For which $a \in \mathbb{R}$ has the system of linear equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ 2x & + & 2y & + & 4z & = & 1 \\ 3x & + & 4y & + & a \cdot z & = & 2 \end{array}$$

no solution / exactly one solution / infinitely many solutions.

(5) **(5 pt.)** Compute

$$\int \cos^2(2x)\, \mathrm{d}x.$$

(6) **(5 pt.)** Compute

$$\int_0^4 x e^{-x} \mathrm{d}x.$$

(7) **(5 pt.)** Solve the following quadratic equation over the complex numbers ($\mathrm{i}^2 = -1$)

$$z^2 + (2 - 2\,\mathrm{i})z - 2\,\mathrm{i} - 1 = 0.$$

(8) **(5 pt.)** Compute the gradiant and the Hessian matrix (second partial derivatives) of

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xyz.$$

(9) **(5 pt.)** Find the quadratic approximation of $\frac{1}{\sqrt{1-x^2}}$ at the point $x_0 = 0$.

(10) **(5 pt.)** For which $a \in \mathbb{R}$ is the following matrix positiv definite:

$$\begin{pmatrix} a^2 - 1 & a \\ a & 2 \end{pmatrix}.$$

(11) **(5 pt.)** Find the (two) stationary points of

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^3 + 3xz + 2y - y^2 - 3z^2.$$

You do not need to check whether there are maxima or minima at these points.