# Mathematical Foundations

(Otto-von-Guericke University Magedeburg, winter term 2014/15)

http://fma2.math.uni-magdeburg.de/∼mathww/mathfound2014/

# 1 Mathematical objects, sets, functions

## 1.1 Preliminaries

### Mathematical Objects

Numbers (integers, rationals,reals, complex); points or vectors; sets; functions or mappings; ...

### Sets

A **set** is just a collection of objects. If $x$ is a mathematical object and $M$ is a set, then either $x \in M$ or $x \notin M$.

**Special (important) sets:**

$\mathbb{N} = \{ 1, 2, 3, \dots \}$ (set of positive integers)
$\mathbb{Z} = \{ 0, \pm 1, \pm 2, \pm 3, \dots \}$ (set of integers)
$\mathbb{R}$ = set of real numbers (real line)
intervals: $[a\,,b]\,, \quad ]a\,,b[\,, \quad [a\,,b[, \quad ]a\,,b]$ (for $a, b \in \mathbb{R}\,,\ a < b$)
also: $[a\,,\infty[\,, \quad ]a\,,\infty[\,, \quad ]-\infty\,,b]\,, \quad ]-\infty\,,b[$
$\mathbb{C}$ = set of complex numbers (complex plane)
$\mathbb{R}^2 = \{ \mathbf{x} = (x_1, x_2) \ : \ x_1, x_2 \in \mathbb{R} \}$ Euclidean plane
$\mathbb{R}^3 = \{ \mathbf{x} = (x_1, x_2, x_3) \ : \ x_1, x_2, x_3 \in \mathbb{R} \}$

**The $n$-dimensional Euclidean space**

$$\mathbb{R}^n = \Big\{ \mathbf{x} = (x_1, x_2, \dots, x_n) \ : \ x_i \in \mathbb{R},\ i = 1, 2, \dots, n \Big\}$$

The elements $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of $\mathbb{R}^n$ are called **points** or **vectors** (in $n$ dimensions).

$x_i$ (a real number) is the $i$-th coordinate or the $i$-th component of $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

### Functions

$$f \ : \ M \longrightarrow N \qquad (f \text{ is a function from } M \text{ to } N)$$

(where $M$ and $N$ are two nonempty sets)

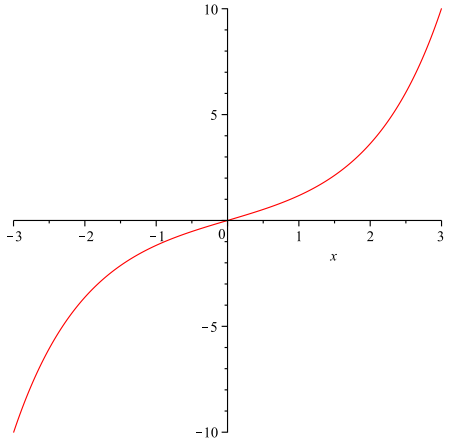
To every $x \in M$ the function $f$ gives a unique value $f(x) \in N$.

$M$ is the **domain** of $f$ ; $N$ is the **set of possible values** of $f$.

Popular case: A real-valued function of one real variable:
$M \subseteq \mathbb{R}$ and $N = \mathbb{R}$ (or $N \subseteq \mathbb{R}$), i.e. $x \in M \subseteq \mathbb{R}$ and $f(x) \in \mathbb{R}$ .

**Graphing a function**

For a *real-valued* function of *one real variable*: Graphical representation is very useful. Here is the graph of the function $f(x) = \frac{1}{2}(e^x - e^{-x})$




## 1.2 Special real-valued functions of one real variable

**Polynomials**

$f(x) \;=\; a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$
(where $a_0, a_1, \ldots, a_n$ are given real numbers)

Polynomials are very important functions since many functions can be "approximated", at least locally, by polynomials. The following two facts are very useful:

1. Given a polynomial $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ and a real number $x_0$, then there are real numbers $b_i$ such that
$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \ldots + b_n(x - x_0)^n.$$
2. Given $n+1$ pairs $(x_i, y_i)$, $i = 1, \ldots, n+1$, with different first coordinates (that is, $x_i \neq x_j$ if $i \neq j$) there is precisely one polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ with $f(x_i) = y_i$ for $i = 1, \ldots, n+1$.

**Trigonometric functions**

$\sin(x)$ , $\cos(x)$ , $\tan(x) = \frac{\sin(x)}{\cos(x)}$ , $\cot(x) = \frac{\cos(x)}{\sin(x)}$

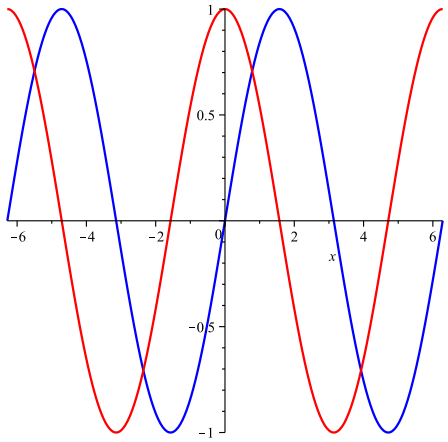
(i) **Sine**
Given an angle $\alpha \in [0, \frac{\pi}{2}]$, we define $\sin\alpha = \frac{\text{length of the leg opposite to angle}}{\text{length of the hypotenuse}}$. If $\alpha \in [\frac{\pi}{2}, \pi]$, then we define $\sin\alpha = \sin(\pi - \alpha)$. If $\alpha \in [\pi, 2\pi]$, we put $\sin\alpha = -\sin(\alpha - \pi)$.

If $x \in \mathbb{R}$, then $x = 2m\pi + \alpha$ with $m \in \mathbb{Z}$, $\alpha \in [0, 2\pi)$, and we put $\sin x = \sin\alpha$. Then the sine is defined for all $x \in \mathbb{R}$. The function is periodic with period $2\pi$, i.e. $\sin(x + 2\pi) = \sin(x)$. The set of all values (range) is $\{y \in \mathbb{R} : -1 \leq y \leq 1\} = [-1, 1]$.

(ii) **Cosine**
Given $\alpha \in [0, \frac{\pi}{2}]$, we define $\cos\alpha = \frac{\text{length of the leg adjacent to angle}}{\text{length of the hypotenuse}}$. If $\alpha \in [\frac{\pi}{2}, \pi]$, then $\cos\alpha = -\cos(\pi - \alpha)$. If $\alpha \in [\pi, 2\pi]$, then $\cos\alpha = -\cos(\alpha - \pi)$.

For $x \in \mathbb{R}$, we write $x = 2m\pi + \alpha$ with $m \in \mathbb{Z}$, $\alpha \in [0, 2\pi)$, and put $\cos x = \cos\alpha$.

Again, the cosine is periodic with period $2\pi$. The set of all values (range) is $[-1, 1]$.




$f(x) = \sin(x)$ (blue), $f(x) = \cos(x)$ (red)

**Some properties of trigonometric functions**

1.

| | $0$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| sin | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| cos | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ |
| tan | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | $---$ |
| cot | $---$ | $\sqrt{3}$ | $1$ | $\frac{\sqrt{3}}{3}$ | $0$ |

2. periodicity:
$$\sin(x+2\pi) = \sin x\,,\ \cos(x+2\pi) = \cos x$$

3. symmetry:
$$\sin(-x) = -\sin x\,,\ \cos(-x) = \cos x$$

4. Pythagorean Theorem: $\sin^2 x + \cos^2 x = 1$.

5. addition theorems
$$\begin{aligned}\sin(x+y) &= \sin x\cdot\cos y + \cos x\cdot\sin y\\ \cos(x+y) &= \cos x\cdot\cos y - \sin x\cdot\sin y\end{aligned}$$

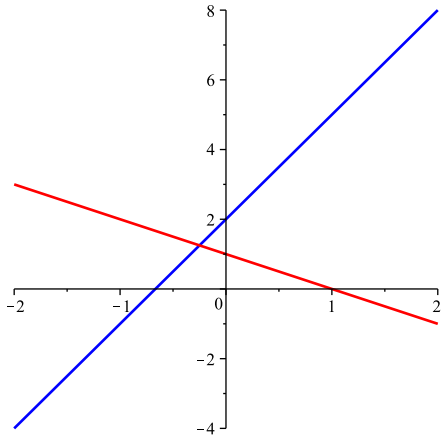
7. shifts about $\pi/2$ und $\pi$:
$$\begin{aligned}\sin(x+\tfrac{\pi}{2}) &= \cos(x)\\ \cos(x-\tfrac{\pi}{2}) &= \sin(x)\\ \sin(x+\pi) &= -\sin(x)\\ \cos(x+\pi) &= -\cos(x)\\ \tan(x+\pi) &= \tan(x)\\ \tan(x+\tfrac{\pi}{2}) &= -\cot(x)\\ \cot(x+\tfrac{\pi}{2}) &= -\tan(x)\\ \cot(x+\pi) &= \cot(x)\\ \tan(x) &= \tfrac{1}{\cot(x)}.\end{aligned}$$

**Note:**

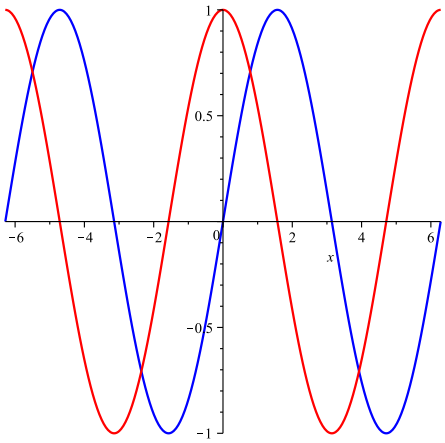
Domain of $\sin(x)$ , $\cos(x)$ is $\mathbb{R}$.
Domain of $\tan(x)$ is $\mathbb{R}\setminus\left\{(k+\frac{1}{2})\,\pi\,:\,k\in\mathbb{Z}\right\}$.
Domain of $\cot(x)$ is $\mathbb{R}\setminus\left\{k\pi\,:\,k\in\mathbb{Z}\right\}$.

**Some examples**

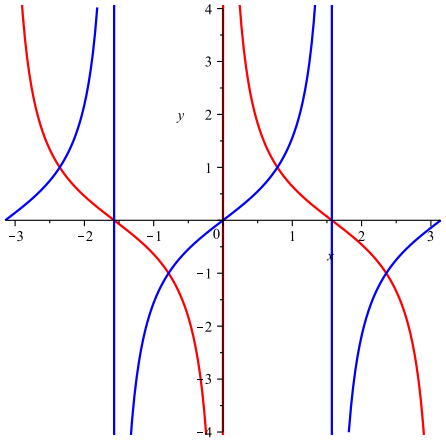
On the following pages, we show the graphs of several functions (polynomials and trigonometric functions):



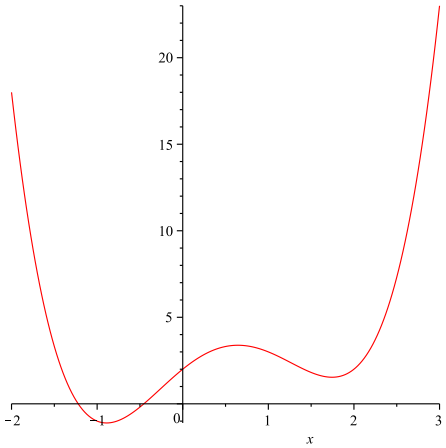

$f(x) = 2x + 1$ (blue), $f(x) = -x + 1$ (red)




$f(x) = \sin(x)$ (blue), $f(x) = \cos(x)$ (red)




$f(x) = \tan(x)$ (blue), $f(x) = \cot(x)$ (red)




$f(x) = x^4 - 2x^3 - 2x^2 + 4x + 2$

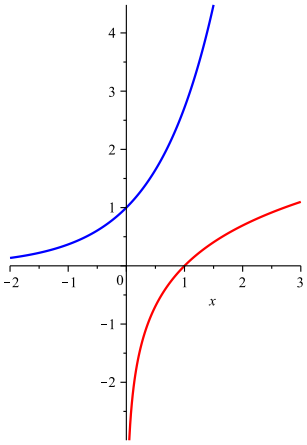
**Exponential function** $\exp(x) = e^x\,, \quad x \in \mathbb{R}$
($e$ the **Euler number**, $e \approx 2.71828$)

**Logarithm:** $\ln(x)\,, \quad x \in ]0\,, \infty[$ , the inverse function of the exponential function; i.e. $\ln(\exp(x)) = x$ for all $x \in \mathbb{R}$ and $\exp(\ln(x)) = x$ for all $x \in ]0, \infty[$.

**Remark:** Other basis $a > 0$, $a \neq 1$:

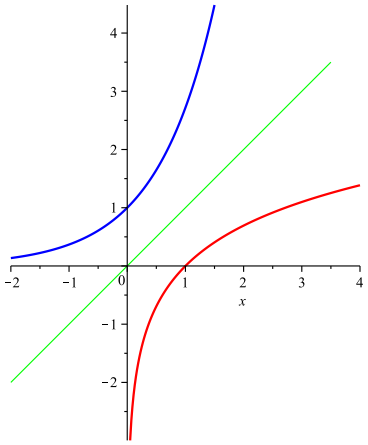
$$\begin{aligned} a^x &= \exp\big(x\,\ln(a)\big), \qquad x \in \mathbb{R} \\ \log_a(x) &= \frac{\ln(x)}{\ln(a)}, \qquad x \in ]0\,, \infty[ \end{aligned}$$

Here are the graphs of the exponential function $e^x$ and $\ln(x)$:




$f(x) = e^x$ (blue), $\quad f(x) = \ln(x)$ (red)

If $f(x)$ is a function, the inverse $g(x)$ of $f(x)$ is a function such that $g(f(x)) = x$ for all $x$ that are allowed for $f$ anf $f(g(x)) = x$ for all $x$ that are allowed for $g$. Not all functions have an inverse, for instance there is no inverse for $f(x) = x^2$. We obtain the equation for $g(x)$ by solving $f(x) = y$ for $x$, then we get an expression $g(y) = x$, and then we simply replace $x$ by $y$. This means that the graph of the inverse fuinction $g(x)$ can be obtained from the graph of $f(x)$ by reflecting it about the line with equation $y = x$.
This can be seen in the following picture for the exponential and logarithmic function:



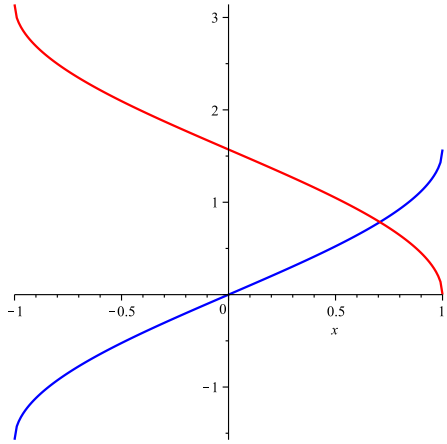

**Inverse trigonometric functions** (with suitable domains)

$\arcsin(x)$ , $\quad x \in [-1\,,\,1]$ , $\qquad$ has its values in $\left[-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right]$

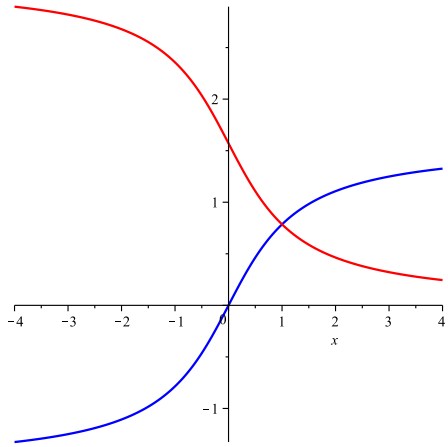
$\arccos(x)\ ,\quad x \in [-1\,,\,1]\ ,\qquad$ has its values in $[0\,,\,\pi]$

$\arctan(x)\ ,\quad x \in \mathbb{R}\ ,\qquad$ has its values in $\left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[$

$\operatorname{arccot}(x)\ ,\quad x \in \mathbb{R}\ ,\qquad$ has its values in $]0\,,\,\pi[$




$f(x) = \arcsin(x)$ (blue), $\quad f(x) = \arccos(x)$ (red)




$f(x) = \arctan(x)$ (blue), $\quad f(x) = \operatorname{arccot}(x)$ (red)

## 1.3 The $n$-dimensional Euclidean space

Note that functions $f : M \to N$ have been defined for arbitrary sets $M$ and $N$. In most applications, $M$ and $N$ are not just real numbers but elements from the Euclidean space.

**Vectors: Addition, multiplication by scalars**

For two vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ und $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, the vector $\mathbf{a} + \mathbf{b}$ (the sum of $\mathbf{a}$ and $\mathbf{b}$) is given by

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1,\, a_2 + b_2,\, \ldots,\, a_n + b_n)\,.$$

For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and a scalar (real number) $\lambda$, the vector $\lambda\,\mathbf{a}$ or $\mathbf{a}\,\lambda$ is given by

$$\lambda\,\mathbf{a} = \mathbf{a}\,\lambda = (\lambda a_1,\, \lambda a_2,\, \ldots, \lambda a_n)\,.$$

**Remark:**

$$\begin{aligned} -\mathbf{b} &= (-1)\mathbf{b} = (-b_1, -b_2, \ldots, -b_n) \\ \mathbf{a} - \mathbf{b} &= \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1,\, a_2 - b_2,\, \ldots,\, a_n - b_n) \qquad \text{(subtraction)} \\ \frac{\mathbf{a}}{\alpha} &= \tfrac{1}{\alpha}\,\mathbf{a} \qquad (\text{if } \alpha \neq 0) \end{aligned}$$

**Obvious properties**

For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^n$ and scalars $\lambda, \mu \in \mathbb{R}$ :

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})\,, \qquad \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}\,,$$
$$\lambda\,(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}\,, \qquad (\lambda + \mu)\,\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}\,,$$
$$\lambda\,(\mu\,\mathbf{a}) = (\lambda\,\mu)\,\mathbf{a}\,,$$

$\mathbf{a} + \mathbf{0} = \mathbf{a}\,, \quad \mathbf{a} - \mathbf{a} = \mathbf{0}\,, \quad$ null-vector $\mathbf{0} = (0, 0, \ldots, 0)\,.$

Sum and linear combinations of $k$ vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_k$ of $\mathbb{R}^n$ :

$$\sum_{i=1}^{k} \mathbf{v}_i = \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_k$$

$$\sum_{i=1}^{k} \lambda_i \mathbf{v}_i = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \ldots + \lambda_k\mathbf{v}_k \qquad (\text{for scalars } \lambda_1, \ldots, \lambda_k)\,.$$

**A simple fact:**

Every vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of $\mathbb{R}^n$ is a linear combination of the elementary unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ , where

$$\mathbf{e}_i = \big(0, \ldots, 0, \underbrace{1}_{i\text{-th}}, 0, \ldots, 0\big) \qquad (i = 1, 2, \ldots, n),$$

since (obviously) :

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \ldots + a_n\mathbf{e}_n$$

**Scalar product** or **inner product**

For $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ :

$$\mathbf{a} \cdot \mathbf{b} \;=\; \sum_{i=1}^{n} a_i b_i \;=\; a_1 b_1 \,+\, a_2 b_2 \,+\, \ldots \,+\, a_n b_n$$

is called the *scalar product* or the *inner product* of the two vectors $\mathbf{a}$ und $\mathbf{b}$.
Note: The result $\mathbf{a} \cdot \mathbf{b}$ is a *scalar* (real number), but *not* a vector.

**Obvious properties:**

$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ , $\quad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} \,=\, \mathbf{a} \cdot \mathbf{c} \,+\, \mathbf{b} \cdot \mathbf{c}$ ,

$(\lambda\, \mathbf{a}) \cdot \mathbf{b} \,=\, \lambda\, (\mathbf{a} \cdot \mathbf{b})$ $\quad$ (where $\lambda \in \mathbb{R}$) ,

$\mathbf{a} \cdot \mathbf{a} \,\geq 0$ , $\quad$ and $\quad \mathbf{a} \cdot \mathbf{a} = 0$ only if $\mathbf{a} = \mathbf{0}$ (the null-vector)

**Length** or **norm of a vector**

For $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ :

$$|\mathbf{a}| \;=\; \sqrt{\mathbf{a} \cdot \mathbf{a}} \;=\; \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

Note: For two points (=vectors) of $\mathbb{R}^n$ ,
$\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ ,
the *distance of* $\mathbf{b}$ *and* $\mathbf{c}$ is given by
$|\mathbf{b} - \mathbf{c}| = \sqrt{(b_1 - c_1)^2 + (b_2 - c_2)^2 + \ldots + (b_n - c_n)^2}$

Properties:

$|\mathbf{a}| \geq 0$ ; $\quad |\mathbf{a}| = 0$ only if $\mathbf{a} = \mathbf{0}$ ;

$|\lambda\, \mathbf{a}| \;=\; |\lambda| \cdot |\mathbf{a}|$ , $\quad$ (where $\lambda \in \mathbb{R}$) ;

$|\mathbf{b} \pm \mathbf{c}| \;\leq\; |\mathbf{b}| \,+\, |\mathbf{c}|$ $\quad$ ("triangle inequality") ;

$|\mathbf{b} \cdot \mathbf{c}| \;\leq\; |\mathbf{b}| \cdot |\mathbf{c}|$ $\quad$ ("Cauchy-Schwarz inequality")

**The angle included by two vectors**

For two vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$
(both not the null-vector) the *angle included by* $\mathbf{a}$ *and* $\mathbf{b}$ is given by

$$\varphi \;=\; \angle(\mathbf{a}, \mathbf{b}) \;= \arccos\Big( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} \Big) \,. \qquad \text{Note: } 0 \leq \varphi \leq \pi \,.$$

Equivalent formulation: $\quad \mathbf{a} \cdot \mathbf{b} \;=\; |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\varphi)$ .

**Orthogonality of two vectors**

Two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ are said to be *orthogonal* (or *perpendicular*)
[abbreviation: $\mathbf{a} \perp \mathbf{b}$ ], if $\quad \mathbf{a} \cdot \mathbf{b} = 0$ .
Note: $\quad \mathbf{a} \perp \mathbf{b} \iff \angle(\mathbf{a}, \mathbf{b}) \,=\, \frac{\pi}{2}$ $\quad$ (if $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$).

# 2 Vectors and matrices

## 2.1 Definitions

Let $p$ and $n$ be positive integers: $p, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ . A (real) $p \times n$ matrix is an arrangement of $p \cdot n$ real numbers in $p$ rows and $n$ columns. Each real number is identified by the pair $(i, j)$ representing the numbers of the row $(i)$ and column $(j)$ in which it occurs, and is called the $(i, j)$-th entry of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pn} \end{pmatrix}$$

Here the $(i, j)$-th entry is $a_{ij} \in \mathbb{R}$, $(i = 1, \ldots, p,\ j = 1, \ldots, n)$.

Short notation: $\boldsymbol{A} = \big(a_{ij}\big)_{\substack{i=1,\ldots,p \\ j=1,\ldots,n}}$ or simply $\boldsymbol{A}$.

**Examples**

Special $2 \times 3$ matrix: $\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{pmatrix}$ ; special $3 \times 2$ matrix: $\boldsymbol{B} = \begin{pmatrix} 1 & -3 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$ .

For a $p \times n$ matrix $\boldsymbol{A} = \big(a_{ij}\big)_{\substack{i=1,\ldots,p \\ j=1,\ldots,n}}$ and a $q \times m$ matrix $\boldsymbol{B} = \big(b_{i,j}\big)_{\substack{i=1,\ldots,q \\ j=1,\ldots,m}}$ :

$$\boldsymbol{A} = \boldsymbol{B} \quad \Longleftrightarrow \quad p = q,\ n = m,\ a_{ij} = b_{ij} \text{ for all } i, j$$

Only *one row*: A $1 \times n$ matrix is called a *row vector* (of dimension $n$),

$$\boldsymbol{A} = \big(a_{11}\ a_{12}\ \ldots\ a_{1n}\big) \text{ , or } \quad \boldsymbol{a} = (a_1, a_2, \ldots, a_n) \text{ .}$$

Only *one column*: A $p \times 1$ matrix is called a *column vector* (of dimension $p$),

$$\boldsymbol{A} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{p1} \end{pmatrix} \text{ , or } \quad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} \text{ .}$$

We distinguish between *row* vectors and *column* vectors. So, e.g., the vectors $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $(3, 5)$ are different.

We write $\mathbb{R}^n$ instead of $\mathbb{R}^{n \times 1}$ (the set of all column vectors of dimension $n$).

Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pn} \end{pmatrix}$

The row vectors of the matrix $\boldsymbol{A}$ are

$\boldsymbol{a}^i = \big(a_{i1}, a_{i2}, \ldots, a_{in}\big)$, *(the $i$-th row of $\boldsymbol{A}$)*, for $i = 1, \ldots, p$,

The column vectors of the matrix $\boldsymbol{A}$ are

$$\boldsymbol{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{pj} \end{pmatrix}, \quad \textit{(the } j\textit{-th column of } \boldsymbol{A}\textit{)}, \text{ for } j = 1, \ldots, n\,.$$

So one may write: $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}^1 \\ \boldsymbol{a}^2 \\ \vdots \\ \boldsymbol{a}^p \end{pmatrix} = \big(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\big)$

**Transposition**

For a row vector (of matrix size $1 \times n$, or of dimension $n$), $\boldsymbol{a} = \big(a_1, a_2, \ldots, a_n\big)$, its *transpose* is:

$\boldsymbol{a}^{\mathrm{t}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ (which is a column vector of size $n \times 1$, i.e. of dimension $n$).

For a column vector (of matrix size $p \times 1$, or of dimension $p$), $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}$, its *transpose* is:

$\boldsymbol{b}^{\mathrm{t}} = \big(b_1, b_2, \ldots, b_p\big)$ (which is a row vector of matrix size $1 \times p$, i.e. of dimension $p$).

For a $p \times n$ matrix $\boldsymbol{A}$ with rows $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^p$ :
The *transpose* $\boldsymbol{A}^{\mathrm{t}}$ has the columns $(\boldsymbol{a}^1)^{\mathrm{t}}, (\boldsymbol{a}^2)^{\mathrm{t}}, \ldots, (\boldsymbol{a}^p)^{\mathrm{t}}$ ;
so $\boldsymbol{A}^{\mathrm{t}}$ is an $n \times p$ matrix.
In other words: The $(i, j)$-th entry of $\boldsymbol{A}^{\mathrm{t}}$ equals the $(j, i)$-th entry of $\boldsymbol{A}$, for all $i = 1, \ldots, n$, $j = 1, \ldots, p$.

**Examples: Some *square* matrices**

$$\boldsymbol{B} = \begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix}, \quad \boldsymbol{B}^{\mathrm{t}} = \begin{pmatrix} 0 & 3 \\ -1 & 1 \end{pmatrix}.$$

$$\boldsymbol{C} = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \quad \boldsymbol{C}^{\mathrm{t}} = \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix}.$$

$$\boldsymbol{D} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \boldsymbol{D}^{\mathrm{t}}, \qquad \boldsymbol{E} = \begin{pmatrix} 9 & 8 & 7 \\ 8 & 5 & 4 \\ 7 & 4 & 1 \end{pmatrix} = \boldsymbol{E}^{\mathrm{t}}.$$

**Symmetric matrices**

A matrix $\boldsymbol{A}$ is called *symmetric*, if $\boldsymbol{A}^{\mathrm{t}} = \boldsymbol{A}$.
In particular: A symmetric matrix must be *a square matrix*, i.e., of size $n \times n$ (for some $n \in \mathbb{N}$).

## 2.2 Matrix algebra

**Addition, multiplication by scalars**

If $\boldsymbol{A} = \big(a_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}$ and $\boldsymbol{B} = \big(b_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}$ are two $p \times n$ matrices:

$$\boldsymbol{A} + \boldsymbol{B} = \big(a_{ij} + b_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}, \quad \text{(again a } p \times n \text{ matrix)}.$$

If $\boldsymbol{A} = \big(a_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}$ is a $p \times n$ matrix and $\lambda \in \mathbb{R}$:

$$\lambda\,\boldsymbol{A} = \boldsymbol{A}\,\lambda = \big(\lambda\, a_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}, \quad \text{(again a } p \times n \text{ matrix)}.$$

1. $-\boldsymbol{B} = (-1)\boldsymbol{B} = \big(-b_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}$
2. $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B}) = \big(a_{ij} - b_{ij}\big)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}$
3. $\boldsymbol{A}/\alpha = (1/\alpha)\boldsymbol{A} \quad$ (if $\alpha \neq 0$)
4. $(\boldsymbol{A} + \boldsymbol{B})^{\mathrm{t}} = \boldsymbol{A}^{\mathrm{t}} + \boldsymbol{B}^{\mathrm{t}}$, $\quad (\lambda\boldsymbol{A})^{\mathrm{t}} = \lambda\boldsymbol{A}^{\mathrm{t}}$

**Multiplication of matrices**

The product $\boldsymbol{AB}$ of two matrices is defined only if they are *compatible* w.r.t. their sizes, i.e. $\boldsymbol{AB}$ requires: $\boldsymbol{A}$ is $q \times p$, $\boldsymbol{B}$ is $p \times n$ (# of columns of $\boldsymbol{A}$ = # of rows of $\boldsymbol{B}$):

If $\boldsymbol{A}$ is $q \times p$ and $\boldsymbol{B}$ is $p \times n$ then then $\boldsymbol{AB}$ is $q \times n$ defined by:

$$\boldsymbol{AB} = \big(\boldsymbol{a}^i \cdot \boldsymbol{b}_j\big)_{\substack{i=1,\dots,q\\ j=1,\dots,n}},$$

where $\boldsymbol{a}^i$ $(i = 1, \dots, q)$ are the rows of $\boldsymbol{A}$,
$\boldsymbol{b}_j$ $(j = 1, \dots, n)$ are the columns of $\boldsymbol{B}$,
$\boldsymbol{a}^i \cdot \boldsymbol{b}_j$ is the scalar product of the two $p$-dimensional row- and column vectors, resp.

More explicitly: $\boldsymbol{a}^i \cdot \boldsymbol{b}_j = \sum_{k=1}^{p} a_{ik}\, b_{kj}$.

**Matrix product motivated by linear functions**

Let $\boldsymbol{B}$ be $p \times n$ and $\boldsymbol{x} \in \mathbb{R}^n$, then $\boldsymbol{Bx}$ is a vector in $\mathbb{R}^p$. Consider $\boldsymbol{x}$ as a *variable* in $\mathbb{R}^n$, while $\boldsymbol{B}$ is fixed. Then the matrix $\boldsymbol{B}$ defines a mapping $\mathbb{R}^n \to \mathbb{R}^p$:

$$\boldsymbol{L_B} : \mathbb{R}^n \longrightarrow \mathbb{R}^p , \quad \boldsymbol{L_B}(\boldsymbol{x}) = \boldsymbol{B}\boldsymbol{x}.$$

Easy to see: $\boldsymbol{L_B}$ is a *linear* function, i.e.,

$$\boldsymbol{L_B}(\lambda \boldsymbol{x}_1 + \mu \boldsymbol{x}_2) = \lambda \boldsymbol{L_B}(\boldsymbol{x}_1) + \mu \boldsymbol{L_B}(\boldsymbol{x}_2).$$

For the *composition* of $\boldsymbol{L_B}$ and $\boldsymbol{L_A}$, we have

$$\boldsymbol{L_A}\big(\boldsymbol{L_B}(\boldsymbol{x})\big) = (\boldsymbol{AB})\boldsymbol{x},$$

i.e., the composition is just $\boldsymbol{L_{AB}}$.

**Examples**

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & 6 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}. \quad \text{Then } \boldsymbol{AB} = \begin{pmatrix} 6 & 4 & 2 \\ 12 & 5 & -2 \end{pmatrix}.$$

Here $\boldsymbol{BA}$ is *not* defined.

Even if two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are such that both products $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are defined, one has in general: $\boldsymbol{AB} \neq \boldsymbol{BA}$:
Both products are defined if and only if: $\boldsymbol{A}$ is $p \times n$ and $\boldsymbol{B}$ is $n \times p$. Then $\boldsymbol{AB}$ is $p \times p$, $\boldsymbol{BA}$ is $n \times n$. So, in case $p \neq n$ *always* $\boldsymbol{AB} \neq \boldsymbol{BA}$. But also in case $p = n$, the two products are different, in general.
E.g., $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$; then:

$$\boldsymbol{AB} = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{BA} = \begin{pmatrix} 2 & 3 \\ 2 & -2 \end{pmatrix}.$$

**Further properties**

| | |
|---|---|
| $\boldsymbol{A}$ $m \times q$, $\boldsymbol{B}$ $q \times p$, $\boldsymbol{C}$ $p \times n$: | $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$ (associativity) |
| $\boldsymbol{A}$ $q \times p$, $\boldsymbol{B}$ $p \times n$: | $(\boldsymbol{AB})^{\mathrm{t}} = \boldsymbol{B}^{\mathrm{t}}\boldsymbol{A}^{\mathrm{t}}$ |
| $\boldsymbol{A}, \boldsymbol{B}$ $q \times p$, $\boldsymbol{C}$ $p \times n$: | $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$ |
| $\boldsymbol{A}$ $q \times p$, $\boldsymbol{B}, \boldsymbol{C}$ $p \times n$: | $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ |
| $\boldsymbol{A}$ $q \times p$, $\boldsymbol{B}$ $p \times n$, $\lambda \in \mathbb{R}$: | $(\lambda \boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(\lambda \boldsymbol{B}) = \lambda(\boldsymbol{AB})$ |

By rule of associativity: $\boldsymbol{ABC}$ is defined, as well as products $\boldsymbol{A}_1\boldsymbol{A}_2 \ldots \boldsymbol{A}_k$ for any number $k$ of matrices (provided the matrices are compatible for multiplication). In particular: For a *square* matrix $\boldsymbol{A}$ $(n \times n)$ : $\boldsymbol{A}^k = \underbrace{\boldsymbol{A} \ldots \boldsymbol{A}}_{k-\text{times}}$, $k$-th power of $\boldsymbol{A}$, again $n \times n$.

## 2.3 Regular matrix, inverse matrix

**Notation:** $\boldsymbol{I}_n$ the $n \times n$ *unit matrix (identity matrix)* which has all its diagonal entries equal to 1 and all its off-diagonal entries equal 0. E.g. $\boldsymbol{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$\boldsymbol{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly: $\boldsymbol{I}_n = \begin{pmatrix} \boldsymbol{e}_1^{\mathrm{t}} \\ \boldsymbol{e}_2^{\mathrm{t}} \\ \vdots \\ \boldsymbol{e}_n^{\mathrm{t}} \end{pmatrix} = \big(\boldsymbol{e}_1\,,\,\boldsymbol{e}_2\,,\,\ldots\,,\,\boldsymbol{e}_n\big)$

$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e_n} \in \mathbb{R}^n$ the elementary unit-(column-)vectors

Easy to see: $\boldsymbol{B}\boldsymbol{I}_n = \boldsymbol{B}$ for all $\boldsymbol{B}$ $p \times n$; $\quad \boldsymbol{I}_n\boldsymbol{C} = \boldsymbol{C}$ for all $\boldsymbol{C}$ $n \times m$, in particular:

$$\boldsymbol{I}_n\boldsymbol{x} = \boldsymbol{x} \text{ for all } \boldsymbol{x} \in \mathbb{R}^n,$$

i.e., the linear function $\boldsymbol{L}_{\boldsymbol{I}_n}(\boldsymbol{x}) = \boldsymbol{I}_n\boldsymbol{x}$ is the identity function (on $\mathbb{R}^n$).

**Regular (square) matrix and its inverse matrix**

An $n \times n$ matrix $\boldsymbol{A}$ is called *regular* (or *nonsingular* or *invertible*) if there exists an $n \times n$ matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$ or, equivalently, $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$. In case that $\boldsymbol{A}$ is regular, a matrix $\boldsymbol{A}^{-1}$ as above is unique and is called the *inverse of* $\boldsymbol{A}$.

In terms of the associated linear function $\boldsymbol{L}_{\boldsymbol{A}}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$\boldsymbol{A}$ is regular if and only if the function $\boldsymbol{L}_{\boldsymbol{A}}$ has an inverse function $\boldsymbol{L}_{\boldsymbol{A}}^{-1}$, in which case $\boldsymbol{L}_{\boldsymbol{A}}^{-1}(\boldsymbol{y}) = \boldsymbol{A}^{-1}\boldsymbol{y}$, $(\boldsymbol{y} \in \mathbb{R}^n)$.

We have $\boldsymbol{A}$, $\boldsymbol{B}$ $n \times n$ regular $\implies$ $\boldsymbol{A}\boldsymbol{B}$ regular, and $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

**Simplest case:** $n = 2$

For a $2 \times 2$ matrix $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$: $\quad \boldsymbol{A}$ regular $\iff$ $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

In case that $\boldsymbol{A}$ is regular, $\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

**Another simple case: diagonal matrix**

$\boldsymbol{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$,

which means that the $(i,j)$-th entry of $\boldsymbol{D}$ is $\begin{cases} \lambda_i & \text{, if } i = j \\ 0 & \text{, if } i \neq j \end{cases}$.

$\boldsymbol{D}$ regular $\iff$ $\lambda_i \neq 0$ for all $i = 1, 2, \ldots, n$, in which case $\boldsymbol{D}^{-1} = \mathrm{diag}\big(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}\big)$.

## 2.4 Linear equation systems

A linear equation system (LES) ($p$ equations and $n$ real unknowns)

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pn}x_n & = & b_p \end{array}$$

can be written in matrix notation as $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the $p \times n$ matrix with entries $a_{ij}$ (the coefficients of (LES)), $\boldsymbol{b} \in \mathbb{R}^p$ the vector with coordinates $b_i$ (the r.h.s. of (LES)), $\boldsymbol{x} \in \mathbb{R}^n$ the vector of the unknown reals $x_i$, $(i = 1, 2, \ldots, n)$. Sometimes: $p = n$ and $\boldsymbol{A}$ regular; then: $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ is the unique solution.
Solution of (LES) / computation of $\boldsymbol{A}^{-1} \longrightarrow$ numerical algorithms (computer)

**Example: An LES with $p = n = 4$**

$$\begin{array}{rcl} 3x_1 - 4x_2 + 4x_3 + 2x_4 & = & 15 \\ -2x_1 + 4x_3 - x_4 & = & -8 \\ -5x_1 + 4x_2 - 5x_3 + 3x_4 & = & 0 \\ 3x_1 + x_2 - 2x_4 & = & -3 \end{array}$$

In matrix notation:

$$\begin{pmatrix} 3 & -4 & 4 & 2 \\ -2 & 0 & 4 & -1 \\ -5 & 4 & -5 & 3 \\ 3 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 15 \\ -8 \\ 0 \\ -3 \end{pmatrix}$$

$$\boldsymbol{A} \qquad\qquad \boldsymbol{x} \quad = \quad \boldsymbol{b}$$

The system of linear equations $\mathbf{Ax} = \mathbf{b}$ is called homogeneous if $\mathbf{b} = \mathbf{0}$, otherwise inhomogeneous. If $\mathbf{v}$ is a solution of

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b},$$

$$L_h = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{0}\} \quad \text{and} \quad L_i = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}\},$$

then

$$\boxed{L_i = L_h + \mathbf{v}}$$

where $L_h + \mathbf{v} = \{\mathbf{x} + \mathbf{v} : \mathbf{x} \in L_h\}$.

**Solution of the homogeneous system**

Try to transform the system $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ into an easier one using the following transformations, which are called *elementary row operations*:

- Interchange two rows.
- Add $\lambda$ times row $i$ to row $j$.
- Multiply a row with $\lambda \neq 0$.

If $\mathbf{B}$ can be obtained from $\mathbf{A}$ via a sequence of elementary row operations, then we call $\mathbf{A}$ and $\mathbf{B}$ row equivalent. Easy to see: $\mathbf{A}$ is row equivalent to a matrix of the following shape:

$$\mathbf{A}' = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & \cdots & \cdots & \cdots & a'_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & \cdots & \cdots & a'_{2n} \\ \vdots & & \ddots & & & & \vdots \\ 0 & 0 & \cdots & a'_{rr} & \cdots & \cdots & a'_{rn} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

If the $i$th row is nonzero, let $a_{i,j_i}$ be its leftmost nonzero entry. We can assume that $j_1 < j_2 < \ldots < j_r$, hence:

$$\mathbf{A}' = \begin{bmatrix} a_{1,j_1} & * & \cdots & * & * & * & \cdots & * & * & * & \cdots & * & * & \cdots & * \\ & & & & a_{2,j_2} & * & \cdots & * & * & * & \cdots & * & * & \cdots & * \\ & & & & & & & & a_{3,j_3} & * & \cdots & * & * & \cdots & * \\ & & & & & & & & & & & \vdots & \vdots & & \vdots \\ & & & & & & & & & & & a_{r,j_r} & * & \cdots & * \\ & & & & & & & & & & & & & & \end{bmatrix}$$

We call such a matrix row reduced. Moreover, we can reach that the columns $j_1, j_2, \ldots, j_r$ are vectors which contain just one nonzero entry 1. Hence we obtain the following form, which is called a row reduced normal form:

$$\mathbf{A}' = \begin{bmatrix} 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & \cdots & 0 & * & \cdots & * \\ & & & & 1 & * & \cdots & * & 0 & * & \cdots & \cdots & 0 & * & \cdots & * \\ & & & & & & & & 1 & * & \cdots & \cdots & 0 & * & \cdots & * \\ & & & & & & & & & & & & \vdots & \vdots & & \vdots \\ & & & & & & & & & & & & 1 & * & \cdots & * \\ & & & & & & & & & & & & & & & \end{bmatrix}.$$

The solutions of $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ are the same as the solutions of $\mathbf{A}' \cdot \mathbf{x} = \mathbf{0}$ if $\mathbf{A}'$ is row equivalent to $\mathbf{A}$.

If $\mathbf{A}'$ is row reduced and has $r$ nonzero rows, we call $r$ the rank of the matrix $\mathbf{A}'$ (and also of $\mathbf{A}$ ).
We are going to describe a method how to transform a given matrix $\mathbf{A}$ into row reduced form:

1. Find the first nonzero column of $\mathbf{A}$, say column $j$ (in most cases $j = 1$).
2. By interchanging appropriate rows, we get $a_{1,j} \neq 0$.
3. Subtract the first row $a_{i,j}/a_{1,j}$ times from row $i$ $(i = 2, \ldots, m)$.

Call the matrix that we obtain by such a *Pivot* step as $\mathbf{A}'$. Now apply the same procedure to the matrix that you obtain by deleting row 1 (or actually by ignoring row 1) of $\mathbf{A}'$. We continue like this until we cannot find a nonzero entry any more.

**Example** Solve the homogeneous LES

$$\begin{array}{rcrcrcl} 3x_1 & + & 2x_2 & - & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \end{array}$$

Put $x_3 = a$. Then $x_2 = -a$. Using this in the first row, we obtain

$$3x_1 - 2a - a = 0,$$

hence $x_1 = a$. The solution set $L$ is

$$L = \left\{ \begin{pmatrix} a \\ -a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

We say that the solution space has one degree of freedom, since we can choose the value of one variable freely.

This example was easy, since the matrix of coefficients

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

was row reduced.

**Example** Solve $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 6 \\ 1 & 1 & -2 \\ 1 & -2 & 8 \end{pmatrix}$$

Divide the first row by 2 and subtract row 1 from the second and third row. We obtain

$$\begin{pmatrix} 1 & -1/2 & 3 \\ 0 & 3/2 & -5 \\ 0 & -3/2 & 5 \end{pmatrix}$$

Now add row 2 to row 3, and multiply row 2 by 2/3:

$$\begin{pmatrix} 1 & -1/2 & 3 \\ 0 & 1 & -10/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we add the second row 1/2-times to the first one:

$$\begin{pmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -10/3 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is row reduced, and its solutions are

$$x_3 = a, \quad x_2 = \frac{10}{3}a, \quad x_1 = -\frac{4}{3}a$$

or

$$L = \left\{ \begin{pmatrix} -(4/3)a \\ (10/3)a \\ a \end{pmatrix} : a \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -4b \\ 10b \\ 3b \end{pmatrix} : b \in \mathbb{R} \right\}$$

As before, we have one degree of freedom. (If there is no degree of freedom, the only solution is $\mathbf{0}$.)

To summarize:

| |
|---|
| $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ with $m$ equations and $n$ variables has at least the solution $\mathbf{0}$. If $r$ is the rank of $\mathbf{A}$, then the system has $n - r$ degrees of freedom. In particular: if $m < n$, then the system has more than just the solution $\mathbf{0}$, since $r \leq m < n$. |

We solve the system by transforming $\mathbf{A}$ into row reduced form (Gaussian elimination).

**The inhomogeneous case**

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

and

$$\mathbf{A}' \cdot \mathbf{x} = \mathbf{b}'$$

have the same solutions if the extended matrix

$$(\mathbf{A}'|\mathbf{b}')$$

can be obtained from

$$(\mathbf{A}|\mathbf{b})$$

via elementary row operations.

**Example**

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

with

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 5 & 2 & 4 \\ 5 & 2 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2,6 \\ 6,7 \\ 7,2 \end{pmatrix}$$

The extended matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 2,6 \\ 5 & 2 & 4 & 6,7 \\ 5 & 2 & 6 & 7,2 \end{array}\right)$$

and we transform it:

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 2,6 \\ 5 & 2 & 4 & 6,7 \\ 0 & 0 & 2 & 0,5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0,5 & 0 & 1,3 \\ 0 & -0,5 & 4 & 0,2 \\ 0 & 0 & 2 & 0,5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0,5 & 0 & 1,3 \\ 0 & -0,5 & 0 & -0,8 \\ 0 & 0 & 1 & 0,25 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0,5 \\ 0 & 1 & 0 & 1,6 \\ 0 & 0 & 1 & 0,25 \end{array}\right)$$

The unique solution is

$$x_1 = 0,5, \quad x_2 = 1,6, \quad x_3 = 0,25; \quad \text{i.e.,} \quad L = \left\{ \begin{pmatrix} 0,5 \\ 1,6 \\ 0,25 \end{pmatrix} \right\}$$

**Example**

$$\left(\begin{array}{ccc|c} 2 & -1 & 6 & 10 \\ 1 & 1 & -2 & 4 \\ 1 & -2 & 8 & 6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1/2 & 3 & 5 \\ 1 & 1 & -2 & 4 \\ 1 & -2 & 8 & 6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1/2 & 3 & 5 \\ 0 & 3/2 & -5 & -1 \\ 0 & -3/2 & 5 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1/2 & 3 & 5 \\ 0 & 1 & -10/3 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 4/3 & 14/3 \\ 0 & 1 & -10/3 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We can choose $x_3 = a$, then

$$\begin{array}{rcl} x_2 & = & -\frac{2}{3} + \frac{10}{3}a \\ x_1 & = & \frac{14}{3} - \frac{4}{3}a \\ x_3 & = & a \end{array}$$

In other words:

$$L_i = \left\{ \begin{pmatrix} \frac{14}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix} + a \cdot \begin{pmatrix} \frac{-4}{3} \\ \frac{10}{3} \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

We see that

$$L_h = \left\{ a \cdot \begin{pmatrix} \frac{-4}{3} \\ \frac{10}{3} \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$

is the solution set of the homogeneous equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$$

and

$$\begin{pmatrix} \frac{14}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix}$$

is one solution of the inhomogeneous system. If the righthand side had been

$$\begin{pmatrix} 10 \\ 4 \\ 5 \end{pmatrix},$$

the system would have had no solution, since it would have implied an equation of the form $0x_1 + 0x_2 + 0x_3 \neq 0$.

The LES $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has a solution, if and only if $\quad \text{rank}(\mathbf{A}) \;=\; \text{rank}(\mathbf{A}|\mathbf{b}) \quad .$

**If** $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ with $m$ equations and $n$ variables has a solution, **then** it has $n - r$ degrees of freedom, where $r$ is the rank of $\mathbf{A}$.

**Example**

$$\left(\begin{array}{cccc|c} 3 & 5 & 2 & -2 & 12 \\ -1 & 0 & 3 & 4 & -2 \\ 2 & -1 & -2 & 1 & -2 \end{array}\right)$$

We start our row operations:

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -4 & 2 \\ 3 & 5 & 2 & -2 & 12 \\ 2 & -1 & -2 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -4 & 2 \\ 0 & 5 & 11 & 10 & 6 \\ 0 & -1 & 4 & 9 & -6 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -4 & 2 \\ 0 & 1 & -4 & -9 & 6 \\ 0 & 5 & 11 & 10 & 6 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -4 & 2 \\ 0 & 1 & -4 & -9 & 6 \\ 0 & 0 & 31 & 55 & -24 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -4 & 2 \\ 0 & 1 & -4 & -9 & 6 \\ 0 & 0 & 1 & 55/31 & -24/31 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -4 & 2 \\ 0 & 1 & 0 & -59/31 & 90/31 \\ 0 & 0 & 1 & 55/31 & -24/31 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 41/31 & -10/31 \\ 0 & 1 & 0 & -59/31 & 90/31 \\ 0 & 0 & 1 & 55/31 & -24/31 \end{array}\right)$$

One solution of the inhomogeneous system is

$$\begin{pmatrix} -10/31 \\ 90/31 \\ -24/31 \\ 0 \end{pmatrix}$$

The solutions of the homogeneous system are

$$a \cdot \begin{pmatrix} -41/31 \\ 59/31 \\ -55/31 \\ 1 \end{pmatrix}, \quad a \in \mathbb{R}\,,$$

hence $L_i$, the solution set of the inhomogeneous system, is

$$L_i = \left\{ \begin{pmatrix} \frac{-10}{31} \\ \frac{90}{31} \\ \frac{-24}{31} \\ 0 \end{pmatrix} + a \cdot \begin{pmatrix} -\frac{41}{31} \\ \frac{59}{31} \\ -\frac{55}{31} \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$

# 3 Further matrix theory

## 3.1 Permutations

A *permutation* of the numbers $1, 2, \ldots, n$ is a function $\sigma : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ such that: the values $\sigma(1), \sigma(2), \ldots, \sigma(n)$ cover all the numbers $1, 2, \ldots, n$. So the vector $\boldsymbol{\sigma} = \big(\sigma(1), \sigma(2), \ldots, \sigma(n)\big)$ has as coordinates the numbers $1, 2, \ldots, n$ in *permuted order*.

The *set of all permutations* of the numbers $1, 2, \ldots, n$ is called the *permutation group of order* $n$, for which the symbol $\mathcal{S}_n$ is used.

**Note:** $\quad \#\,\mathcal{S}_n \,=\, n! \,=\, 1 \cdot 2 \cdot \cdots \cdot n$ .

**Example:** $\mathcal{S}_2$ and $\mathcal{S}_3$ in vector notation:

$n = 2$: $\quad \mathcal{S}_2 \,=\, \big\{(1,2)\,,\,(2,1)\big\}$

$n = 3$: $\quad \mathcal{S}_3 \,=\, \big\{(1,2,3)\,,\,(1,3,2)\,,\,(2,1,3)\,,\,(2,3,1)\,,\,(3,1,2)\,,\,(3,2,1)\big\}$

**Sign of a permutation**

Given $\sigma \in \mathcal{S}_n$, consider the vector $\boldsymbol{\sigma} = \big(\sigma(1), \sigma(2), \ldots, \sigma(n)\big)$.

Obviously: By successive interchanges of two coordinates at a time the vector $\boldsymbol{\sigma}$ can be transformed to the *identity permutation vector* $(1, 2, \ldots, n)$ .

**Example:**

$n = 7$, $\sigma \in \mathcal{S}_7$ such that $\boldsymbol{\sigma} \,=\, (3, 2, 5, 4, 1, 7, 6)$.

Perform successive interchanges of two coordinates:

$(3, 2, 5, 4, 1, 7, 6) \longrightarrow (1, 2, 5, 4, 3, 7, 6) \longrightarrow (1, 2, 3, 4, 5, 7, 6) \longrightarrow (1, 2, 3, 4, 5, 6, 7)$

We needed $3$ interchanges. We can do it in a different way:

$(3, 2, 5, 4, 1, 7, 6) \longrightarrow (2, 3, 5, 4, 1, 7, 6) \longrightarrow (2, 3, 4, 5, 1, 7, 6) \longrightarrow (2, 3, 4, 1, 5, 7, 6) \longrightarrow$
$(2, 3, 1, 4, 5, 7, 6) \longrightarrow (2, 1, 3, 4, 5, 7, 6) \longrightarrow (1, 2, 3, 4, 5, 7, 6) \longrightarrow (1, 2, 3, 4, 5, 6, 7)$

which required $7$ interchanges.

Note: 3 and 7 are clearly different, but they are *both odd*. Irrespectively *which* sequence of interchanges is used to achieve transformation to $(1, 2, \ldots, n)$, the number of interchanges performed is either *always even* or it is *always odd*.

One defines: $\quad \mathrm{sgn}(\sigma) = \begin{cases} +1 & \text{in the 'even' case} \\ -1 & \text{in the 'odd' case} \end{cases}$

**Example:** $\quad n = 2$:

| $\sigma$ | $(1,2)$ | $(2,1)$ |
|---|---|---|
| $\mathrm{sgn}(\sigma)$ | $+1$ | $-1$ |

$n = 3$:

| $\sigma$ | $(1,2,3)$ | $(2,3,1)$ | $(3,1,2)$ | $(3,2,1)$ | $(1,3,2)$ | $(2,1,3)$ |
|---|---|---|---|---|---|---|
| $\mathrm{sgn}(\sigma)$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ | $-1$ |

## 3.2 Determinants

For an $n \times n$ matrix $\boldsymbol{A} = \big(a_{ij}\big)_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$ the *determinant* of $\boldsymbol{A}$ is defined by

$$\det \boldsymbol{A} \;=\; \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\; a_{1\,\sigma(1)}\, a_{2\,\sigma(2)} \;\cdots\; a_{n\,\sigma(n)}$$

**Example** $n = 2$:

| $\sigma$ | $(1,2)$ | $(2,1)$ |
|---|---|---|
| $\operatorname{sgn}(\sigma)$ | $+1$ | $-1$ |

$$\det \boldsymbol{A} \;=\; a_{11}a_{22} - a_{12}a_{21}$$

$n = 3$:

| $\sigma$ | $(1,2,3)$ | $(2,3,1)$ | $(3,1,2)$ | $(3,2,1)$ | $(1,3,2)$ | $(2,1,3)$ |
|---|---|---|---|---|---|---|
| $\operatorname{sgn}(\sigma)$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ | $-1$ |

$$\begin{aligned}\det \boldsymbol{A} \;=\; & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

*Multilinearity* (w.r.t. rows) of determinants:

Given an $n \times n$-matrix $\boldsymbol{A}$ with rows $\boldsymbol{a}^1, \boldsymbol{a}^2, \ldots, \boldsymbol{a}^n$, and given a (row number) $k \in \{1, 2, \ldots, n\}$. If $\boldsymbol{a}^k = \lambda \boldsymbol{b} + \mu \boldsymbol{c}$, where $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{1\times n}$ and $\lambda, \mu \in \mathbb{R}$, then:

$$\det \begin{pmatrix} \boldsymbol{a}^1 \\ \vdots \\ \boldsymbol{a}^{k-1} \\ \lambda\boldsymbol{b} + \mu\boldsymbol{c} \\ \boldsymbol{a}^{k+1} \\ \vdots \\ \boldsymbol{a}^n \end{pmatrix} \;=\; \lambda \det \begin{pmatrix} \boldsymbol{a}^1 \\ \vdots \\ \boldsymbol{a}^{k-1} \\ \boldsymbol{b} \\ \boldsymbol{a}^{k+1} \\ \vdots \\ \boldsymbol{a}^n \end{pmatrix} + \mu \det \begin{pmatrix} \boldsymbol{a}^1 \\ \vdots \\ \boldsymbol{a}^{k-1} \\ \boldsymbol{c} \\ \boldsymbol{a}^{k+1} \\ \vdots \\ \boldsymbol{a}^n \end{pmatrix}$$

*Alternation property* (w.r.t. rows) of determinants:

If $\boldsymbol{A}$ is an $n \times n$-matrix, and one interchanges two rows (say $i_1$ and $i_2$, where $i_1 \neq i_2$) of $\boldsymbol{A}$ to obtain another matrix $\tilde{\boldsymbol{A}}$, say, then:

$$\det \tilde{\boldsymbol{A}} \;=\; -\det \boldsymbol{A}.$$

**Further properties of determinants**

Let $\boldsymbol{A}$ be an $n \times n$ matrix: (i) $\det \boldsymbol{A}^{\mathrm{t}} \;=\; \det \boldsymbol{A}$. (So, as a consequence: multilinearity and alternation property do also hold *w.r.t. columns.*)
(ii) $\boldsymbol{A}$ is *regular* if and only if $\det \boldsymbol{A} \neq 0$.

(iii) If $\boldsymbol{B}$ is $n \times n$ as well, then: $\det \boldsymbol{AB} \;=\; \det \boldsymbol{A} \det \boldsymbol{B}$.

(iv) If $\boldsymbol{A}$ is a diagonal matrix, then the determinant is the product of the diagonal entries!

Consequences for *elementary row- or column operations*:

- If the $i$-th row of $\boldsymbol{A}$ is replaced by $\tilde{\boldsymbol{a}}^i = \boldsymbol{a}^i + \lambda \boldsymbol{a}^h$ (where $\boldsymbol{a}^i$ and $\boldsymbol{a}^h$ are the $i$-th and $h$-th row of $\boldsymbol{A}$, resp., $i \neq h$, and $\lambda$ is a scalar), then for the

new matrix $\tilde{\boldsymbol{A}}$ obtained one has: $\det \tilde{\boldsymbol{A}} = \det \boldsymbol{A}$. In other words: If we add a multiple of the $h$'th row to the $i$'th row of $\boldsymbol{A}$, we do not change the determinant (if $i \neq h$).

- If the $i$-th and $h$-th rows ($i \neq h$) of $\boldsymbol{A}$ are interchanged, then for the new matrix $\tilde{\boldsymbol{A}}$ obtained one has: $\det \tilde{\boldsymbol{A}} = -\det \boldsymbol{A}$.
- If one row of $\boldsymbol{A}$ is multiplied with a scalar $\lambda$, then for the new matrix $\tilde{\boldsymbol{A}}$ obtained one has: $\det \tilde{\boldsymbol{A}} = \lambda \det \boldsymbol{A}$.

Analogous statements hold for *columns* instead of rows.

**Expansion of a determinant along a row or a column**

Notation: If $\boldsymbol{A}$ is an $n \times n$-matrix, $k, \ell \in \{1, 2, \ldots, n\}$ are the chosen row- and column numbers, resp., then we denote:

$\boldsymbol{A}^{\setminus k}_{\setminus \ell}$ the $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the $k$-th row and the $\ell$-th column.

**Example**

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & -2 & 0 & 1 \end{pmatrix}; \qquad \boldsymbol{A}^{\setminus 2}_{\setminus 3} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$\det(\boldsymbol{A}) = \sum_{\ell=1}^{n} (-1)^{k+\ell} a_{k\ell} \det\big(\boldsymbol{A}^{\setminus k}_{\setminus \ell}\big) \qquad \text{(expansion along } k\text{-th row)}.$$

$$\det(\boldsymbol{A}) = \sum_{k=1}^{n} (-1)^{k+\ell} a_{k\ell} \det\big(\boldsymbol{A}^{\setminus k}_{\setminus \ell}\big) \qquad \text{(expansion along } \ell\text{-th column)}.$$

**Example:**

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$

We expand its determinant along the first row:

$$\begin{aligned} \det(\boldsymbol{A}) &= 2(-1)^{1+1} \det\big(\boldsymbol{A}^{\setminus 1}_{\setminus 1}\big) + 0(-1)^{1+2} \det\big(\boldsymbol{A}^{\setminus 1}_{\setminus 2}\big) \\ &\quad + 1(-1)^{1+3} \det\big(\boldsymbol{A}^{\setminus 1}_{\setminus 3}\big) + (-1)(-1)^{1+4} \det\big(\boldsymbol{A}^{\setminus 1}_{\setminus 4}\big) \\ &= 2 \det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{pmatrix} + \det \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix} + \det \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \\ &= 2(-9) + 19 + 4 = 5 . \end{aligned}$$

**Note:** If the $\ell$-th column of $\boldsymbol{A}$ has only one nonzero entry $a_{k\ell}$, say, (while $a_{i\ell} = 0$ for all $i \neq k$) then, by the column expansion formula: $\det \boldsymbol{A} = (-1)^{k+\ell} a_{k\ell} \det\big(\boldsymbol{A}^{\setminus k}_{\setminus \ell}\big)$ .

Particular case: $\ell = 1$ and $k = 1$

$$\boldsymbol{A} = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \dots & a_{1n} \\ \hline 0 & & & \\ \vdots & & \boldsymbol{A}^{\setminus 1}_{\setminus 1} & \\ 0 & & & \end{array}\right), \quad \det \boldsymbol{A} = a_{11} \det\big(\boldsymbol{A}^{\setminus 1}_{\setminus 1}\big)$$

**Example:**

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$

By row operations $\tilde{\boldsymbol{a}}^2 = \boldsymbol{a}^2 - \frac{3}{2}\boldsymbol{a}^1$ , $\tilde{\boldsymbol{a}}^4 = \boldsymbol{a}^4 - \boldsymbol{a}^1$ :

$$\det \boldsymbol{A} = \det \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -1 & 2 \end{pmatrix} = 2 \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & 1 & 2 \\ -2 & -1 & 2 \end{pmatrix} .$$

To the latter $3 \times 3$ matrix row operations $\tilde{\boldsymbol{r}}^2 = \boldsymbol{r}^2 - \boldsymbol{r}^1$ , $\tilde{\boldsymbol{r}}^3 = \boldsymbol{r}^3 + 2\boldsymbol{r}^1$ give:

$$\det \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & 1 & 2 \\ -2 & -1 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 5 \end{pmatrix} = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 5 \end{pmatrix} = \frac{5}{2} .$$

So the result is: $\det \boldsymbol{A} = 2 \cdot \frac{5}{2} = 5$ .

## 3.3 Eigenvalues

For an $n \times n$ *square* matrix $\boldsymbol{A}$, a (real) *eigenvalue* of $\boldsymbol{A}$ is a real number $\lambda$ such that there is a *nonzero* vector $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$ satisfying the equation

$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \quad (\boldsymbol{x} \neq \boldsymbol{0})$, (eigenvalue/eigenvector equation).

If $\lambda$ is an eigenvalue of $\boldsymbol{A}$, then every vector $\boldsymbol{x} \neq \boldsymbol{0}$ satisfying the eigenvalue/eigenvector equation is called an *eigenvector* of $\boldsymbol{A}$ to the eigenvalue $\lambda$.

**A result on eigenvalues**

A real number $\lambda$ is an eigenvalue of $\boldsymbol{A}$ if and only if
$\det\big(\boldsymbol{A} - \lambda \boldsymbol{I}_n\big) = 0$.

Note: This can be utilized for calculating eigenvalues
(see next: *characteristic polynomial*).

**Characteristic polynomial**

For an $n \times n$-matrix $\boldsymbol{A}$, the *polynomial of degree* $n$ (in a real variable $x$),

$$P_{\boldsymbol{A}}(x) \,=\, \det\bigl(\boldsymbol{A} - x\boldsymbol{I}_n\bigr)\;, \quad x \in \mathbb{R}\,,$$

is called the *characteristic polynomial* of $\boldsymbol{A}$.

The zeros of $P_{\boldsymbol{A}}(x)$ are the (real) eigenvalues of $\boldsymbol{A}$.

So there are at most $n$ eigenvalues of $\boldsymbol{A}$; it may happen that there are less than $n$ eigenvalues and even that there is *no* (real) eigenvalue of $\boldsymbol{A}$.

In fact: $P_{\boldsymbol{A}}(x)$ is polynomial of degree $n$, since

$$\det\bigl(\boldsymbol{A} - x\boldsymbol{I}_n\bigr) \;=\; \det\begin{pmatrix} a_{11}-x & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-x \end{pmatrix}$$

$$= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\,(a_{1\,\sigma(1)} - \delta_{1\,\sigma(1)}x)\,(a_{2\,\sigma(2)} - \delta_{2\,\sigma(2)}x)\,\cdots\cdot\,(a_{n\,\sigma(n)} - \delta_{n\,\sigma(n)}x)\,,$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$ (the so-called Kronecker delta).

More precisely, $P_{\boldsymbol{A}}(x) \,=\, c_n x^n + c_{n-1}x^{n-1} + \,\dots\, + c_1 x + c_0$ , where

$$c_n = (-1)^n\;,\quad c_{n-1} = (-1)^{n-1}\sum_{i=1}^{n} a_{ii}\;,\quad c_0 = \det(\boldsymbol{A})\;;$$

the remaining coefficients $c_j$ $(j = 1, \dots, n-2)$ are determined by the entries of $\boldsymbol{A}$ in a complicated way.

**Special case:** $n = 2$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $P_{\boldsymbol{A}}(x) = x^2 - (a_{11}+a_{22})x + \det(\boldsymbol{A})$.

Abbreviate: $\Delta \,=\, \frac{(a_{11}+a_{22})^2 - 4\det(\boldsymbol{A})}{4} \,=\, \frac{(a_{11}+a_{22})^2}{4} - \det(\boldsymbol{A})$.

Zeros of $P_{\boldsymbol{A}}(x)$, i.e., (real) eigenvalues of $\boldsymbol{A}$:

Case 1. If $\Delta > 0$, two zeros $\lambda_{1,2} = \frac{1}{2}(a_{11}+a_{22}) \pm \sqrt{\Delta}$ .

Case 2. If $\Delta = 0$, one zero $\lambda \,=\, \frac{1}{2}(a_{11}+a_{22})$ .

Case 3. If $\Delta < 0$, no zero.

**Special case:** $n = 3$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\;,\quad \boldsymbol{A} - x\boldsymbol{I}_3 \,=\, \begin{pmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{pmatrix}$$

$$\begin{aligned} P_{\boldsymbol{A}}(x) \,&=\, (a_{11}-x)\,(a_{22}-x)\,(a_{33}-x) \,+\, a_{12}a_{23}a_{31} \,+\, a_{13}a_{21}a_{32} \\ &\quad - a_{13}(a_{22}-x)\,a_{31} \,-\, (a_{11}-x)\,a_{23}a_{32} \,-\, a_{12}a_{21}(a_{33}-x) \\ &=\, -x^3 \,+\, (a_{11}+a_{22}+a_{33})\,x^2 \,+\, c_1 x \,+\, \det(\boldsymbol{A})\,, \end{aligned}$$

where $c_1 \,=\, -a_{22}a_{33} - a_{11}a_{33} - a_{11}a_{22} + a_{13}a_{31} + a_{23}a_{32} + a_{12}a_{21}$ .

Note: A 3rd degree polynomial has at least one zero.

**Special case:** Diagonal matrix, $\boldsymbol{D} = \operatorname{diag}\big(d_1, d_2, \ldots, d_n\big)$ ,

$$\begin{aligned} P_{\boldsymbol{D}}(x) &= (d_1 - x)(d_2 - x) \cdots (d_n - x) \\ &= (-1)^n \, (x - d_1) \, (x - d_2) \cdots (x - d_n) \, . \end{aligned}$$

So the eigenvalues of $\boldsymbol{D}$ are the diagonal entries $d_1, d_2, \ldots, d_n$ .

$\boldsymbol{e}_j$ (the $j$th elementary unit vector) is an eigenvector of $\boldsymbol{D}$ to the eigenvalue $d_j$ :
$\boldsymbol{D}\boldsymbol{e}_j = d_j \boldsymbol{e}_j \;\; (j = 1, 2, \ldots, n)$.

**Example:** $\boldsymbol{A} = \begin{pmatrix} 16 & 0 & 12 \\ 0 & 5 & 0 \\ 12 & 0 & 9 \end{pmatrix}$ .

$$P_{\boldsymbol{A}}(x) = \det \begin{pmatrix} 16 - x & 0 & 12 \\ 0 & 5 - x & 0 \\ 12 & 0 & 9 - x \end{pmatrix} = (5 - x) \det \begin{pmatrix} 16 - x & 12 \\ 12 & 9 - x \end{pmatrix}$$

$$= (5 - x)\big[\,(16 - x)(9 - x) - 144\big] = -(x - 5)\, x \, (x - 25) \, .$$

So $\boldsymbol{A}$ has three eigenvalues $\lambda_1 = 0$ , $\lambda_2 = 5$ , and $\lambda_3 = 25$ .

Find an eigenvector of $\boldsymbol{A}$ to the eigenvalue $\lambda_3 = 25$ !

Solve $\boldsymbol{A}\boldsymbol{x} = 25\boldsymbol{x}$ , i.e., $(\boldsymbol{A} - 25\boldsymbol{I}_3)\boldsymbol{x} = \boldsymbol{0}$ (hom. LES), i.e.,

$$\begin{pmatrix} -9 & 0 & 12 \\ 0 & -20 & 0 \\ 12 & 0 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \text{ solutions: } \boldsymbol{x} = a \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} .$$

Similarly, eigenvectors of $\boldsymbol{A}$ to $\lambda_2 = 5$ and to $\lambda_1 = 0$ are found:

for $\lambda_2 = 5$ : $\boldsymbol{x} = a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ; for $\lambda_1 = 0$ : $\boldsymbol{x} = a \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ .

Eigenvectors are obtained for $a \in \mathbb{R}$, $a \neq 0$.

## 3.4 Quadratic forms

### Quadratic form of a symmetric matrix

For a *symmetric* $n \times n$ matrix $\boldsymbol{A}$ its *quadratic form* is the following function of a vector variable $\boldsymbol{x} \in \mathbb{R}^{n \times 1}$ ,

$Q_{\boldsymbol{A}}(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{t}} \boldsymbol{A} \boldsymbol{x}$ , which is a *real valued* function.

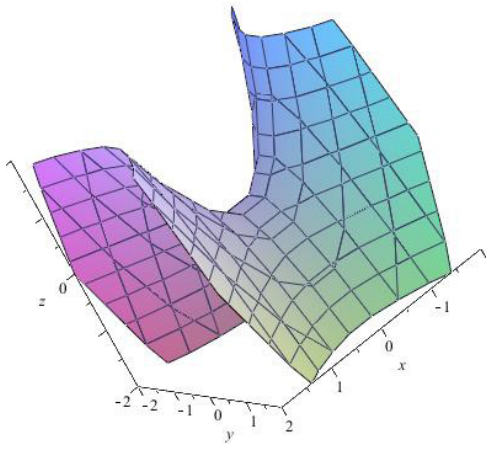
In fact: The matrix product $\boldsymbol{x}^{\mathrm{t}} \boldsymbol{A} \boldsymbol{x}$ is $1 \times 1$ , i.e., a real number;
it is the scalar product of the vectors $\boldsymbol{x}$ and $\boldsymbol{A}\boldsymbol{x}$ ;

more explicitly: $\boldsymbol{x}^{\mathrm{t}} \boldsymbol{A} \boldsymbol{x} = \sum\limits_{i,j=1}^{n} a_{ij} x_i x_j$ .

**Example:** $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

$$Q_{\boldsymbol{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1, x_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 - x_2^2 .$$

Here is an illustration:




**Positive definite matrix**

A *symmetric* $n \times n$ matrix $\boldsymbol{A}$ is called *positive definite* if its quadratic form is positive on $\mathbb{R}^{n\times 1} \setminus \{\boldsymbol{0}\}$ , i.e.,

$\boldsymbol{x}^{\mathrm{t}}\boldsymbol{A}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ .

A *symmetric* $n \times n$ matrix $\boldsymbol{B}$ is called *negative definite* if the matrix $\boldsymbol{A} = -\boldsymbol{B}$ is positive definite (or, equivalently, if $\boldsymbol{x}^{\mathrm{t}}\boldsymbol{B}\boldsymbol{x} < 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ ).

| A symmetric matrix is positive definite if and only if all its eigenvalues are positive. |
|---|

| For a symmetric $n \times n$ matrix $\boldsymbol{A}$ :<br>$\boldsymbol{A}$ is positive definite if and only if there is a *regular* $n \times n$ matrix $\boldsymbol{W}$ such that $\boldsymbol{A} = \boldsymbol{W}\boldsymbol{W}^{\mathrm{t}}$ .<br>Moreover: If $\boldsymbol{A}$ is positive definite then there exists a unique *positive definite* $n \times n$ matrix $\boldsymbol{A}^{1/2}$ such that $\boldsymbol{A} = \left(\boldsymbol{A}^{1/2}\right)^2$ . |
|---|

**Cholesky factorization**

If a symmetric $n \times n$ matrix $\boldsymbol{A}$ is positive definite, then there exists a *lower triangular* matrix $\boldsymbol{L}$ such that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^{\mathrm{t}}$ .

Note: An $n \times n$ matrix $\boldsymbol{L} = \left(\ell_{ij}\right)_{i,j=1,2,\ldots,n}$ is said to be *lower triangular* if $\ell_{ij} = 0$ for all $i < j$ .

**Positive definite $\iff$ principal minors are positive**

Notation: For an $n \times n$ matrix $\boldsymbol{A} = \left(a_{ij}\right)_{i,j=1,\ldots,n}$, we denote

$$\boldsymbol{A}^{1,\ldots,k}_{1,\ldots,k} = \left(a_{ij}\right)_{i,j=1,2,\ldots,k} , \quad \text{where } k \in \{1, 2, \ldots, n\} .$$

(This is the upper left $k \times k$ submatrix of $\boldsymbol{A}$ .)

For a symmetric $n \times n$ matrix $\boldsymbol{A}$ :
$\boldsymbol{A}$ is positive definite if and only if

$$\det\left(\boldsymbol{A}_{1,\ldots,k}^{1,\ldots,k}\right) > 0 \quad \text{for all } k = 1, 2, \ldots, n\,.$$

Remark: These (sub-)determinants are called the *principal minors* of $\boldsymbol{A}$.

**Special case** $n = 2$.

For a symmetric $2 \times 2$ matrix $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, the result says:
$\boldsymbol{A}$ is positive definite if and only if $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$.

# 4 Complex numbers

## 4.1 The first view

$\mathbb{C}$ (set of complex numbers) is the same as $\mathbb{R}^2$ (Euclidean plane), up to different notation:

(i) Write the real number 1 instead of the first elementary unit vector $\boldsymbol{e}_1 = (1,0)$, and (consequently) write the real number $\alpha$ instead of $\alpha\boldsymbol{e}_1 = (\alpha,0)$;

(ii) write $\boldsymbol{i}$ (the imaginary unit) instead of the second elementary unit vector $\boldsymbol{e}_2 = (0,1)$, and (consequently) for $\beta \in \mathbb{R}$: write $\beta\boldsymbol{i}$ instead of $\beta\boldsymbol{e}_2 = (0,\beta)$;

(iii) for $\alpha, \beta \in \mathbb{R}$: write $\alpha + \beta\boldsymbol{i}$ instead of $\alpha\boldsymbol{e}_1 + \beta\boldsymbol{e}_2 = (\alpha,\beta)$.

So: $\mathbb{C} = \{\alpha + \beta\boldsymbol{i} : \alpha, \beta \in \mathbb{R}\} = \mathbb{R}^2$;
for $z = \alpha + \beta\boldsymbol{i}$ one calls $\alpha$ the *real part* of $z$ and $\beta$ the *imaginary part* of $z$, for short: $\alpha = \mathrm{Re}(z)$ and $\beta = \mathrm{Im}(z)$.

## 4.2 The field of complex numbers: algebraic structure

Besides the vector space structure

$$(\alpha_1 + \beta_1\boldsymbol{i}) + (\alpha_2 + \beta_2\boldsymbol{i}) = (\alpha_1 + \alpha_2) + (\beta_2 + \beta_2)\boldsymbol{i} \quad \text{(addition)},$$
$$\lambda(\alpha + \beta\boldsymbol{i}) = (\alpha + \beta\boldsymbol{i})\lambda = (\lambda\alpha) + (\lambda\beta)\boldsymbol{i} \quad \text{(multiplication by a real)},$$

one defines: **multiplication** in $\mathbb{C}$; **reciprocal** and **division**.

$$(\alpha_1 + \beta_1\boldsymbol{i})(\alpha_2 + \beta_2\boldsymbol{i}) = (\alpha_1\alpha_2 - \beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)\boldsymbol{i}$$

Comment: We have formally multiplied out and used $\boldsymbol{i}^2 = -1$.

If $z = \alpha + \beta\boldsymbol{i} \neq 0$ (i.e., *not* $\alpha = \beta = 0$), then

$$\frac{1}{z} = \frac{1}{\alpha^2 + \beta^2}\big(\alpha - \beta\boldsymbol{i}\big),$$

i.e., with this $\frac{1}{z}$ one has $\frac{1}{z}\, z = 1$.

So the *division* of any two complex numbers (but, of course, the denominator must be nonzero) is given by:

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

if $z_2 \neq 0$. Multiplication in $\mathbb{C}$ extends the well-known multiplication in $\mathbb{R}$ preserving its properties (associativity, commutativity, distributivity):

$$(z_1 z_2)z_3 = z_1(z_2 z_3), \quad z_1 z_2 = z_2\, z_1, \quad (z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3\,.$$

Recall: *absolute value* $|z| = \sqrt{\alpha^2 + \beta^2}$ for $z = \alpha + \beta\boldsymbol{i}$.
Properties: $|z_1 + z_2| \leq |z_1| + |z_2|$, $|z_1 z_2| = |z_1|\,|z_2|$.

## 4.3 Conjugation

For $z = \alpha + \beta\boldsymbol{i}$, one defines $\overline{z} = \alpha - \beta\boldsymbol{i}$, and calls $\overline{z}$ the *conjugated complex* of $z$.

The function $z \mapsto \overline{z}$ from $\mathbb{C}$ to $\mathbb{C}$ is an *automorphism*, i.e.:
$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z}_1\,\overline{z_2}$.

Interesting to note: $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$, $\mathrm{Im}(z) = \frac{1}{2\boldsymbol{i}}(z - \overline{z})$,
$z\overline{z} = |z|^2$ and $\frac{1}{z} = \frac{1}{|z|^2}\,\overline{z}$.

## 4.4 Complex exponential function

Continuation of the *real* exponential function $\exp(x) = e^x$, $x \in \mathbb{R}$, to a *complex-valued* function of a *complex* variable,
$\exp(z) = e^z$, $z \in \mathbb{C}$, by:

$$e^z = e^\alpha \big(\cos(\beta) + \sin(\beta)\boldsymbol{i}\big)$$

for all $z = \alpha + \beta\boldsymbol{i}$. Using well-known properties of the *real* exponential function and of the sine and cosine functions we see: basic property (power law) of exp is preserved:

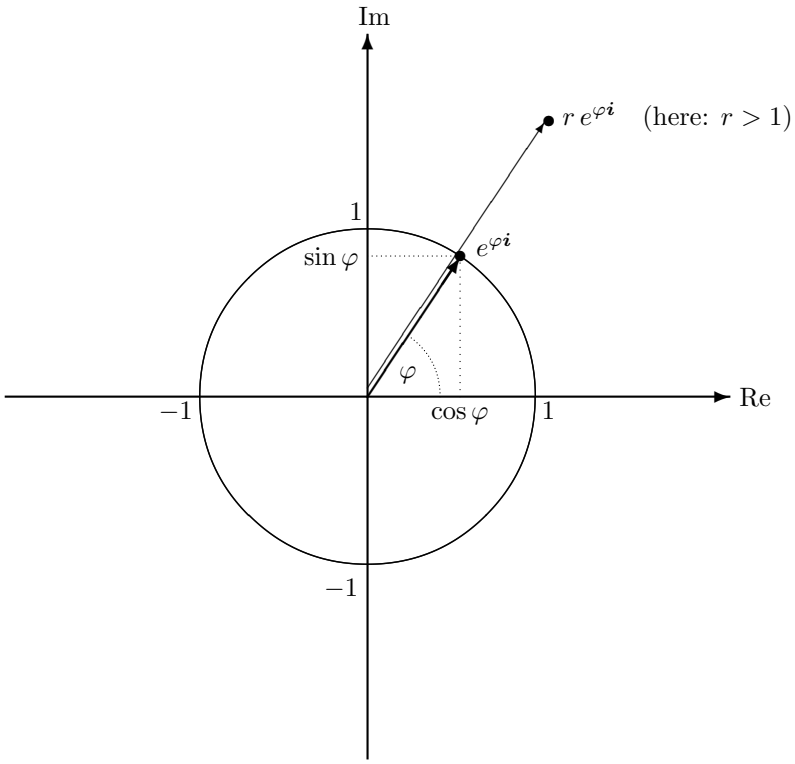
$$e^{z_1 + z_2} = e^{z_1}\, e^{z_2}$$

for all $z_1, z_2 \in \mathbb{C}$.
The complex exponential function for *purely imaginary* arguments:
$e^{\varphi\boldsymbol{i}} = \cos(\varphi) + \sin(\varphi)\boldsymbol{i}$, for any $\varphi \in \mathbb{R}$,
and these complex numbers (as $\varphi$ varies over $\mathbb{R}$ or only over an interval of length $2\pi$) constitute the *unit circle* $\{z \in \mathbb{C} : |z| = 1\}$.

**Euler representation** of a complex number $z = \alpha + \beta\boldsymbol{i}$:

$$z = r\, e^{\varphi\boldsymbol{i}}, \quad \text{where } r = |z| \text{ and } \varphi \in [0\,,\, 2\pi[$$

(corresponds to *polar coordinates* of points in the Euclidean plane).




Note: A geometric interpretation of complex multiplication

For $z_1, z_2 \in \mathbb{C}$, write $z_1 = r_1 e^{\varphi_1 \boldsymbol{i}}$ and $z_2 = r_2 e^{\varphi_2 \boldsymbol{i}}$, then

$$z_1 z_2 \;=\; r_1 r_2\, e^{(\varphi_1+\varphi_2)\boldsymbol{i}}.$$

**Complex representations of sine and cosine**

$$\cos\varphi \;=\; \mathrm{Re}\big(e^{\varphi \boldsymbol{i}}\big) \;=\; \tfrac{1}{2}\left(e^{\varphi \boldsymbol{i}} + e^{-\varphi \boldsymbol{i}}\right),$$
$$\sin\varphi \;=\; \mathrm{Im}\big(e^{\varphi \boldsymbol{i}}\big) \;=\; \tfrac{1}{2\boldsymbol{i}}\left(e^{\varphi \boldsymbol{i}} - e^{-\varphi \boldsymbol{i}}\right).$$

Note: $\overline{e^{\varphi \boldsymbol{i}}} \;=\; \cos(\varphi) - \sin(\varphi)\boldsymbol{i} \;=\; \cos(-\varphi) + \sin(-\varphi)\boldsymbol{i} \;=\; e^{-\varphi \boldsymbol{i}}$.

**Example:** Deriving trigonometric formula

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta):$$

$$\begin{aligned}
\text{l.h.s.} \;&=\; \mathrm{Re}\big(e^{(\alpha+\beta)\boldsymbol{i}}\big) \;=\; \mathrm{Re}\big(e^{\alpha \boldsymbol{i}} e^{\beta \boldsymbol{i}}\big) \\
&=\; \mathrm{Re}\Big(\big(\cos(\alpha) + \sin(\alpha)\boldsymbol{i}\big)\big(\cos(\beta) + \sin(\beta)\boldsymbol{i}\big)\Big) \\
&=\; \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \;= \text{r.h.s.}
\end{aligned}$$

**Example:** One-dimensional harmonic oscillation

$$s(t) \;=\; a\,\cos\big(\omega t + \varphi_0\big)\,, \quad t \text{ a real variable (time)},$$

($a > 0$ amplitude, $\omega > 0$ angular frequency, $\varphi_0 \in [0\,,\,2\pi[$ phase shift).

Complex representation: $S(t) = A\,e^{\omega t \boldsymbol{i}}$ , where $A = ae^{\varphi_0 \boldsymbol{i}}$ the *complex amplitude*.

**Note:** $s(t) = \mathrm{Re}\big(S(t)\big)$ , since $a\cos(\omega t+\varphi_0) = \mathrm{Re}\big(ae^{(\omega t+\varphi_0)\boldsymbol{i}}\big) = \mathrm{Re}\big(\underbrace{ae^{\varphi_0 \boldsymbol{i}}}_{=A} e^{\omega t \boldsymbol{i}}\big)$.

## 4.5 Complex polynomials

Well-known: A *real* polynomial of degree $n$

$P(x) = c_n x^n + c_{n-1}x^{n-1} + \ \ldots\ + c_1 x + c_0$ , $x \in \mathbb{R}$,

(with real coefficients $c_0, c_1, \ldots, c_n$ , where $c_n \neq 0$)

may not have any zero, i.e., there may not exist an $x_0 \in \mathbb{R}$ with $P(x_0) = 0$.

**Example:** A second degree polynomial $P(x) = x^2 + px + q$ , where $p, q \in \mathbb{R}$ are such that $\Delta = \left(\frac{p}{2}\right)^2 - q < 0$ , does not have any (real) zeros.

However, for a *complex polynomial* of degree $n$ ,

$P(z) = c_n z^n + c_{n-1}z^{n-1} + \ \ldots\ + c_1 z + c_0$ , $z \in \mathbb{C}$,

(with complex coefficients $c_0, c_1, \ldots, c_n$ , where $c_n \neq 0$)

the situation is completely different.

**Fundamental theorem on complex polynomials**

If $P(z)$ is a complex polynomial of degree $n \geq 1$, then there exists a zero of $P(z)$, i.e., a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$;
and, moreover:
There exist $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that $P(z)$ factorizes to

$P(z) = c_n(z - z_1)(z - z_2) \cdots (z - z_n)$ , for all $z \in \mathbb{C}$,

(where $c_n \neq 0$ is the highest coefficient of $P(z)$ ).

Note: The complex numbers $z_1, z_2, \ldots, z_n$ (the zeros of $P(z)$) need not be all distinct.

**Example:** The above 2nd degree polynomial, but now with a *complex* variable:
$P(z) = z^2 + pz + q$ , $z \in \mathbb{C}$ ,
where $p, q \in \mathbb{R}$ such that $\Delta = \left(\frac{p}{2}\right)^2 - q < 0$ .
Not difficult to check:

$P(z) = (z - z_1)(z - z_2)$ , where $z_{1,2} = -\frac{p}{2} \pm \sqrt{|\Delta|}\,\boldsymbol{i}$ .

# 5 Derivatives of functions of one real variable

## 5.1 Basics on derivatives

Derivative of $f$ at $x_0$ :

$$f'(x_0) \;=\; \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Other symbols for the derivative $f'(x_0)$ :

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) \quad \text{or} \quad \frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x=x_0}, \quad \text{etc.}$$

**Local linear approximation of $f(x)$ :**

The *tangent* as a (linear) function approximates $f(x)$ if $x$ is close to $x_0$ :

$$f(x) \;\approx\; f(x_0) \,+\, f'(x_0)\,(x - x_0).$$

$f$ is called *differentiable* on its domain (usually an interval $I \subseteq \mathbb{R}$), if the derivative $f'(x_0)$ exists for each $x_0 \in I$ . Then $x = x_0 \in I$ is a *variable*, and we have *the function* $f'(x)$ or shortly $f'$ , *the derivative of* $f$. The derivative of $f'$ at $x_0$ (if it exists) is called the *second derivative of* $f$ *at* $x_0$ and is denoted by $f''(x_0)$ . If $f'$ is differentiable on its domain $I$ then we have the function $f''(x)$ or shortly $f''$ , *the second derivative of* $f$.

**Local quadratic approximation of $f(x)$ :** If $x$ is close to $x_0$ , then

$$f(x) \;\approx\; f(x_0) \,+\, f'(x_0)\,(x - x_0) \,+\, \frac{1}{2} f''(x_0)\,(x - x_0)^2.$$

In particular, when $f'(x_0)$ is nearly zero, then the local quadratic approximation is considerably better than the local *linear* approximation.

| function $f$ | derivative $f'$ | domain $I$ |
|---|---|---|
| $c$ (constant) | $0$ | $\mathbb{R}$ |
| $x^m$ , $(m \in \mathbb{N})$ | $m\,x^{m-1}$ | $\mathbb{R}$ |
| $x^m$ , $(m \in \mathbb{Z},\, m \le -1)$ | $m\,x^{m-1}$ | $]\,0\,,\,\infty\,[$ or $]-\infty\,,\,0\,[$ |
| $x^\alpha$ , $(\alpha \in \mathbb{R})$ | $\alpha\,x^{\alpha-1}$ | $]\,0\,,\,\infty[$ |
| $\ln x$ | $\frac{1}{x}$ | $]\,0\,,\,\infty[$ |
| $\ln(\lvert x\rvert)$ | $\frac{1}{x}$ | $]\,0\,,\,\infty\,[$ or $]-\infty\,,\,0\,[$ |
| $e^x$ | $e^x$ | $\mathbb{R}$ |
| $\sin x$ | $\cos(x)$ | $\mathbb{R}$ |
| $\cos x$ | $-\sin(x)$ | $\mathbb{R}$ |
| $\tan x$ | $\frac{1}{\cos^2(x)}$ | $\big]\,(k-\frac{1}{2})\pi\,,\,(k+\frac{1}{2})\pi\,\big[\,,\ (k \in \mathbb{Z})$ |
| $\cot x$ | $-\frac{1}{\sin^2(x)}$ | $\big]\,k\pi\,,\,(k+1)\pi\,\big[\,,\ (k \in \mathbb{Z})$ |
| $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ | $]\,-1\,,\,1\,[$ |
| $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ | $]\,-1\,,\,1\,[$ |
| $\arctan x$ | $\frac{1}{1+x^2}$ | $\mathbb{R}$ |

**Useful rules for derivatives**

For functions $f$ and $g$ on a common domain $I$ :

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[\,\alpha f(x)+\beta g(x)\,\big] \;=\; \alpha f'(x)+\beta g'(x)\;,\quad (\text{for } \alpha,\beta \in \mathbb{R})\;;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[\,f(x)\,g(x)\,\big] \;=\; f'(x)\,g(x)\;+\;f(x)\,g'(x)\;;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{f(x)}{g(x)} \;=\; \frac{f'(x)\,g(x)\;-\;f(x)\,g'(x)}{\big(\,g(x)\,\big)^2}\;,\quad (\text{if } g(x)\neq 0 \text{ for all } x \in I).$$

If $\varphi=\varphi(x)$ and $f=f(y)$ are two functions such that the *composition* $f\big(\,\varphi(x)\,\big)$ exists, then the *chain rule* says:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,f\big(\,\varphi(x)\,\big) \;=\; f'(y)\,\varphi'(x) \quad \text{with} \quad y=\varphi(x)\;.$$

**Examples:** $\frac{\mathrm{d}}{\mathrm{d}x}\,e^{\lambda x} \;=\; \lambda\,e^{\lambda x}$ , $\frac{\mathrm{d}}{\mathrm{d}x}\,\exp\big(-c\,x^2\big) \;=\; -2c\,x\,\exp\big(-c\,x^2\big)$

$\frac{\mathrm{d}}{\mathrm{d}x}\,\cosh x \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\Big(e^x+e^{-x}\Big)\right] \;=\; \frac{1}{2}\Big(e^x\;-\;e^{-x}\Big) \;=\; \sinh x\;;$

$\frac{\mathrm{d}}{\mathrm{d}x}\big[x\;-\;\cos(x)\sin(x)\,\big] \;=\; 1\;+\;\sin^2(x)\;-\cos^2(x) \;=\; 2\sin^2(x)\;;$

$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{x^2-1}{x^2+1} \;=\; \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} \;=\; \frac{4x}{(x^2+1)^2}\;;$

$\frac{\mathrm{d}}{\mathrm{d}x}\,\arctan\sqrt{x} \;=\; \frac{1}{1+(\sqrt{x})^2}\,\frac{1}{2}\,x^{-\frac{1}{2}} \;=\; \frac{1}{2(1+x)\sqrt{x}}\;.$

## 5.2 Derivatives of vector-valued functions

**Vector-valued functions of one real variable**

$\boldsymbol{f}(x) = \big(f_1(x), f_2(x), \ldots, f_p(x)\big)$ , $x \in I \subseteq \mathbb{R}$, i.e., $\boldsymbol{f}(x) \in \mathbb{R}^p$. The $f_i$ are called the *component functions*. Derivative of $\boldsymbol{f}$ at $x_0 \in I$: the derivative $\boldsymbol{f}'(x_0)$ is explained as

$$\boldsymbol{f}'(x_0) = \lim_{x_1 \to x_0} \frac{\boldsymbol{f}(x_1) - \boldsymbol{f}(x_0)}{x_1 - x_0}.$$

Easy to see: $\boldsymbol{f}'(x_0) = \big(f_1'(x_0), f_2'(x_0), \ldots, f_p'(x_0)\big)$. If the derivatives $\boldsymbol{f}'(x_0)$ exist *at each* $x_0 = x \in I$, then the derivative $\boldsymbol{f}'$ of $\boldsymbol{f}$ is the $\mathbb{R}^p$-valued function

$$\boldsymbol{f}'(x) = \big(f_1'(x), f_2'(x), \ldots, f_p'(x)\big), \quad x \in I.$$

If, furthermore, the derivatives of $\boldsymbol{f}'$ exist on $I$, then we may also define a second derivative:

$$\boldsymbol{f}''(x) = \big(f_1''(x), f_2''(x), \ldots, f_p''(x)\big), \quad x \in I.$$

Other notations, e.g.: $\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{f}(x) = \boldsymbol{f}'(x)$ ; $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{f}(x) = \boldsymbol{f}''(x)$ .

**Local linear approximation of $\boldsymbol{f}(x)$ :**

$$\boldsymbol{f}(x) \approx \boldsymbol{f}(x_0) + \boldsymbol{f}'(x_0)\,(x - x_0),$$

for $x$ close to $x_0$.

**Local quadratic approximation of $\boldsymbol{f}(x)$ :**

$$\boldsymbol{f}(x) \approx \boldsymbol{f}(x_0) + \boldsymbol{f}'(x_0)\,(x - x_0) + \frac{1}{2}\boldsymbol{f}''(x_0)\,(x - x_0)^2,$$

for $x$ close to $x_0$.

**Example**

$\boldsymbol{f}(t) = r \cdot \big(\cos(\omega t), \sin(\omega t)\big)$ , $t \in [0, \infty[$. The variable $t$ is interpreted as *time*; $r > 0$, $\omega > 0$ given constants. $\boldsymbol{f}(t)$ models *uniform circular motion* of a particle.

*Velocity:* $\boldsymbol{v}(t) = \boldsymbol{f}'(t) = r\,\omega\,\big(-\sin(\omega t), \cos(\omega t)\big)$ ;
note: $|\boldsymbol{v}(t)| = r\omega$ , $\boldsymbol{v}(t) \perp \boldsymbol{f}(t)$ .

*Acceleration:* $\boldsymbol{a}(t) = \boldsymbol{f}''(t) = -r\,\omega^2\big(\cos(\omega t), \sin(\omega t)\big)$ ;
note: $|\boldsymbol{a}(t)| = r\omega^2$ , $\boldsymbol{a}(t)$ has opposite direction as $\boldsymbol{f}(t)$ .

**Example: a complex-valued function of one real variable.**

$f(x) = e^{c\,x}$ , $x \in \mathbb{R}$ , where $c \in \mathbb{C}$ a *complex* constant.

$c = \gamma_1 + \gamma_2 \boldsymbol{i}$ , so $cx = \gamma_1 x + \gamma_2 x \boldsymbol{i}$ , and:

$$
\begin{aligned}
f_1(x) \;&=\; \mathrm{Re}\big(f(x)\big) \;=\; e^{\gamma_1 x}\,\cos(\gamma_2 x)\;, \\
f_2(x) \;&=\; \mathrm{Im}\big(f(x)\big) \;=\; e^{\gamma_1 x}\,\sin(\gamma_2 x)\;; \quad \text{so:} \\
f'(x) \;&=\; f_1'(x) + f_2'(x)\boldsymbol{i} \;=\; \gamma_1 e^{\gamma_1 x}\cos(\gamma_2 x) \;-\; e^{\gamma_1 x}\,\gamma_2\sin(\gamma_2 x) \\
&\qquad\qquad + \big[\gamma_1 e^{\gamma_1 x}\sin(\gamma_2 x) \;+\; e^{\gamma_1 x}\,\gamma_2\cos(\gamma_2 x)\big]\,\boldsymbol{i} \\
&= e^{\gamma_1 x}\,\big(\gamma_1 + \gamma_2\boldsymbol{i}\big)\,\big(\cos(\gamma_2 x) + \sin(\gamma_2 x)\boldsymbol{i}\big) \;=\; c\,e^{cx}\;.
\end{aligned}
$$

| Result: $\frac{\mathrm{d}}{\mathrm{d}x}\,e^{cx} \;=\; c\,e^{cx}$ (same formula for *complex* $c$ as for *real* $c$) |
|---|

## 5.3 First order ordinary differential equations

Given a real-valued function of *two* real variables defined on some domain $M \subseteq \mathbb{R}^2$ : $\quad G(x,y)\;,\;\; (x,y) \in M \subseteq \mathbb{R}^2\;;$ shortly: $\;G : M \longrightarrow \mathbb{R}\,.$

Then an *ordinary diffential equation* (ODE) is given by

$$y' \;=\; G(x,y),$$

which is to mean: find a (real-valued) function $f(x)$ of *one* real variable $x \in I\,,$ where $I \subseteq \mathbb{R}$ is some interval, such that:

$$\big(x\,,\, f(x)\,\big) \in M$$

and

$$f'(x) \;=\; G\big(x\,,\, f(x)\,\big)$$

for all $x \in I\,.$

Common notation: symbol $y \;=\; y(x)$ instead of $f(x)$ , although somewhat ambiguous: $y$ denotes also a real variable in $G = G(x,y)\,.$

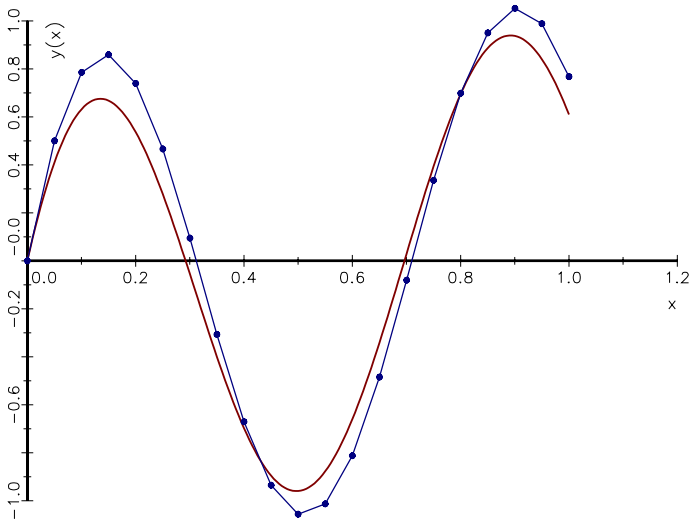
**Example:** ODE $y' = 1 + y^2$ , i.e., $G(x,y) = 1 + y^2$ and $M = \mathbb{R}^2.$
Solutions: $y_c(x) = \tan(x-c)$ , $x \in \,\big]c - \frac{\pi}{2}\,,\, c + \frac{\pi}{2}\big[$ , for $c \in \mathbb{R}.$
In general: An ODE $y' = G(x,y)$ has *many* solutions $y = y(x)\,.$ Selection of *one* solution by further requirements:

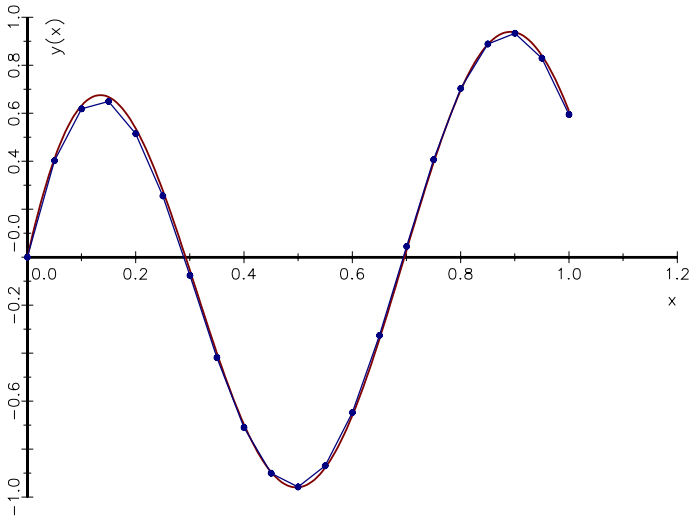
- *Initial values* prescribed: given a point $(x_0, y_0) \in M\,,$
  the solution $y(x)$ of the ODE has to satisfy $y(x_0) = y_0\,,$
  and of course: $x_0$ must belong to the domain $I$ of $y(x).$
- *Asymptotic behaviour*, e.g., $y(x) \to 0$ as $x \to \pm\infty$
  (in case that solutions on $I = \mathbb{R}$ exist).

**Computing approximate solution** to: $\;y' \;=\; G(x,y)\;,\; y(x_0) = y_0$
Choose step size $h\,,$ choose total number $K$ of steps; starting with $(x_0, y_0)\,,$ generate points $(x_k, y_k)\,,\; k = 1, 2, \ldots, K$:

$x_k = x_{k-1} + h\;,\quad y_k = y_{k-1} + h \cdot G\big(x_{k-1}, y_{k-1}\big)$ (Euler method).

The points $(x_k, y_k)$ $(k = 0, 1, \ldots, K)$ belong to the graph of an *approximate solution* $y(x)\,,$ (use linear interpolation).
Method based on local linearization $y(x) \;\approx\; y(x_{k-1}) \;+\; y'(x_{k-1})\,(x - x_{k-1})\;;$
here: $\;x = x_{k-1} + h\;,\quad y'(x_{k-1}) = G\big(x_{k-1}, y(x_{k-1})\big)\,.$


y(x)
1.0
0.8
0.6
0.4
0.0
−0.2
−0.6
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x


y(x)
1.0
0.8
0.6
0.4
0.0
−0.2
−0.6
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
x

The graphs show (approximate) solutions to the ODE $y' = -7y + 10\cos(8x)$, $y(0) = 0$ .
In the first graph, we used Euler approximation with $K = 20$, $h = 0.05$, in the second graph we used an improved Euler method with $K = 20$, $h = 0.05$. The improved Euler works as follows:

$\tilde{x}_k = x_{k-1} + \frac{h}{2}$ , $\tilde{y}_k = y_{k-1} + \frac{h}{2} G(x_{k-1}, y_{k-1})$ ,
$x_k = x_{k-1} + h$ , $y_k = y_{k-1} + h\, G(\tilde{x}_k, \tilde{y}_k)$ .

Exact solution:

$$y(x) = \frac{10}{\sqrt{113}} \cos(8x - \beta) - \frac{70}{113} e^{-7x}, \quad \beta = \arctan\left(\frac{8}{7}\right).$$

**1st order ODE for *vector-valued* functions**

$$\boldsymbol{y}' = \boldsymbol{G}(x, \boldsymbol{y}), \qquad \boldsymbol{y} = \boldsymbol{y}(x)$$

$\mathbb{R}^p$-valued, where $\boldsymbol{G} = \boldsymbol{G}(x, \boldsymbol{y})$ a given $\mathbb{R}^p$-valued function of $p + 1$ real variables.

• The numerical solution methods from above carry over "word by word" to the vector-valued case.

• A higher order ODE (for real-valued functions) can formally be written as a 1st order ODE for vector-valued functions.

Consequence: Numerical solution methods for higher order ODEs (for real-valued functions) with initial values.

**Example:** 2nd order ODE $y'' = -c \sin(y)$ , ($c > 0$ given), can formally be written as a 1st order ODE for an $\mathbb{R}^2$-valued function $\boldsymbol{y}(x) = \big(y_1(x), y_2(x)\big)$ :

$$y_1' = y_2, \quad y_2' = -c \sin(y_1),$$

and, in vector notation,

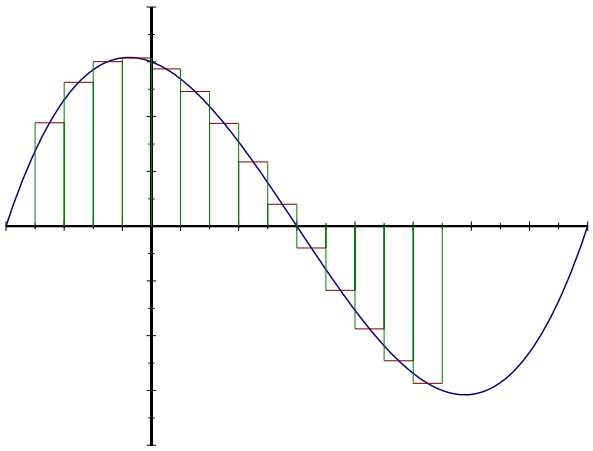
$$\boldsymbol{y}' = \boldsymbol{G}(x, \boldsymbol{y}),$$

where

$$\boldsymbol{G}(x, y_1, y_2) = \big(\, y_2 \,,\, -c \sin(y_1) \,\big).$$

# 6 Integrals of functions of one real variable

## 6.1 Basics on integrals



Discretization of the interval $[a\,,\,b]$ :

$$\mathcal{D}_k \;=\; \begin{pmatrix} x_0 & x_1 & \dots & x_{k-1} & x_k \\ & z_1 & \dots & z_{k-1} & z_k \end{pmatrix}$$

$x_0 = a \,<\, x_1 \,<\, \ldots \,<\, x_{k-1} \,<\, x_k = b$ and $z_i \in [\,x_{i-1}\,,\,x_i\,]\;\; (i = 1, \ldots, k)\,.$

$$S(f, \mathcal{D}_k) \;=\; \sum_{i=1}^{k} f(z_i)\,(x_i - x_{i-1}) \approx \int_a^b f(x)\,\mathrm{d}x$$

In the limit, as $k \to \infty$ and $|\mathcal{D}_k| \to 0\,,$ where $|\mathcal{D}_k| \;=\; \max\limits_{i=1,\ldots,k}(x_i - x_{i-1})$ :

$$\lim_{k\to\infty,\;|\mathcal{D}_k|\to 0} S(f, \mathcal{D}_k) \;=\; \int_a^b f(x)\,\mathrm{d}x$$

(which is a real number).

**Properties: linearity, monotonicity, additivity**

- $\int\limits_a^b \big(\,\alpha f(x) \,+\, \beta g(x)\,\big)\,\mathrm{d}x \;=\; \alpha \int\limits_a^b f(x)\,\mathrm{d}x \,+\, \beta \int\limits_a^b g(x)\,\mathrm{d}x.$
- If $f(x) \le g(x)$ for all $x \in [\,a\,,\,b\,]$, then $\;\int\limits_a^b f(x)\,\mathrm{d}x \;\le\; \int\limits_a^b g(x)\,\mathrm{d}x.$

- $\int_a^b f(x)\,\mathrm{d}x \;=\; \int_a^c f(x)\,\mathrm{d}x \;+\; \int_c^b f(x)\,\mathrm{d}x \qquad \text{for } a < c < b\,.$

**Extensions to *unbounded* integration intervals:**

$$\int_a^\infty f(x)\,\mathrm{d}x \;=\; \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x\;, \qquad \int_{-\infty}^b f(x)\,\mathrm{d}x \;=\; \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x,$$

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x \;=\; \int_{-\infty}^c f(x)\,\mathrm{d}x \;+\; \int_c^\infty f(x)\,\mathrm{d}x.$$

**Examples:** $\int_0^\infty e^{-x}\,\mathrm{d}x \;=\; 1\;, \qquad \int_{-\infty}^\infty \exp(-\tfrac{1}{2}x^2)\,\mathrm{d}x \;=\; \sqrt{2\pi}.$

**Indefinite integral or antiderivative**

For a function $f(x)\,,\ x \in I$, an *indefinite integral* or an *antiderivative* of $f$ is a function $F(x)\,,\ x \in I$, such that $F'(x) \;=\; f(x)$ for all $x \in I\,;$ then one writes

$$F(x) \;=\; \int f(x)\,\mathrm{d}x\;.$$

If $F(x)$ is an indefinite integral of $f(x)$, then so is $F(x) + c$ for any constant $c$. Conversely: if $F_1$ and $F_2$ are indefinite integrals of $f$, then there is a constant $c$ such that $F_1(x) - F_2(x) \;= c$ for all $x \in I$. So 'the indefinite integral' $\int f(x)\,\mathrm{d}x$ is not a unique function, but rather a *set of functions* any two of which differ only by an additive constant.

**Main Theorem.** If $F(x)\,,\ x \in I$, is an indefinite integral of $f(x)\,,\ x \in I\,,$ then for any subinterval $[a\,,\,b] \subseteq I\,:$

$$\int_a^b f(x)\,\mathrm{d}x \;=\; F(b) - F(a)\;.$$

| function $f$ | indefinite integral $F$ | domain $I$ |
|---|---|---|
| $c$ (constant) | $cx$ | $\mathbb{R}$ |
| $x^m\ ,\quad (m \in \mathbb{N})$ | $\frac{1}{m+1}\, x^{m+1}$ | $\mathbb{R}$ |
| $x^m\ ,\quad (m \in \mathbb{Z},\, m \leq -2)$ | $\frac{1}{m+1}\, x^{m+1}$ | $]\,0\,,\,\infty\,[\quad$ or $\quad]-\infty\,,\,0\,[$ |
| $x^\alpha\ ,\quad (\alpha \in \mathbb{R}\,,\, \alpha \neq -1)$ | $\frac{1}{\alpha+1}\, x^{\alpha+1}$ | $]\,0\,,\,\infty[$ |
| $\frac{1}{x}$ | $\ln x$ | $]\,0\,,\,\infty[$ |
| $\frac{1}{x}$ | $\ln(\lvert x\rvert)$ | $]\,0\,,\,\infty\,[\quad$ or $\quad]-\infty\,,\,0\,[$ |
| $e^x$ | $e^x$ | $\mathbb{R}$ |
| $\sin x$ | $-\cos(x)$ | $\mathbb{R}$ |
| $\cos x$ | $\sin(x)$ | $\mathbb{R}$ |
| $\frac{1}{\sin^2 x}$ | $-\cot x$ | $\big]\, k\pi\,,\,(k+1)\pi\,\big[\ ,\ (k \in \mathbb{Z})$ |
| $\frac{1}{\cos^2 x}$ | $\tan x$ | $\big]\,(k-\frac{1}{2})\pi\,,\,(k+\frac{1}{2})\pi\,\big[\ ,\ (k \in \mathbb{Z})$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin x$ | $]\,-1\,,\,1\,[$ |
| $\frac{1}{1+x^2}$ | $\arctan x$ | $\mathbb{R}$ |

It is easy to see that the indefinite integral is linear, i.e.,

$$\int \big(\alpha f(x) + \beta g(x)\big)\,\mathrm{d}x \;=\; \alpha \int f(x)\,\mathrm{d}x \;+\; \beta \int g(x)\,\mathrm{d}x \;+\; c \quad (x \in I).$$

**Useful rule: integration by parts.**

If $u(x)$ and $v(x)$ , $x \in I$ , are two differentiable functions, then:

$$\int u'(x)\, v(x)\,\mathrm{d}x \;=\; u(x)\, v(x) \;-\; \int u(x)\, v'(x)\,\mathrm{d}x \;+\; c \quad (x \in I)\ .$$

Thus, for any subinterval $[a\,,\,b] \subseteq I$ :

$$\int_a^b u'(x)\, v(x)\,\mathrm{d}x \;=\; \big[u(x)\, v(x)\big]_a^b \;-\; \int_a^b u(x)\, v'(x)\,\mathrm{d}x\ ,$$

where $\big[u(x)\, v(x)\big]_a^b \;=\; u(b)v(b) - u(a)v(a)$.

**Example:** $\int x\, e^{-x}\,\mathrm{d}x$ ? We choose $u'(x) = e^{-x}$ and $v(x) = x$, and thus $u(x) = -e^{-x}$ and $v'(x) = 1$ , which yields:

$$\int e^{-x}\, x\,\mathrm{d}x \;=\; -e^{-x}\, x \;-\; \int -e^{-x}\, 1\,\mathrm{d}x \;=\; -e^{-x}x \;-\; e^{-x} = -(x+1)e^{-x}.$$

**Useful rule: substitution formula.**

If $\varphi(x)$ , $x \in I$ , and $f(y)$ , $y \in J$ , are two functions such that the composition $f\big(\varphi(x)\big)$ , $x \in I$ , is defined and $\varphi$ is differentiable, then

$$\int f\big(\varphi(x)\big)\,\varphi'(x)\,\mathrm{d}x \;=\; \int f(y)\,\mathrm{d}y \;+c \quad \text{with } y=\varphi(x), \quad x \in I.$$

In other words (where in (2) we assume that $\varphi^{-1}(y)$ exists):
(1) If $F(y)$ is an indefinite integral of $f(y)$, then the composition $F\big(\varphi(x)\big)$ is an indefinite integral of the function $f\big(\varphi(x)\big)\,\varphi'(x)$ .
(2) If $G(x)$ is an indefinite integral of the function $f\big(\varphi(x)\big)\,\varphi'(x)$, then $F(y) = G\big(\varphi^{-1}(y)\big)$ is an indefinite integral of $f(y)$.
Furthermore:

$$\text{For } [a\,,\,b] \subseteq I: \quad \int_a^b f\big(\varphi(x)\big)\,\varphi'(x)\,\mathrm{d}x \;=\; \int_{\varphi(a)}^{\varphi(b)} f(y)\,\mathrm{d}y\ .$$

**Convention for $c \geq d$:** $\int_c^d f(y)\,\mathrm{d}y \;=\; -\int_d^c f(y)\,\mathrm{d}y$ ; $\int_c^c f(y)\,\mathrm{d}y \;=\; 0$ .

**Example:** $\int e^{a\,x}\,\mathrm{d}x$ (where $a \neq 0$) ?
Use $\varphi(x) = ax$ , hence $\varphi'(x) = a$ , and $f(y) = e^y$ ; so:
$\int e^{ax}\,\mathrm{d}x \;=\; \frac{1}{a}\int a\,e^{ax}\,\mathrm{d}x \;=\; \frac{1}{a}\int e^y\,\mathrm{d}y \;=\; \frac{1}{a}\,e^y \;+c$ , with $y = ax$ .

Result: $\int e^{a\,x}\,\mathrm{d}x \;=\; \frac{1}{a}\,e^{a\,x} \;+c$ .

**Example:** $\int \tan(x)\,\mathrm{d}x$ , $x \in I \;=\; \left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[$ .
$\tan x \;=\; \frac{\sin x}{\cos x}$ . Choose $\varphi(x) \;=\; \cos x$ , hence $\varphi'(x) \;=\; -\sin x$ , and thus: $\tan x \;=\; -\frac{\varphi'(x)}{\varphi(x)} \;=\; f\big(\varphi(x)\big)\,\varphi'(x)$ with $f(y) \;=\; -\frac{1}{y}$ . Since $F(y) \;=\; -\ln(y)$ is an indefinite integral of $-\frac{1}{y}$ , it follows: $F\big(\varphi(x)\big) = -\ln\big(\cos x\big)$ is an indefinite integral of $\tan x$ .

Result: $\int \tan x\,\mathrm{d}x \;=\; -\ln(\cos x) \;+c$ , $x \in \left]-\frac{\pi}{2}\,,\,\frac{\pi}{2}\right[$ .

## 6.2 Approximate calculation of integrals

One approximation procedure is given directly by the definition of the integral:

$$\int_a^b f(x)\,\mathrm{d}x \;\approx\; S(f,\mathcal{D}_k) \;=\; \sum_{i=1}^{k} f(z_i)\,(x_i - x_{i-1})\ ,$$

for $\mathcal{D}_k \;=\; \begin{pmatrix} x_0 & x_1 & \ldots & x_{k-1} & x_k \\ & z_1 & \ldots & z_{k-1} & z_k \end{pmatrix}$ , a discretization of $[a\,,\,b]$ , where:

$$x_0 = a \;<\; x_1 \;<\; \ldots \;<\; x_{k-1} \;<\; x_k = b\ , \quad z_i \in [\,x_{i-1}\,,\,x_i\,]\ , \ (i = 1,\ldots,k)\,.$$

Another view of this and more general idea: approximate $f(x) \;\approx\; f_k(x)$ ,

$x \in [a\,,\,b]$ , where $f_k(x)$ is a function simple enough that its integral over $[a\,,\,b]$ is easy to calculate; then use:

$$\int_a^b f(x)\,\mathrm{d}x \;\approx\; \int_a^b f_k(x)\,\mathrm{d}x.$$

The above approximation, implied by definition of the integral, employs *step functions* $f_k(x)$ , $x \in [a\,,\,b]$ :

$f_k(x) \;=\; c_i$ for all $x \in [x_{i-1},\, x_i[$ , $(i = 1, 2, \ldots, k)$ , and $f(b) = c_k$ ,
where $c_i = f(z_i)$ , $(i = 1, 2, \ldots, k)$.

$\int_a^b f_k(x)\,\mathrm{d}x \;=\; S(f, \mathcal{D}_k) \;=\; \sum_{i=1}^{k} c_i\,(x_i - x_{i-1})$ (easy to calculate) .

Better approximation by *linear interpolation:* again, partition of the integration interval $[a\,,\,b]$ is used:

$x_0 = a \,<\, x_1 \,<\, \ldots \,<\, x_{k-1} \,<\, x_k = b$ .

Calculate the function values $y_i = f(x_i)$ , $(i = 1, 2, \ldots, k)$ , and use the *linear interpolating* function $f_k(x)$ , $x \in [a, b]$ :

$f_k(x) \;=\; \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\,(x - x_{i-1}) \,+\, y_{i-1}$ for $x \in [\,x_{i-1},\, x_i]$ .

Then

$$\int_a^b f_k(x)\,\mathrm{d}x \;=\; \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} f_k(x)\,\mathrm{d}x \;=\; \ldots \;=\; \tfrac{1}{2} \sum_{i=1}^{k} (y_i + y_{i-1})\,(x_i - x_{i-1})\,,$$

which is easy to calculate.

# 7 Particular classes of ODEs

## 7.1 First order ODE with separated variables

This is the case that the function $G$ has the form

$$G(x,y) \;=\; u(x)\,v(y), \quad x \in I, \quad y \in J,$$

with intervals $I, J \subseteq \mathbb{R}$. So the ODE is

$$y'(x) \;=\; u(x)\,v\big(y(x)\big).$$

We will consider the ODE with initial values $y(x_0) = y_0$ , where $x_0 \in I$ and $y_0 \in J$ are given.

**Example.** Chemical reaction A + B $\longrightarrow$ AB . Interesting: concentrations $a(t)$ and $b(t)$ of substances A and B, resp., as functions of time $t \geq 0$. They obey the chemical law

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}b(t) \;=\; -k\,a(t)\,b(t),$$

where $k > 0$ is a given constant. If $a(0) = a_0 > 0$, $b(0) = b_0 > 0$, then an ODE with separated variables results:

$$a'(t) \;=\; -k\,a(t)\,\big(a(t) + \Delta_0\big),\; a(0) = a_0$$

where $\Delta_0 = b_0 - a_0$, since $b(t) = a(t) + c_0$ for some constant $c_0$. The constant can be determined using $a(0) = a_0$, $b(0) = b_0$.

Let $u(x)$ , $x \in I$, and $v(y)$ , $y \in J$, be continuous functions, and $y' = u(x)v(y)$ , $y(x_0) = y_0$ . Assume $v(y_0) \neq 0$ . The solution $y(x)$ , $x \in I_0$ (subinterval of $I$ containing $x_0$), is obtained by the following steps.

(i) Check $v(y_0) \neq 0$ and determine the largest subinterval $J_0 \subseteq J$ such that $y_0 \in J_0$ and $v(y) \neq 0$ on $J_0$ .
[If $v(y_0) = 0$, then the constant function $y(x) = y_0$, $x \in I$, solves the problem.]

(ii) Find indefinite integrals $U(x) = \int u(x)\,\mathrm{d}x$, $x \in I$, and $V(y) \;=\; \int \frac{1}{v(y)}\,\mathrm{d}y$, $y \in J_0$ .

(iii) For any $x \in I$ consider the equation (with $y \in J_0$ as the unknown)

$$V(y) \;=\; U(x) - c_0,\; y \in J_0,$$

where $c_0 = U(x_0) - V(y_0)$ . Let $I_0$ be the largest subinterval of $I$ that contains $x_0$ and such that the last equation has a solution $y = y(x) \in J_0$ for all $x \in I_0$. This gives the solution $y = y(x)$ , $x \in I_0$ .

**Example:** $y' = 1 + y^2$ , $y(x_0) = y_0$ , with some fixed $x_0, y_0 \in \mathbb{R}$.
Here: $u(x) = 1$ , $x \in I = \mathbb{R}$ , and $v(y) = 1 + y^2$ , $y \in J = \mathbb{R}$ .

(i) $v(y_0) > 0$ and $J_0 = \mathbb{R}$ .

(ii) $U(x) = \int 1 \mathrm{d}x = x$ , $V(y) = \int \frac{1}{1+y^2} \mathrm{d}y = \arctan(y)$ .

(iii) The equation $V\big(y\big) = U(x) - c_0$ , where $c_0 = U(x_0) - V(y_0)$, reads as

$$\arctan(y) = x - c_0, \quad \text{with } c_0 = x_0 - \arctan(y_0).$$

We get: $I_0 = \,]-\frac{\pi}{2} + c_0, \frac{\pi}{2} + c_0[$ , and

$$y(x) = \tan(x - c_0) , \quad x \in I_0 .$$

**Example:** Chemical reaction.

$$y' = -k\, y\, (y + \Delta_0), \; y(0) = y_0,$$

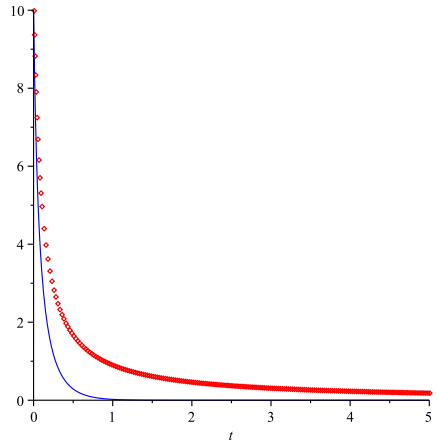
where $k > 0$ , $y_0 = a_0$ , $\Delta_0 = b_0 - a_0$ , $a_0 > 0$, $b_0 > 0$ . Assume $\Delta_0 \geq 0$.
Here: $u(x) = -k$ on $I = [0\,, \infty[$ ; $v(y) = y(y + \Delta_0)$ , $y \in J = \mathbb{R}$ .

(i) $v(y_0) = a_0 b_0 > 0$ , and $J_0 = \,]\,0\,, \infty[$ .

(ii) $U(x) = \int -k \,\mathrm{d}x = -kx$ , $V(y) = \int \frac{1}{y(y+\Delta_0)} \mathrm{d}y$ , $y > 0$ .
Case 1: $\Delta_0 = 0$ ; $V(y) = -\frac{1}{y}$ , $y > 0$ .
Case 2: $\Delta_0 > 0$ ; $V(y) = -\frac{1}{\Delta_0} \ln\big(1 + \frac{\Delta_0}{y}\big)$ , $y > 0$ .

(iii) Eq. $V(y) = U(x) - c_0$ , $y > 0$ , where $c_0 = U(0) - V(y_0) = -V(a_0)$.
Case 1: $-\frac{1}{y} = -kx - \frac{1}{a_0}$ , $y > 0$; so: $I_0 = [0\,, \infty[$, $y(x) = \frac{1}{kx + a_0^{-1}}$ , $x \geq 0$ .
Case 2: $-\frac{1}{\Delta_0} \ln\big(1 + \frac{\Delta_0}{y}\big) = -kx - \frac{1}{\Delta_0} \ln\big(\frac{b_0}{a_0}\big)$ , $y > 0$ ; so: $I_0 = [0\,, \infty[$ , $y(x) = \frac{\Delta_0}{\frac{b_0}{a_0}\, e^{\Delta_0 kx} - 1}$ , $x \geq 0$ .

The following graph shows the solutions for $k = 1$ with $a_0 = b_0 = 10$ (i.e. $\Delta_0 = 0$, the dotted red graph) and with $a_0 = 10, b_0 = 15$ (the blue graph):



## 7.2 First order linear ODE

$$\boxed{y'(x) \,+\, a(x)\,y(x) \,=\, \psi(x),}$$

where $a(x)$ and $\psi(x)$ , $x \in I \subseteq \mathbb{R}$ , are given (continuous) functions.

**Note.** Equivalently: $y'(x) \,=\, G\big(x, y(x)\big)$, where $G(x,y) \,=\, \psi(x) - a(x)y$ (for $x \in I$ and $y \in \mathbb{R}$).

**Structure of the solution set**

Consider the above ODE $y' + a(x)y \,=\, \psi(x)$, which is an *inhomogeneous* ODE with corresponding *homogeneous* ODE $y' + a(x)y \,=\, 0$. Denote by $\mathcal{F}_{\rm ih}$ and by $\mathcal{F}_{\rm h}$ the sets of all functions $y = y(x)$ on $I$ which solve the inhomogeneous ODE and the homogeneous ODE, respectively. If $y^* = y^*(x)$ is *one particular* solution of the inhomogeneous ODE, then we have:

$$\mathcal{F}_{\rm ih} \;=\; \Big\{ y^* \,+\, y \;:\; y \in \mathcal{F}_{\rm h} \Big\} \;=\; y^* + \mathcal{F}_{\rm h}.$$

Note that $\mathcal{F}_{\rm h}$ is a *linear space* of functions, i.e., the constant 0 is in $\mathcal{F}_{\rm h}$ and if $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are in $\mathcal{F}_{\rm h}$ then so is $c_1 y_1 + c_2 y_2$ for any $c_1, c_2 \in \mathbb{R}$. The solution set $\mathcal{F}_{\rm h}$ of the homogeneous ODE $y' + a(x)y \,=\, 0$ consists of the functions

$$y(x) \,=\, c\, e^{-A(x)}, \quad x \in I,$$

where $A(x) = \int a(x)\,{\rm d}x$ , $x \in I$ , and $c \in \mathbb{R}$ is any constant.

*One particular* solution $y^*(x)$ , $x \in I$ , of the *inhomogeneous* linear first order ODE $y' + a(x)y \,=\, \psi(x)$:

Often by guessing, *if $a(x) = a_0$ is a constant function.* **Guessing** a solution works as follows: If the right-hand side $\psi(x)$ is a polynomial of degree $n$, we **guess** that a particular solution is a polynomial of degree $n$. And if the right-hand side is a linear combination $a\sin(\omega x) + b\cos(\omega x)$, we try an expression $A\sin(\omega x) + B\cos(\omega x)$. Similarly, if the right-hand side is an exponential function $ae^{\lambda x}$, we try an exponential function $Ae^{\lambda x}$.

A systematic way (*for general $a(x)$*) is as follows. Choose $A(x) = \int a(x)\,{\rm d}x$ , $x \in I$, as above. Find $W(x) = \int \psi(x)\, e^{A(x)}\,{\rm d}x$ , $x \in I$ . Then

$$y^*(x) \,=\, W(x)\, e^{-A(x)}, \;\; x \in I,$$

is a solution of the inhomogeneous ODE.

**Example.** $a(x) = a_0$ constant, $a_0 \neq 0$ , $\quad \psi(x) = p_0$ constant.

Consider the inhomogeneous ODE $y' + a_0 y \,=\, p_0$. Guess a *constant* $y^*(x) = c_0$ ; in fact this works with $c_0 = \frac{p_0}{a_0}$ . Here $A(x) = \int a_0\,{\rm d}x \,= a_0 x$ , and we get:

$$y(x) \,=\, \frac{p_0}{a_0} \,+\, c\, e^{-a_0 x}, \;\; x \in I\,,$$

where $c \in \mathbb{R}$ is any constant.

**Example:** $a(x) = a_0$ as above, $\psi(x) = b_0 \cos(\omega x)$, $(b_0 > 0\,,\, \omega > 0)$,

$$y' + a_0 y \;=\; b_0 \cos(\omega x).$$

Again, the solution of the homogeneous ODE is $c\, e^{-a_0 x}$. For a particular solution $y^*(x)$ of the inhomogeneous ODE, we try

$$y^*(x) = A \cdot \cos(\omega x) + B \cdot \sin(\omega x).$$

Plug this into the inhomogeneous ODE:

$$-\omega\, A \sin(\omega x) \;+\; \omega\, B \cos(\omega x) \;+\; a_0\, A \cos(\omega x) \;+\; a_0\, B \sin(\omega x) = b_0 \cos(\omega x).$$

Comparing coefficients of sin and cos, we get

$$\begin{aligned} -\omega A + a_0 B &= 0, \\ a_0 A + \omega B &= b_0. \end{aligned}$$

We obtain the two solutions

$$A = \frac{a_0 b_0}{\omega^2 + a_0^2}, \quad B = \frac{b_0 \omega}{\omega^2 + a_0^2},$$

hence

$$y^*(x) = \frac{b_0}{\sqrt{\omega^2 + a_0^2}} \left( \frac{a_0}{\sqrt{\omega^2 + a_0^2}} \cos(\omega x) + \frac{\omega}{\sqrt{\omega^2 + a_0^2}} \sin(\omega x) \right).$$

Using the addition formulae for the trigonometric functions, one obtains

$$\cos(\omega x - \beta) = \cos(\beta)\cos(\omega x) + \sin(\beta)\sin(\omega x),$$

which gives

$$\cos(\beta) = \frac{a_0}{\sqrt{\omega^2 + a_0^2}}, \quad \sin(\beta) = \frac{\omega}{\sqrt{\omega^2 + a_0^2}} \qquad \left( \tan(\beta) = \frac{\omega}{a_0} \right).$$

An example with initial values: solve $y' + 7y = 10 \cos(8x)$, $y(0) = 0$.

$$y(x) \;=\; \frac{10}{\sqrt{113}} \cos(8x - \beta) \;+\; c\, e^{-7x},$$

$\beta = \arctan\left(\frac{8}{7}\right)$, $c$ such that $y(0) = 0$, i.e.: $\frac{10}{\sqrt{113}} \cos(-\beta) + c \;=\; 0$, $c = -\frac{70}{113}$.

## 7.3 Second order linear ODE with constant coefficients

Here we consider ODE's

$$\boxed{y''(x) \;+\; a\, y'(x) \;+\; b\, y(x) \;=\; \psi(x),}$$

where $a, b \in \mathbb{R}$ and $\psi(x)$ , $x \in I \subseteq \mathbb{R}$ , a (continuous) function are given.

**Structure of the solution set**

Consider the above ODE $y'' + ay' + by = \psi(x)$, with the corresponding *homogeneous* ODE $y'' + ay' + by = 0$. Denote by $\mathcal{F}_{\rm ih}$ and by $\mathcal{F}_{\rm h}$ the sets of all functions $y = y(x)$ on $I$ which solve the inhomogeneous and the homogeneous ODE, respectively. If $y^* = y^*(x)$ is *one particular* solution of the inhomogeneous ODE, then we have:

$$\mathcal{F}_{\rm ih} \;=\; \Big\{ y^* \,+\, y \;:\; y \in \mathcal{F}_{\rm h} \Big\} \;=\; y^* + \mathcal{F}_{\rm h}.$$

As before, $\mathcal{F}_{\rm h}$ is a *linear space* of functions.

**Solutions of the *homogeneous* 2nd order linear ODE**

The solution set $\mathcal{F}_{\rm h}$ of $y'' + ay' + by = 0$ consists of the linear combinations $c_1 y_1(x) + c_2 y_2(x)$ , $(c_1, c_2 \in \mathbb{R})$, of two *basic solutions* $y_1(x)$ and $y_2(x)$ , $(x \in I)$, which are as follows:

(i) If $\left(\frac{a}{2}\right)^2 > b$, then $y_1(x) = e^{\lambda_1 x}$ , $\quad y_2(x) = e^{\lambda_2 x}$ , where $\lambda_{1,2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$ are the roots of $\lambda^2 + a\lambda + b$.

(ii) If $\left(\frac{a}{2}\right)^2 = b$, then $y_1(x) = e^{\lambda x}$ , $\quad y_2(x) = x e^{\lambda x}$ , where $\lambda = -\frac{a}{2}$ .

(iii) If $\left(\frac{a}{2}\right)^2 < b$, then $y_1(x) = \cos(\omega x)\, e^{\lambda x}$ , $y_2(x) = \sin(\omega x)\, e^{\lambda x}$ , where $\lambda = -\frac{a}{2}$ , $\quad \omega = \sqrt{b - \left(\frac{a}{2}\right)^2}$ .

In case (iii) the solution set $\mathcal{F}_{\rm h}$ may also be represented as

$$y(x) \;=\; c\, \cos(\omega x - \beta)\, e^{\lambda x}$$

with any constants $c \in \mathbb{R}$ and $\beta \in [0\,,\, 2\pi\,[$ .

**One particular solution of the *inhomogeneous* ODE**

We consider

$$\boxed{y''(x) + ay'(x) + by(x) \;=\; \psi(x),}$$

where $x \in I$ (and suppose that we can't guess a solution). We use the basic solutions $y_1(x)$ and $y_2(x)$ of the homogeneous ODE. We define:

$$w(x) \;=\; \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \;= y_1(x) y_2'(x) - y_1'(x) y_2(x)\,.$$

Find indefinite integrals:

$$W_1(x) = \int -\frac{y_2(x)\psi(x)}{w(x)}\,\mathrm{d}x, \quad W_2(x) = \int \frac{y_1(x)\psi(x)}{w(x)}\,\mathrm{d}x.$$

Then a particular solution of the inhomogeneous ODE is

$$y^*(x) \;=\; W_1(x)y_1(x) + W_2(x)y_2(x).$$

More explicitly, the *Wronski determinant* $w(x)$ reads as:

$$w(x) \;=\; \begin{cases} (\lambda_2 - \lambda_1)\, e^{(\lambda_1+\lambda_2)x} & \text{in case (i): } (\frac{a}{2})^2 > b, \\ e^{2\lambda x} & \text{in case (ii): } (\frac{a}{2})^2 = b, \\ \omega\, e^{2\lambda x} & \text{in case (iii): } (\frac{a}{2})^2 < b. \end{cases}$$

**Example:** $y''(x)+ay'(x)+by(x) = c\,\cos(\omega x)$ with constants $c > 0$ and $\omega > 0$. Find one particular solution!
We introduce *complex-valued* functions $Y(x)$ and $\Psi(x) = c\,e^{\boldsymbol{i}\omega x}$, and consider

$$Y''(x) + aY'(x) + bY(x) = c\,e^{\boldsymbol{i}\omega x}.$$

This is a complex inhomogeneous ODE. Obviously: If $Y^*(x)$, $x \in I$, is a solution of the complex inhomogeneous ODE, then $y^*(x) = \mathrm{Re}\big(Y(x)\big)$, is a solution of the real inhomogeneous ODE. Guess:

$$Y^*(x) \;=\; K\,e^{\boldsymbol{i}\omega x}$$

with some constant $K \in \mathbb{C}$.
**Calculation:**

$$\begin{aligned} (Y^*)'(x) &= K\boldsymbol{i}\omega e^{\boldsymbol{i}\omega x} \\ (Y^*)''(x) &= -K\omega^2 e^{\boldsymbol{i}\omega x}, \end{aligned}$$

so:

$$(Y^*)''(x) + a(Y^*)'(x) + bY^*(x) \;=\; K\left(-\omega^2 + \boldsymbol{i}a\omega + b\right) e^{\boldsymbol{i}\omega x}.$$

Equating this to $\Psi(x) = c\,e^{\boldsymbol{i}\omega x}$ gives:

$$K = \frac{c}{b - \omega^2 + \boldsymbol{i}a\omega} = \frac{c}{d^2}(b - \omega^2 - \boldsymbol{i}a\omega)$$

where

$$d = \sqrt{(b-\omega^2)^2 + a^2\omega^2}.$$

Euler representation of $K$ is

$$\frac{c}{d} \cdot \frac{b - \omega^2 - \boldsymbol{i}a\omega}{d} \;=\; \frac{c}{d}\, e^{-\boldsymbol{i}\beta}$$

with $\beta \in [0\,,\, 2\pi[$, since the absolute value of

$$\frac{b-\omega^2}{d} - \boldsymbol{i}\frac{a\omega}{d}$$

is 1.

**Result:** $y^*(x) = \mathrm{Re}\big(\frac{c}{d}\, e^{\boldsymbol{i}(\omega x - \beta)}\big) = \frac{c}{d}\,\cos\big(\omega x - \beta\big)$,
(under the assumption $b \neq \omega^2$ or $a \neq 0$).

# 8 Derivatives of functions of several real variables

## 8.1 Definitions and examples

Here we consider a *real-valued function of n real variables:*

$$f : D \longrightarrow \mathbb{R}, \quad \text{where } D \subseteq \mathbb{R}^n \ \ (D\text{: the domain of } f).$$

We write: $f(x_1, x_2, \ldots, x_n)$ , where $(x_1, x_2, \ldots, x_n) \in D$ ; or shorter: $f(\boldsymbol{x})$ , $\boldsymbol{x} \in D$ .

**Example:** $n = 2$ , $f(x_1, x_2) = x_1^{x_2}$ with domain $D = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\big\}$, which is the the right half plane.

We often prefer the notation $(x, y)$ instead of $(x_1, x_2)$, hence in the above example $f(x, y) = x^y$ , $(x, y) \in D$, i.e., $x > 0$, $y \in \mathbb{R}$ .

**Example:** $n = 3$ , $f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_3$ with domain $D = \mathbb{R}^3$ . Using notation $(x, y, z)$ instead of $(x_1, x_2, x_3)$:

$$f(x, y, z) = x^2 + 3y^2 - z^2 + 2xz, \quad (x, y, z) \in \mathbb{R}^3.$$

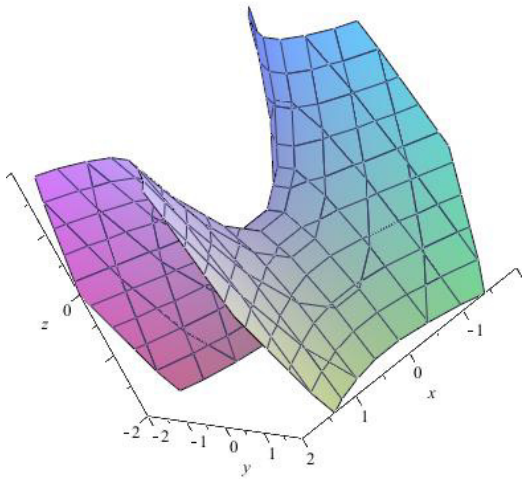
**Example:** $n$ arbitrary,

$$f(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|}, \quad \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}.$$

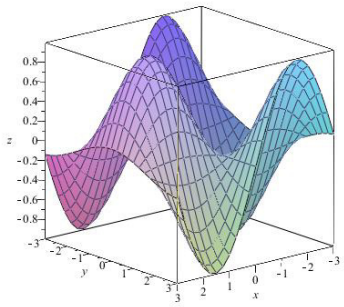
In coordinates: $f(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}}$ .

**Example. Graph of a function of two variables ($n = 2$)**

The following is the graph of $f(x, y) = x^2 - y^2$:



Here is the graph of $f(x,y) = \sin(x) \cdot \cos(y)$:



## 8.2 Partial derivatives and gradient: the case $n = 2$

Given a function of *two* variables: $f : D \longrightarrow \mathbb{R}$ , $D \subseteq \mathbb{R}^2$ ; for $(x_0, y_0) \in D$ , we define

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{x_1 \to x_0} \frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0}, \\ \frac{\partial f}{\partial y}(x_0, y_0) &= \lim_{y_1 \to y_0} \frac{f(x_0, y_1) - f(x_0, y_0)}{y_1 - y_0} \end{aligned}$$

**Note:** $\frac{\partial f}{\partial x}(x_0, y_0)$ equals the derivative of the function of *one* variable, where $y = y_0$ is fixed and $x \longmapsto f(x, y_0)$ is the function in one variable, and we consider its derivative at the point $x_0$; similarly for the case $\frac{\partial f}{\partial y}(x_0, y_0)$.

The *gradient* of $f$ at $(x_0, y_0)$:

$$\boldsymbol{\nabla} f(x_0, y_0) \;=\; \Big( \frac{\partial f}{\partial x}(x_0, y_0) \,,\; \frac{\partial f}{\partial y}(x_0, y_0) \Big) \;\in \mathbb{R}^2 \,.$$

We may use the gradient for a local linear approximation of $f(x, y)$ at $(x_0, y_0)$ :

$$f(x, y) \;\approx\; f(x_0, y_0) \;+\; \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) \;+\; \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0),$$

or shortly: $f(\boldsymbol{x}) \;\approx\; f(\boldsymbol{x}_0) + \boldsymbol{\nabla} f(\boldsymbol{x}_0) \cdot (\boldsymbol{x} - \boldsymbol{x}_0)$ , where $\boldsymbol{x} = (x, y)$ , $\boldsymbol{x}_0 = (x_0, y_0)$.

**Example.** We consider $f(x, y) = x^2 y^3 + xy^2 + 2y$, then

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= 2xy^3 + y^2, \\ \frac{\partial f}{\partial y}(x, y) &= 3x^2 y^2 + 2xy + 2. \end{aligned}$$

The gradient is

$$\boldsymbol{\nabla} f(x,y) = (2xy^3 + y^2, 3x^2y^2 + 2xy + 2).$$

The local approximation of $f$ at the point $(0,1)$ is

$$f(x,y) \approx 2 + (1,2)\cdot(x-0, y-1) = x + 2y.$$

If partial derivatives $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist *for all* $(x_0,y_0) \in D$, then we call $f$ (partially) **differentiable**. Then we write $(x,y)$ instead of $(x_0,y_0)$ and the partial derivatives of $f$ are *functions*

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial x}(x,y), \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial y}(x,y), \end{aligned}$$

each of which is a real-valued function of two real variables, and the gradient is an $\mathbb{R}^2$-valued function of two real variables:

$$\boldsymbol{\nabla} f = \boldsymbol{\nabla} f(x,y) = \Big( \frac{\partial f}{\partial x}(x,y)\,,\, \frac{\partial f}{\partial y}(x,y) \Big), \quad (x,y) \in D.$$

**Example:** $f(x,y) = x^y = \exp\big(y \ln(x)\big)\,, \quad x > 0,\, y \in \mathbb{R}\,.$

We calculate: $\frac{\partial f}{\partial x}(x,y) = y\, x^{y-1}\,, \qquad \frac{\partial f}{\partial y}(x,y) = \ln(x)\, x^y.$

## 8.3 Partial derivatives and gradient; general case: $n \geq 2$

Given a function of $n$ real variables: $f : D \longrightarrow \mathbb{R}\,, \quad D \subseteq \mathbb{R}^n$ and a point $\boldsymbol{x}_0 = (x_{01}, x_{02}, \ldots, x_{0n}) \in D$, we define for each $i = 1, 2, \ldots, n$:

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0) = \lim_{x_i \to x_{0i}} \frac{f(x_{01}, \ldots, x_{0\,i-1},\, x_i\,, x_{0\,i+1}, \ldots, x_{0n}) - f(\boldsymbol{x}_0)}{x_i - x_{0i}}.$$

**Note:** $\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0)$ equals the derivative of a function of *one* variable $x_i$, where the other variables are considered to be fixed. The function is

$$x_i \longmapsto f(x_{01}, \ldots, x_{0\,i-1},\, x_i\,, x_{0\,i+1}, \ldots, x_{0n}),$$

and we compute the derivative at $x_{0i}$, which is the $i$-th coordinate of $\boldsymbol{x}_0$. The **gradient** of $f$ at $\boldsymbol{x}_0$:

$$\boldsymbol{\nabla} f(\boldsymbol{x}_0) = \Big( \frac{\partial f}{\partial x_1}(\boldsymbol{x}_0)\,,\, \frac{\partial f}{\partial x_2}(\boldsymbol{x}_0)\,,\, \ldots\,,\, \frac{\partial f}{\partial x_n}(\boldsymbol{x}_0) \Big) \in \mathbb{R}^n.$$

**Local linear approximation** of $f(\boldsymbol{x})$ at $\boldsymbol{x}_0$, when $\boldsymbol{x}$ is close to $\boldsymbol{x}_0$ :

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_0) \cdot (x_i - x_{0i})$$

for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, or, shorter

$$f(\boldsymbol{x}) \;\approx\; f(\boldsymbol{x}_0) \,+\, \boldsymbol{\nabla} f(\boldsymbol{x}_0)\cdot(\boldsymbol{x} - \boldsymbol{x}_0)$$

Let us consider

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0) = \lim_{h\to 0} \frac{f(\boldsymbol{x}_0 + h\boldsymbol{e}_i) \,-\, f(\boldsymbol{x}_0)}{h}$$

again. Here $\boldsymbol{e}_i$ is the $i$-th elementary unit vector in $\mathbb{R}^n$ , $i = 1, 2, \ldots, n$. This gives rise to the more general concept of **directional derivatives**:

For *any* vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ with $|\boldsymbol{v}| = 1$, we define

$$\boxed{\mathrm{D}_{\boldsymbol{v}} f(\boldsymbol{x}_0) \;=\; \lim_{h\to 0} \tfrac{f(\boldsymbol{x}_0 + h\boldsymbol{v}) \,-\, f(\boldsymbol{x}_0)}{h},}$$

which is **the directional derivative** of $f$ at $\boldsymbol{x}_0$ in the direction $\boldsymbol{v}$. The directional derivative is closely related to the gradient:

$$\mathrm{D}_{\boldsymbol{v}} f(\boldsymbol{x}_0) \;=\; \sum_{i=1}^{n} \tfrac{\partial f}{\partial x_i}(\boldsymbol{x}_0)\, v_i \;=\; \boldsymbol{\nabla} f(\boldsymbol{x}_0) \cdot \boldsymbol{v}.$$

The Cauchy-Schwarz inequality shows, that

$$\boldsymbol{v}^* \;=\; \frac{\boldsymbol{\nabla} f(\boldsymbol{x}_0)}{|\boldsymbol{\nabla} f(\boldsymbol{x}_0)|}$$

maximizes $\mathrm{D}_{\boldsymbol{v}} f(\boldsymbol{x}_0)$ among all $\boldsymbol{v} \in \mathbb{R}^n$ with $|\boldsymbol{v}| = 1$, hence it is the direction of steepest ascent, and

$$\boldsymbol{v}^{**} \;=\; -\frac{\boldsymbol{\nabla} f(\boldsymbol{x}_0)}{|\boldsymbol{\nabla} f(\boldsymbol{x}_0)|}$$

minimizes $\mathrm{D}_{\boldsymbol{v}} f(\boldsymbol{x}_0)$, hence it is the direction of steepest descent.

If the partial derivatives $\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0)$ , $i = 1, 2, \ldots, n$, exist *for all* $\boldsymbol{x}_0 \in D$, then $f$ is called (partially) **differentiable**. Then we write $\boldsymbol{x}$ instead of $\boldsymbol{x}_0$ and the partial derivatives of $f$ are functions:

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i}(\boldsymbol{x}), \quad \boldsymbol{x} \in D\,,$$

is for each $i$ again a real-valued function of $n$ real variables. The gradient is an $\mathbb{R}^n$-valued function of $n$ real variables:

$$\boldsymbol{\nabla} f = \boldsymbol{\nabla} f(\boldsymbol{x}) \;=\; \Big( \frac{\partial f}{\partial x_1}(\boldsymbol{x})\,,\, \frac{\partial f}{\partial x_2}(\boldsymbol{x})\,,\, \ldots\,,\, \frac{\partial f}{\partial x_n}(\boldsymbol{x}) \Big), \quad \boldsymbol{x} \in D.$$

**Example.** $n = 3\,, \quad f(\boldsymbol{x}) \;=\; x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_3$ , $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= 2x_1 + 2x_3, \\
\frac{\partial f}{\partial x_2} &= 6x_2, \\
\frac{\partial f}{\partial x_3} &= -2x_3 + 2x_1.
\end{aligned}$$

**Example.** Often one uses notation $(x, y, z)$ instead of $(x_1, x_2, x_3)$:

$$f(x, y, z) \,=\, x^2 + 3y^2 - z^2 + 2xz, \quad (x, y, z) \in \mathbb{R}^3,$$

and the partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 2z. \\ \frac{\partial f}{\partial y} &= 6y, \\ \frac{\partial f}{\partial z} &= -2z + 2x. \end{aligned}$$

**Example:** $n \geq 2$ arbitrary,

$$f(\boldsymbol{x}) \,=\, \frac{1}{|\boldsymbol{x}|} \,=\, \big(x_1^2 + x_2^2 + \ldots + x_n^2\big)^{-1/2}, \quad \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}.$$

We have

$$\frac{\partial f}{\partial x_i} \,=\, -\frac{1}{2}\big(x_1^2 + x_2^2 + \ldots + x_n^2\big)^{-3/2} \cdot 2x_i \,=\, -\frac{x_i}{|\boldsymbol{x}|^3}$$

for $i = 1, 2, \ldots, n$ and

$$\boldsymbol{\nabla} f \,=\, -\Big(\, \frac{x_1}{|\boldsymbol{x}|^3}\,,\, \frac{x_2}{|\boldsymbol{x}|^3}\,,\, \ldots\,,\, \frac{x_n}{|\boldsymbol{x}|^3}\,\Big) \,=\, -\frac{1}{|\boldsymbol{x}|^3}\,\boldsymbol{x}.$$

**Example:** $n = 4$,

$$f(x, y, z, t) \,=\, \exp\Big(-\frac{1}{2}(x^2 + y^2 + z^2) \,-\, t\Big), \quad (x, y, z, t) \in \mathbb{R}^4.$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= -x\, f(x, y, z, t), \\ \frac{\partial f}{\partial y} &= -y\, f(x, y, z, t), \\ \frac{\partial f}{\partial z} &= -z\, f(x, y, z, t), \\ \frac{\partial f}{\partial t} &= -\, f(x, y, z, t). \end{aligned}$$

## 8.4 Second order partial derivatives

Let $f(\boldsymbol{x})$ , $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in D \subseteq \mathbb{R}^n$ , be a differentiable function with partial derivatives $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ , $i = 1, 2, \ldots, n$. If each of these partial derivatives is again differentiable, we say that $f$ *is twice differentiable*; then, for the partial derivatives of the partial derivatives of $f$ one uses notations

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{x})\,,$$

which is the partial derivative w.r.t. $x_j$ of $\frac{\partial f}{\partial x_i}$, $i,j = 1,2,\ldots,n$. In case $i = j$ one writes $\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{x})$ instead of $\frac{\partial^2 f}{\partial x_i \partial x_i}(\boldsymbol{x})$ .

There are $n^2$ second order partial derivatives of $f$ , that may be arranged in an $n \times n$ matrix, which is called the **Hessian matrix** of $f$ at $\boldsymbol{x}$:

$$\boldsymbol{H}_f(\boldsymbol{x}) \;=\; \Big(\frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{x})\Big)_{i,j=1,\ldots,n}, \quad \boldsymbol{x} \in D.$$

One can show that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{x}) \;=\; \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x})$$

for all $i,j$ , i.e., $\boldsymbol{H}_f(\boldsymbol{x})$ is symmetric! In order to prove this, one needs the condition that the second partial derivatives are continuous, which is the true in almost all applications.

**Example** $n = 2$:

$$f(x,y) \;=\; x^y, \quad x > 0, y \in \mathbb{R}.$$

We have

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= y\, x^{y-1}, \\
\frac{\partial f}{\partial y}(x,y) &= \ln(x)\, x^y.
\end{aligned}$$

The second derivatives are

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x,y) &= y(y-1)\, x^{y-2}, \\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \big(\ln(x)\big)^2 x^y, \\
\frac{\partial^2 f}{\partial y \partial x}(x,y) &= x^{y-1} \,+\, y \ln(x)\, x^{y-1} \;=\; \big(1 + y\ln(x)\big)\, x^{y-1}, \\
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{1}{x}\, x^y \,+\, \ln(x)\, y\, x^{y-1} \;=\; \big(1 + y\ln(x)\big)\, x^{y-1}.
\end{aligned}$$

The Hessian is

$$\boldsymbol{H}_f(x,y) \;=\; \begin{pmatrix} y(y-1)\, x^{y-2} & \big(1 + y\ln(x)\big)\, x^{y-1} \\ \big(1 + y\ln(x)\big)\, x^{y-1} & \big(\ln(x)\big)^2 x^y \end{pmatrix}.$$

**Example** $n = 3$: $f(x,y,z) \;=\; x^2 + 3y^2 - z^2 + 2xz$ , $(x,y,z) \in \mathbb{R}^3$. The partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= 2x + 2z, \\
\frac{\partial f}{\partial y}(x,y,z) &= 6y, \\
\frac{\partial f}{\partial z}(x,y,z) &= -2z + 2x.
\end{aligned}$$

One calculates:

$$\boldsymbol{H}_f(x,y,z) \;=\; \begin{pmatrix} 2 & 0 & 2 \\ 0 & 6 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$

**Local quadratic approximation**

If $f(\boldsymbol{x})$ , $\boldsymbol{x} \in D \subseteq \mathbb{R}^n$ , is twice differentiable, and $\boldsymbol{x}_0 \in D$ given, then

$$f(\boldsymbol{x}) \;\approx\; f(\boldsymbol{x}_0) \,+\, \boldsymbol{\nabla} f(\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0) \,+\, \frac{1}{2}\,(\boldsymbol{x}-\boldsymbol{x}_0)^{\mathrm{t}}\boldsymbol{H}_f(\boldsymbol{x}_0)\,(\boldsymbol{x}-\boldsymbol{x}_0)$$

for $\boldsymbol{x}$ close to $\boldsymbol{x}_0$ . Here vectors $\boldsymbol{x}$ and $\boldsymbol{x}_0$ are written as *column* vectors; so

$$(\boldsymbol{x}-\boldsymbol{x}_0)^{\mathrm{t}}\boldsymbol{H}_f(\boldsymbol{x}_0)\,(\boldsymbol{x}-\boldsymbol{x}_0) \;=\; \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}_0)\,(x_i - x_{0i})(x_j - x_{0j}),$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{t}}$ , $\boldsymbol{x}_0 = (x_{01}, x_{02}, \ldots, x_{0n})^{\mathrm{t}}$ .

In particular, in case $n = 2$:

$$\begin{aligned} f(x,y) \;\;\approx\;\; & f(x_0,y_0) \,+\, \frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0) \,+\, \frac{\partial f}{\partial y}(x_0,y_0)\,(y-y_0) \\ & + \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\,(x-x_0)^2 \,+\, \frac{1}{2}\,\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\,(y-y_0)^2 \\ & + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\,(x-x_0)(y-y_0) \end{aligned}$$

for $(x,y)$ close to $(x_0,y_0)$ .

## 8.5 Local minimum and maximum points

Recall case $n = 1$:

- If $x^*$ local minimum/maximum point of $f$, then $f'(x^*) = 0$ .
- If $f'(x^*) = 0$ and $f''(x^*) > 0$, then $x^*$ is a local minimum point of $f$.
- If $f'(x^*) = 0$ and $f''(x^*) < 0$, then $x^*$ is a local maximum point of $f$.

The latter statements assume that $x^*$ is in the *interior* of the domain $I$ of $f$, i.e., there is a small intervall $]x^* - \epsilon, x^* + \epsilon[$ fully contained in the domain of $f$.

Now we are going to generalize this to the case of arbitrary dimension. We begin with the notion of a *local minimum/local maximum point*. In the case $n = 1$, a real number $x^*$ is said to be a *local minimum [maximum] point* of the function $f(x)$ , $x \in I \subseteq \mathbb{R}$ , if there exists an open subinterval $I^* = ]\,\alpha\,,\,\beta\,[ \subseteq I$ with $x^* \in I^*$ and such that $f(x^*) \le f(x)$ $\big[\, f(x^*) \ge f(x) \,\big]$ for all $x \in I^*$ . In the general case: given a real-valued function $f(\boldsymbol{x})$ , where $\boldsymbol{x} \in D \subseteq \mathbb{R}^n$, a point $\boldsymbol{x}^* \in D$ is a *local minimum [maximum] point* of the function $f(\boldsymbol{x})$ , if

there exists an *n-dimensional open interval* $I^*$ with $x^* \in I^* \subseteq D$ and such that $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \quad \big[\, f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}) \,\big] \quad$ for all $\boldsymbol{x} \in I^*$. Here an *n-dimensional open interval* $I^*$ is a subset of $\mathbb{R}^n$ of the form

$$I^* \,=\, \big\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,:\; \alpha_i < x_i < \beta_i \;\text{ for all } i = 1, 2, \ldots, n \big\},$$

with real constants $\alpha_i < \beta_i \;\; (i = 1, \ldots, n)$.

**Local minimum/maximum point via partial derivatives**

Given a function $f(\boldsymbol{x})$ , $\boldsymbol{x} \in D \subseteq \mathbb{R}^n$ , (twice) differentiable, and given $\boldsymbol{x}^* \in D$ (more precisely: $\boldsymbol{x}^*$ in the interior of $D$), then the following hold:

- If $x^*$ is a local minimum/maximum point of $f$ , then $\;\boldsymbol{\nabla} f(\boldsymbol{x}^*) = \boldsymbol{0}$.
- If $\boldsymbol{\nabla} f(\boldsymbol{x}^*) = \boldsymbol{0}$ and $\boldsymbol{H}_f(\boldsymbol{x}^*)$ is positive [negative] definite, then $x^*$ is a local minimum [maximum] point of $f$ .

**Example:** $\;f(x, y) \,=\, x^3 - 3x^2 - 9x - y^3 - 6y^2 + 100\,.$

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= 3x^2 - 6x - 9, \\ \frac{\partial f}{\partial y}(x, y) &= -3y^2 - 12y, \end{aligned}$$

hence $\frac{\partial f}{\partial x}(x, y) = 0 \;\leftrightarrow\; x = 1 \pm 2$ and $\frac{\partial f}{\partial y}(x, y) = 0 \;\leftrightarrow\; y = -2 \pm 2$. Hence the *stationary* points $(x^*, y^*)$, which are defined by $\boldsymbol{\nabla} f(x^*, y^*) = (0, 0)$ , are

$$(-1\,,\,-4),\;\; (-1\,,\,0),\;\; (3\,,\,-4) \;\text{ and }\; (3\,,\,0).$$

We compute the Hessian matrix

$$\boldsymbol{H}_f(x, y) \,=\, \begin{pmatrix} 6x - 6 & 0 \\ 0 & -6y - 12 \end{pmatrix}.$$

The stationary points give the following four matrices:

$$\boldsymbol{H_1} = \begin{pmatrix} -12 & 0 \\ 0 & 12 \end{pmatrix}, \quad \boldsymbol{H_2} = \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix},$$

$$\boldsymbol{H_3} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}, \quad \boldsymbol{H_4} = \begin{pmatrix} 12 & 0 \\ 0 & -12 \end{pmatrix}.$$

Hence $(-1\,,\,0)$ is a local maximum point, since $\boldsymbol{H_2}$ is negative definite, and $(3, -4)$ is a local minimum point, because $\boldsymbol{H_3}$ is positive definite. The matrices $\boldsymbol{H_1}$ and $\boldsymbol{H_4}$ are indefinite (i.e., both have positive as well as negative eigenvalues). This tells that $(-1\,,\,-4)$ and $(3\,,\,0)$ are neither minimum nor maximum points, and therefore are called *saddle points*.