Regular Curves, Singular Graphs: Cantor Parts and the Relaxed Willmore Energy

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Abstract

One might expect that finite relaxed elastic energy rules out diffuse singularities in the derivative, leaving only absolutely continuous and jump parts. This is suggested by the role of SBV in free-discontinuity problems and by interpreting jumps as vertical segments of limiting graphs. We show that it fails for the relaxed one-dimensional Willmore energy. We construct a continuous function $u \in BV((0,1))$ with $D^c u \neq 0$ and $\overline{\mathcal{W}}(u) < \infty$, so finite relaxed Willmore energy does not imply $u \in SBV((0,1))$. The idea is to concentrate the Cantor part exactly where the absolutely continuous slope blows up. There the singular diffuse measure meets the blow-up condition of the relaxation theorem, while the weighted curvature term stays integrable.

Geometrically, the example shows that Cantor parts of BV-graph derivatives need not be intrinsic singularities of the underlying curve. The graph has an arc-length parametrization of class $C^1 \cap W^{2,2}$, and a suitable rotation turns it into a Lipschitz graph whose derivative has no singular part. The construction also rescales to make the relaxed Willmore energy arbitrarily small, and it extends to relaxed L^p-curvature energies for all $p > 1$.

1 Introduction

Curvature energies for graphs form a simple but rich class of geometric variational problems. If $u \in W^{2,p}((0,1))$, the p-curvature energy for $p > 1$ of its graph is

$$\mathcal{W}_p(u) := \int_{\mathrm{graph}[u]} |\kappa_u|^p \,\mathrm{d}s_u = \int_0^1 \frac{|u''(x)|^p}{\big(1+u'(x)^2\big)^{(3p-1)/2}} \,\mathrm{d}x$$

where

$$\kappa_u(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = \frac{u''(x)}{\big(1+u'(x)^2\big)^{3/2}}, \qquad \mathrm{d}s_u = \sqrt{1+u'(x)^2}\,\mathrm{d}x.$$

Thus the one-dimensional Willmore, also called elastic energy, is $\mathcal{W}(u) = \mathcal{W}_2(u)$. For $p > 1$ and $u \in L^1((0,1))$, we define the lower semicontinuous envelope of $\mathcal{W}_p$ with respect to strong L^1-convergence by

$$\overline{\mathcal{W}}_p(u) := \inf\left\{ \liminf_{j\to\infty} \mathcal{W}_p(u_j) \,\middle|\, u_j \in W^{2,p}((0,1)),\ u_j \to u \text{ in } L^1(0,1) \right\}.$$

We refer to $\overline{\mathcal{W}}_p$ as the relaxed p-elastic energy. In particular, $\overline{\mathcal{W}} := \overline{\mathcal{W}}_2$ denotes the relaxed one-dimensional Willmore energy.

In the graph setting, minimizing sequences for Willmore-type energies may develop vertical parts in the limit. This naturally leads to BV-limits. For the two-dimensional Willmore graph problem and its L^1-relaxation we refer to Deckelnick, Grunau and Röger [DGR17]. The one-dimensional weighted relaxation theorem of Dal Maso, Fonseca, Leoni and Morini [DMFLM09] is another central ingredient for the present work, together with standard facts on BV-functions [AFP00].

For $u \in BV((0,1))$, the distributional derivative decomposes as

$$Du = (u')^a \mathcal{L}^1 + D^j u + D^c u,$$

where $D^j u$ is the jump part and $D^c u$ is the Cantor part. Jump parts have a clear geometric interpretation as vertical segments in graph limits. Cantor parts are more elusive: they are diffuse, meaning atomless, singular measures and one might expect them to be excluded by finite relaxed curvature energy. Recall that $SBV((0,1))$ denotes the subspace of $BV((0,1))$ consisting of functions whose Cantor part vanishes.

The main purpose of this paper is to show that this expectation is false. We construct

$$u \in BV((0,1)) \cap C^0([0,1]) \quad \text{such that} \quad D^c u \neq 0, \qquad \overline{\mathcal{W}}(u) < \infty.$$

Hence finite relaxed one-dimensional Willmore energy does not imply $u \in SBV((0,1))$. The same mechanism also applies to the relaxed p-curvature energies $\overline{\mathcal{W}}_p$ for every $p > 1$. Thus diffuse singular parts are not excluded by finite relaxed L^p-curvature energy either. We also want to give a geometrical interpretation of u, and study the regularity of the graph of u.

Relaxation of curvature-dependent energies has been studied from several viewpoints. For planar curvature functionals and elastica-type energies we refer to Bellettini, Dal Maso and Paolini, and to Bellettini and Mugnai [BDMP93, BM04, BM07]. For Cartesian curves and p-elastic energies, the work of Acerbi and Mucci [AM17a, AM17b] provides a geometric relaxation theory in which, for suitable continuous curves, the relaxed elastic energy is described by length plus p-curvature. In the case $p = 1$, remarkably, the relaxed total curvature of a continuous Cartesian curve does *not* depend on the Cantor part of the derivative [AM17a]: Cantor parts are energetically neutral. Weak notions of elastic energy for irregular rectifiable curves have been developed further by Mucci and Saracco [MS23], who showed that the relaxed p-elastic energy of a rectifiable curve is finite if and only if its arc-length parametrization belongs to $W^{2,p}$, see also the recent extension to curves in the sphere [MSS26]. Our example fits precisely into this picture: the graph constructed below is an intrinsically tame $C^1 \cap W^{2,2}$-curve in the sense of this theory, while the Cantor part appears only in one particular graph projection.

In the one-dimensional setting, the Willmore functional is the elastic energy of a curve. Boundary value problems and stability questions for the one-dimensional Willmore equation in graph form were studied by Deckelnick and Grunau [DG07, DG09]. Related boundary value, obstacle, threshold and p-elastic problems for curves and elastic graphs are treated, for instance, by Mandel [Man15], as well as in [Mül19, MR23, GO23, DMOY24, GO25]. Closely related Willmore problems for surfaces of revolution, which reduce to one-dimensional profile curve problems, go back to Langer and Singer [LS84] and are studied, for instance, in [DDG08, DFGS11, EG19, Sch24, DMRS25]. The corresponding geometric flow is the elastic flow, for existence, computation, and long-time behaviour we refer to Dziuk, Kuwert and Schätzle [DKS02], Lin [Lin12], Dall'Acqua, Lin, Pozzi and Spener [DLP17, DPS16], and to the survey of Mantegazza, Pluda and Pozzetta [MPP21].

The expectation that finite energy excludes diffuse singular parts is also suggested by the theory of free discontinuity problems. In the framework of De Giorgi and Ambrosio [DGA88], functionals such as the Mumford-Shah energy [MS89] are posed in SBV, and by Ambrosio's compactness theorem [AFP00, Thm. 4.8] energy-bounded sequences cannot develop a Cantor part in the limit. The same philosophy persists for second-order functionals such as the Blake-Zisserman energy [BZ87, CLT97]. In contrast, the relaxed Willmore energy of graphs is a geometrically natural second-order functional whose finiteness does *not* exclude a nontrivial Cantor part.

The mechanism in this work is that the Cantor part is concentrated exactly on the set where the absolutely continuous slope blows up. Thus the singular diffuse part is compatible with the relaxation theory. Geometrically, this also shows that the Cantor part of a graph derivative is projection-dependent: after a suitable rotation, the same curve may be represented by a Lipschitz graph without any singular derivative. This should be compared with the rectifiability theory for curvature varifolds. In particular, by second-order rectifiability, proved by Menne in [Men13], a graph with finite Willmore energy represented by a varifold is, up to an area null set, covered by a countable collection of C^2-manifolds. All this makes the following main theorem very surprising, since geometrically one would rather expect infinite relaxed Willmore energy because of the non-vanishing Cantor part. It seems that by naively applying projection techniques, one can get these irregular sets.

1.1 Main Results

We now state the condensed versions of the main results of the paper. We begin with the Willmore case for two reasons. First, this is the case from which the original motivation for the problem arose. Second, it is the simplest and geometrically most transparent instance of the construction, and therefore provides a convenient model before the same idea is formulated in the general weighted setting and, in particular, for L^p-curvature energies. It is developed from an idea in unpublished notes of H.-Ch. Grunau and relies heavily on the relaxation theory of [DMFLM09, Prop. 2.3 and Thm. 3.4]. Part of these results already appeared in the PhD thesis [Gul24] of B. Gulyak.

1 Theorem (Finite relaxed Willmore energy with Cantor part)
There exists a function $u \in BV((0,1)) \cap C([0,1])$ with $\overline{\mathcal{W}}(u) < \infty$ so that $|D^c u|((0,1)) > 0$ and in particular $u \notin SBV((0,1))$.

The graph associated with u admits an arc-length parametrization of class $C^1 \cap W^{2,2}$, and its squared curvature energy equals $\mathcal{W}^a(u)$. Nevertheless, the classical curvature on the smooth graph pieces is unbounded near every point of the Cantor trace. Hence the graph is not locally a regular C^2-curve at any Cantor point.

Proof: For the full result, see Thm. 5 for the construction and the relaxed-energy estimate, and Thm. 9 for the geometric interpretation. □

We also record several consequences and geometric interpretations of the construction. First, as explained in Corollary 6, there is no positive energy threshold for the pure relaxed Willmore term which excludes Cantor parts. The example of Thm. 1 can be rescaled so that the relaxed Willmore energy becomes arbitrarily small while the Cantor part remains nontrivial.

Second, the Cantor part is not an intrinsic singularity of the underlying curve. It is a feature of the chosen projection direction. As discussed in Remark 8, after a suitable clockwise rotation, the same geometric curve can be represented as a Lipschitz graph. The derivative of this rotated graph function has no singular part.

There is also a complementary C^2-variant of the construction, see Remark 10. By modifying the singular profile, one can arrange that the curvature of the smooth graph pieces extends continuously across the Cantor set by setting it equal to zero there. In that case the graph is a one-dimensional C^2-submanifold of $\mathbb{R}^2$, although the original graph function still satisfies $D^c u \neq 0$. Thus Cantor parts in graph projections may occur even for geometrically C^2-regular curves.

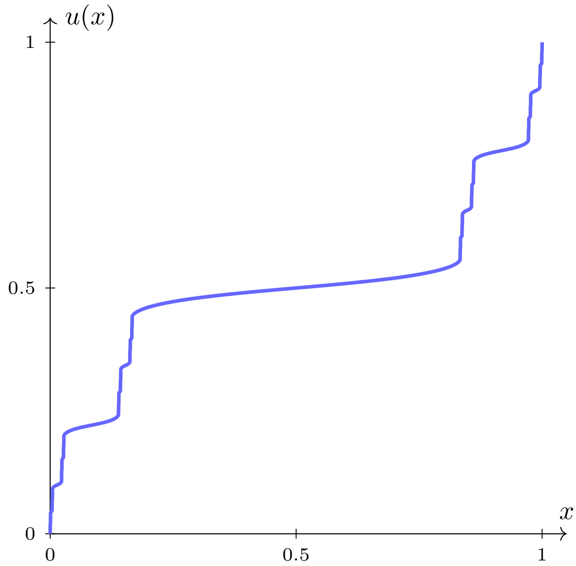


Figure 1: A rescaled version of the example, $u(x) = \frac{1}{2}U(x)/U(1) + \frac{1}{2}f_\delta(x)$, where U and f_δ are constructed in the proof of Thm. 5. The parameters are $s = \frac{6}{5}$, $\delta = \frac{1}{6}$, and $\beta = \frac{2}{5}$. By Corollary 6 this rescaled example still has finite relaxed Willmore energy.

This should be compared with Menne's second-order rectifiability theorem [Men13, Thm. 3.6], see Remark 11. There is no contradiction: the theorem gives a measure-theoretic C^2-covering of the associated varifold, whereas the Cantor part appears in a particular graph projection.

The analytic mechanism behind the construction is more general. We prove the following weighted version, extending the Cantor-part examples from [DMFLM09, Prop. 2.3 and Remark 3.2 (iv)] to all exponents $p > 1$.

2 Theorem (General weighted version)
Let $p > 1$, and let $\psi : \mathbb{R} \to (0, +\infty)$ be a bounded Borel weight satisfying the standard assumptions of the relaxation theory. If, for some $\alpha > 2p - 1$,

$$\psi(t) \leq Ct^{-\alpha} \qquad \textit{for all } t \geq 1,$$

then there exists

$$u \in X^p_\psi((0,1)) \quad \textit{such that} \quad D^c u \neq 0.$$

Proof: For the notation see Section 2. For the proof, see Thm. 12. □

Thm. 2 extends [DMFLM09, Prop. 2.3 and Remark 3.2 (iv)], where such examples were obtained only for exponents $p \in (1, 2)$ sufficiently close to 1, to all exponents $p > 1$.

The construction applies to relaxed L^p-curvature energies of $\mathcal{W}_p$ for $p > 1$. Indeed, the corresponding weight has decay exponent $3p - 1 > 2p - 1$, so the Cantor-profile construction is compatible with the relaxation theorem for $\mathcal{F}_p$. Consequently, finite relaxed L^p-curvature energy does not exclude a nontrivial Cantor part of the distributional derivative in some projection. That means that one cannot avoid these phenomena just by choosing the power of the curvature contribution high enough.

3 Corollary (Cantor parts for L^p-curvature energies)
Let $p > 1$. Then there exists a function $u \in BV((a,b)) \cap C^0([a,b])$ such that

$$D^c u \neq 0 \qquad \textit{and} \qquad \overline{\mathcal{W}}_p(u) < \infty.$$

In particular, finite relaxed L^p-curvature energy does not imply $u \in SBV((a,b))$.

Proof: See the proof of Corollary 13 $\square$

Geometrically, as in the Willmore case, the graph associated with the constructed function admits an arc-length parametrization and its weak scalar curvature belongs to $L^p(0, L)$. Hence the Cantor part in the derivative is compatible with finite L^p-curvature of the underlying curve. By Remark 14, the graph can be chosen as a C^2-curve, although the corresponding graph function still has $D^c u \neq 0$.

Finally, the C^2-variant has a surface-of-revolution consequence. After adding a sufficiently large constant to make the profile strictly positive, one may revolve the resulting C^2-curve around the x-axis. This gives a C^2-surface of revolution with finite Willmore energy, while the generating radius function belongs to $BV \cap C$ and still has a nontrivial Cantor part in its distributional derivative.

1.2 Further research

To the authors' knowledge, it is still open whether *Willmore minimizers* may have a nonzero Cantor part. More precisely, one may ask whether there are boundary conditions, obstacle constraints, or lower-order terms for which a minimizer of the relaxed one-dimensional Willmore functional has a nonvanishing Cantor part. Conversely, it would be important to characterize additional assumptions under which finite relaxed energy, together with minimality, forces $D^c u = 0$, or even implies higher regularity away from jump points.

This question is particularly relevant in connection with the *two-dimensional graph problem* studied by Deckelnick, Grunau and Röger [DGR17]. In that setting, minimizing sequences for the Willmore functional may develop vertical parts, and the relaxed functional naturally acts on BV-limits. It remains unclear whether genuinely diffuse singular parts can occur in minimizers, or whether some variational structure rules them out. Although our construction is one-dimensional, it can be embedded into the two-dimensional graph setting by a product construction. Let $u\colon [0,1] \to \mathbb{R}$ have finite one-dimensional Willmore energy as a graph, then we can extend it to a function $\bar{u}\colon [0,1]^2 \to \mathbb{R}$ with finite two-dimensional Willmore energy as a graph. We simply set for all $(x,y) \in [0,1]^2 : \bar{u}(x,y) = u(x)$. Then $\mathrm{graph}[\bar{u}]$ is the product of the planar graph of u with $[0,1]$. Its principal curvatures are κ_u and 0. Hence, with the convention $H = (k_1 + k_2)/2$,

$$\mathcal{W}(\bar{u}) = \int_{\mathrm{graph}[\bar{u}]} H^2 \,\mathrm{d}A = \frac{1}{4}\int_{\mathrm{graph}[u]} \kappa_u^2 \,\mathrm{d}s = \frac{1}{4}\mathcal{W}(u).$$

By the slicing theorem for BV-functions [AFP00, Theorems 3.107 and 3.108], or directly by testing against smooth compactly supported vector fields, one obtains $D^c\bar{u} = D^c u \otimes \mathcal{L}^1 \llcorner (0,1)$. In particular, if $D^c u \neq 0$, then $D^c \bar{u} \neq 0$.

A related question is whether the present construction can be modified so that the regular pieces of the graph are not only finite-energy pieces, but *critical points of the smooth Willmore functional*. In the present proof, the singular profile in the absolutely continuous part is chosen in order to guarantee $w \in L^1((0,1))$ and $\Psi_2 \circ w \in W^{1,2}((0,1))$, and to make the Cantor part compatible with the blow-up set of w. It would be interesting to know whether one can choose profiles w on the complementary intervals of the Cantor set such that the corresponding graph pieces solve the stationary Willmore equation and still satisfy $w(x) \to +\infty$ at the endpoints of the intervals. Such a construction would produce a one-dimensional relaxed Willmore critical point with nontrivial Cantor part.

Another natural direction is to characterize the singular continuous measures that may arise as Cantor parts of distributional derivatives of functions with finite pure relaxed Willmore energy. Finally, to the best of our knowledge, the exact relaxation of the pure Willmore term remains open. The relaxation formula of Dal Maso, Fonseca, Leoni and Morini [DMFLM09] applies to the augmented functional containing the total variation term, but it does not directly yield a formula for the lower semicontinuous envelope of the pure Willmore energy alone.

2 Preliminaries

In this section we collect the background for the proof of Thm. 5 below. We first recall the theory of BV functions of one variable, with emphasis on the decomposition of the distributional derivative into absolutely continuous, jump and Cantor part. We then present the relaxation theorem of Dal Maso, Fonseca, Leoni and Morini [DMFLM09], which for $p = 2$ identifies the relaxation of the one-dimensional Willmore energy augmented by a total variation term and is the central tool of our construction.

Functions of bounded variation

We briefly recall the theory of BV functions of one variable, following [DMFLM09, Subsection 2.1] and [AFP00]. A function $u \in L^1((a,b))$ belongs to $BV((a,b))$ if and only if its *total variation*

$$\mathrm{V}(u,(a,b)) := \sup\left\{ \int_a^b u\,\varphi'\,\mathrm{d}x \;\middle|\; \varphi \in C_0^1((a,b)) \text{ and } \|\varphi\|_\infty \leq 1 \right\}$$

is finite. In this case the distributional derivative Du of u is a bounded Radon measure on (a,b), and $|Du|$ denotes its total variation measure. In particular, by [AFP00, Proposition 3.6] it holds $|Du|((a,b)) = \mathrm{V}(u,(a,b))$. By the Lebesgue decomposition with respect to $\mathcal{L}^1$, the measure Du splits into an absolutely continuous part and a singular part,

$$Du = (u')^a\,\mathcal{L}^1 + D^s u = (u')^a\,\mathcal{L}^1 + D^j u + D^c u,$$

where $D^j u$ denotes the jump part and $D^c u$ the Cantor part of Du, which we characterize below. Every function $u \in BV((a,b))$ is $\mathcal{L}^1$-a.e. differentiable in (a,b), and for $\mathcal{L}^1$-a.e. $x \in (a,b)$ the classical derivative coincides with $(u')^a(x)$. Moreover, for every $u \in BV((a,b))$ the left and right approximate limits

$$u_-(y) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{y-\varepsilon}^{y} u(x)\,\mathrm{d}x \quad \text{and} \quad u_+(y) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{y}^{y+\varepsilon} u(x)\,\mathrm{d}x$$

exist at every point $y \in (a,b)$ and define a left-continuous and a right-continuous function, respectively. The functions u_- and u_+ coincide $\mathcal{L}^1$-a.e. in (a,b). The exceptional set where they differ,

$$S_u := \left\{ y \in (a,b) \,\middle|\, u_-(y) \neq u_+(y) \right\},$$

is called the set of essential discontinuities, or jump set, of u, it is at most countable. With the counting measure $\mathcal{H}^0$ concentrated on S_u, the singular part can be written as

$$D^s u = (u_+ - u_-)\,\mathcal{H}^0 \llcorner S_u + D^c u,$$

that is, the jump part $D^j u = (u_+ - u_-)\,\mathcal{H}^0 \llcorner S_u$ is a purely atomic measure, while the Cantor part $D^c u$ is a singular diffuse measure.

There is also a pointwise notion of variation for functions defined everywhere. We recall that $u : (a,b) \to \mathbb{R}$ has bounded pointwise variation $\mathrm{pV}(u,(c,d))$ on an interval $(c,d) \subset (a,b)$ if

$$\mathrm{pV}(u,(c,d)) := \sup \sum_{i=1}^{k} |u(y_i) - u(y_{i-1})| < +\infty,$$

where the supremum is taken over all finite families of points $(y_0, y_1, \dots, y_k)$ with $c < y_0 < y_1 < \cdots < y_k < d$ and $k \in \mathbb{N}$. The approximate limits u_- and u_+ defined above are in fact precise and good representatives of $u \in BV((a,b))$, in the sense that for every interval $(c,d) \subset (a,b)$

$$|Du|((c,d)) = \mathrm{pV}(u_-, (c,d)) = \mathrm{pV}(u_+, (c,d)).$$

Conversely, if a function $u \in L^1((a,b))$ has bounded pointwise variation in (a,b), then it belongs to $BV((a,b))$, with $|Du|((c,d)) \le \mathrm{pV}(u,(c,d))$ for every interval $(c,d) \subset (a,b)$.

Finally, we recall a compactness result for BV spaces. If $\{u_k\}_{k=1}^{\infty}$ is a sequence in $BV((a,b))$ satisfying the condition

$$\sup_{k\in\mathbb{N}} \left\{ \|u_k\|_{L^1((a,b))} + |Du_k|((a,b)) \right\} < \infty,$$

then there exist a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and a limit function $u \in BV((a,b))$ such that

$$u_{k_j} \to u \ \text{ in } L^1((a,b)) \text{ as } j \to \infty \quad \text{ and } \quad \lim_{j\to\infty} \int_a^b \varphi \, \mathrm{d}Du_{k_j} = \int_a^b \varphi \, \mathrm{d}Du \ \text{ for all } \varphi \in C_0^0((a,b)),$$

i.e. the derivatives Du_{k_j} converge weakly-$*$ to Du in the sense of measures.

The relaxation theorem of Dal Maso-Fonseca-Leoni-Morini

Let us now focus on some results from [DMFLM09]. There the authors consider the following functional, arising from a total variation-based model for image restoration involving a second-order term that eliminates the staircase effect. For an exponent $p \in (1, +\infty)$ let $\mathcal{F}_p : L^1((a,b)) \to [0, +\infty]$ be defined by

$$\mathcal{F}_p(u) := \begin{cases} \displaystyle\int_a^b |u'| \, \mathrm{d}x + \int_a^b \psi(u') \, |u''|^p \, \mathrm{d}x & \text{if } u \in W^{2,p}((a,b)), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\psi : \mathbb{R} \to (0, +\infty)$ is a bounded Borel function to be specified below. In particular, we extend $\mathcal{F}_p$ to $L^1((a,b))$ by setting $\mathcal{F}_p(u) := +\infty$ if $u \in L^1((a,b)) \setminus W^{2,p}((a,b))$. Using the theory of relaxation, one identifies the lower semicontinuous envelope of $\mathcal{F}_p$ with respect to strong L^1-convergence: for every $u \in L^1((a,b))$ we set

$$\overline{\mathcal{F}}_p(u) := \inf \left\{ \liminf_{k\to\infty} \mathcal{F}_p(u_k) \,\middle|\, u_k \to u \text{ in } L^1((a,b)) \right\}, \tag{1}$$

where the infimum is taken over all sequences $\{u_k\}_{k\in\mathbb{N}}$ in $L^1((a,b))$ with $u_k \to u$ in $L^1((a,b))$.

At this point, let us briefly compare $\mathcal{F}_p$ with the Willmore energy. For $p = 2$ we obtain

$$\mathcal{F}_2(u) = \mathcal{W}(u) + \int_a^b |u'| \, \mathrm{d}x, \quad \text{ with } \quad \psi(\tau) := \frac{1}{(1+\tau^2)^{5/2}} \quad \text{ for all } \tau \in \mathbb{R}. \tag{2}$$

Thus, for $p = 2$, the second-order term in $\mathcal{F}_2$ is exactly the one-dimensional Willmore energy. The relaxation theorem for $\mathcal{F}_2$ therefore provides a useful framework for constructing finite-energy approximating sequences, although the pure functional $\mathcal{W}$ does not contain the total variation term $\int_0^1 |u'| \, \mathrm{d}x$.

In this context, it is important to notice that, even under the boundary conditions, finiteness of $\mathcal{W}(u)$ alone does not yield an $L^1((0,1))$-bound on u'. Such a bound follows only under an additional smallness assumption on $\mathcal{W}(u)$. Without this assumption, the graph in the relaxed setting may contain vertical segments of arbitrarily large length that are not penalized by $\mathcal{W}(u)$.

The underlying obstruction is classical in the theory of free elastica and is often treated as folklore; for a convenient modern reference, see [DGN+25]. The first nontrivial stationary free elastica are given by half-periods of Euler's rectangular elastica and occur at the scale-invariant energy level 2π. In their explicit arc-length parametrization, the horizontal velocity vanishes at the midpoint of a half-period. Thus the graphical parametrization develops a vertical tangent at this first nontrivial threshold.

But first, we impose two additional conditions on the bounded Borel function $\psi : \mathbb{R} \to (0, +\infty)$, namely

$$M := \int_{-\infty}^{+\infty} (\psi(t))^{1/p} \, \mathrm{d}t < +\infty \quad \text{and} \quad \inf_{t \in K} \psi(t) > 0 \text{ for every compact set } K \subset \mathbb{R}. \tag{3}$$

If we now define $\Psi_p : \overline{\mathbb{R}} \to [0, M]$ as the primitive of $\psi^{1/p}$ by

$$\Psi_p(t) := \int_{-\infty}^{t} (\psi(s))^{1/p} \, \mathrm{d}s, \tag{4}$$

and while its inverse function is denoted by $\Psi_p^{-1} : [0, M] \to \overline{\mathbb{R}}$, then for every $u \in W^{2,p}((a,b))$ we obtain

$$\mathcal{F}_p(u) = \int_a^b |u'| \, \mathrm{d}x + \int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}x} \Big(\Psi_p \circ u' \Big)(x) \right|^p \mathrm{d}x. \tag{5}$$

In [DMFLM09, Theorem 3.4] the authors characterize the relaxation of the functional $\mathcal{F}_p$ with respect to strong convergence in $L^1((a,b))$. To this end, they introduce the subspace of L^1-functions that can be approximated by $\mathcal{F}_p$-bounded sequences, which they call $X^p_\psi((a,b))$ [DMFLM09, Definition 3.1 and Remark 3.2] and which we recall now. In view of (5), one of the properties of a function $u \in X^p_\psi((a,b))$ has to be $v := \Psi_p \circ (u')^a \in W^{1,p}((a,b))$. Since Ψ_p^{-1} is continuous and $v \in C^0([a,b])$ by Sobolev embedding, it follows that $\Psi_p^{-1}(v)$ is continuous on $[a,b]$ with values in $\overline{\mathbb{R}}$. Therefore, there is a continuous $\overline{\mathbb{R}}$-valued representative of the absolutely continuous density $(u')^a$ induced by $\Psi_p^{-1}(v)$. Hence, in all pointwise statements for $(u')^a$ below, we use the chosen continuous representative of the absolutely continuous so-called Radon-Nikodym density $(u')^a$.

Next,the sets where the continuous $\overline{\mathbb{R}}$-valued representative of the absolutely continuous part $(u')^a$ blows up are denoted by

$$Z^+ \left[(u')^a \right] := \left\{ x \in (a,b) \;\middle|\; (u')^a(x) = +\infty \right\}, \quad Z^- \left[(u')^a \right] := \left\{ x \in (a,b) \;\middle|\; (u')^a(x) = -\infty \right\}.$$

Then we define

$$X^p_\psi((a,b)) := \left\{ u \in BV((a,b)) \;\middle|\; \Psi_p \circ (u')^a \in W^{1,p}((a,b)),\ (D^s u)^\pm \text{ is concentrated on } Z^\pm \left[(u')^a \right] \right\}.$$

With the set $X^p_\psi((a,b))$ at hand, the authors of [DMFLM09, Theorem 3.4] were able to characterize the relaxation of $\mathcal{F}_p$ with respect to strong convergence in $L^1((a,b))$ defined in (1):

4 Theorem (Dal Maso-Fonseca-Leoni-Morini)
It holds

$$\overline{\mathcal{F}}_p(u) = \begin{cases} |Du|\,((a,b)) + \displaystyle\int_a^b \left| v' \right|^p \mathrm{d}x & \text{if } u \in X^p_\psi((a,b)), \\ +\infty & \text{otherwise,} \end{cases}$$

where $v := \Psi_p \circ (u')^a$, so that the higher-order term depends only on $(u')^a$.

As in [DMFLM09, Remark 3.2.], let us discuss some properties of functions $u \in X^p_\psi((a,b))$. For every jump point x_0 with $u_+(x_0) - u_-(x_0) > 0$ it holds $\lim_{x\to x_0}(u')^a(x) = +\infty$, and for every jump point x_0 with $u_+(x_0) - u_-(x_0) < 0$ we have $\lim_{x\to x_0}(u')^a(x) = -\infty$. Furthermore, if Du has a non-vanishing Cantor or jump part, then $(u')^a$ cannot be bounded. In particular, piecewise constant functions with jumps and the Cantor function are excluded from $X^p_\psi((a,b))$, since for them $Z^\pm[(u')^a] = \varnothing$. The same applies to polygons. For the case $p = 2$, this means that the Cantor function (for its definition, see the proof of Thm. 5 and Fig. 2) and piecewise smooth graphs with corners have infinite relaxed $\mathcal{F}_2$ energy.

In view of the Cantor function example, one might expect that all functions with non-trivial Cantor part are excluded from $X^p_\psi((a,b))$. Surprisingly, in [DMFLM09, Remark 3.2 (iv)] Dal Maso, Fonseca, Leoni and Morini constructed functions with non-trivial Cantor part in $X^p_\psi((a,b))$ for exponents $p \in (1,2)$ sufficiently close to 1, provided that ψ satisfies

$$\psi(t) \le c\,t^{-\alpha} \quad \text{for all } t \ge 1$$

and some $c > 0$, $\alpha > 1$. The function ψ corresponding to the Willmore term in $\mathcal{F}_2$, defined in (2), satisfies this condition with $\alpha = 5$, as well as condition (3). In this paper, we extend this result to $p = 2$, hence to the relaxed Willmore energy $\overline{\mathcal{W}}$, by constructing a function u with $\overline{\mathcal{W}}(u) < +\infty$ and non-vanishing Cantor part. This is indeed a surprising result, since $p = 2$ is not close to 1 and is therefore not covered by [DMFLM09, Remark 3.2 (iv)].

3 Finiteness of the Relaxed Willmore Energy Does Not Imply SBV

By the relaxation theorem of Dal Maso-Fonseca-Leoni-Morini (see Subsection 2), a singular part of Du is admissible only if it is concentrated on the blow-up set of the absolutely continuous slope $(u')^a$. The pure Cantor function fails this criterion, since its absolutely continuous derivative vanishes. We therefore superpose a generalized Cantor function f_δ with a continuous monotone function U whose derivative tends to $+\infty$ precisely on the Cantor set, while $\Psi_2 \circ (u')^a$ remains in $W^{1,2}((0,1))$. For $u := U + f_\delta$ this yields $\overline{\mathcal{W}}(u) \le \mathcal{W}^a(u) < \infty$ despite $D^c u \ne 0$. Two remarks complement the theorem: the example can be rescaled so that its relaxed Willmore energy becomes arbitrarily small (Corollary 6), and the Cantor part is a projection effect that disappears after rotating the graph (Remark 8).

5 Theorem
There exists a function $u \in BV((0,1))$ such that $\overline{\mathcal{W}}(u) < \infty$ and $|D^c u|((0,1)) > 0$. In particular, $u \notin SBV((0,1))$ with

$$\overline{\mathcal{W}}(u) \le \mathcal{W}^a(u) = \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}x} \Psi_2((u')^a(x)) \right|^2 \mathrm{d}x.$$

Proof: By the characterization in [DMFLM09], the Cantor function $f_\delta : [0,1] \to [0,1]$, introduced in Step 2, cannot serve as our example. Indeed, its positive singular derivative is concentrated on a Cantor set $\mathbb{D}_\delta$, defined in (6), whereas $Z^+[(f'_\delta)^a] = \varnothing$ and therefore $\overline{\mathcal{F}}_2(f_\delta) = +\infty$.

To overcome this difficulty, we add a continuous function $U : [0,1] \to \mathbb{R}$, constructed in Steps 3-6, and define $u := f_\delta + U$. The function U is chosen in such a way that $\mathbb{D}_\delta \subset Z^+[(u')^a]$, therefore obeying $(U')^a = +\infty$ on the Cantor set, while at the same time its one-dimensional Willmore energy remains finite. Thus, f_δ generates the Cantor part of the derivative, whereas U ensures that this singular part is compatible with finite relaxed Willmore energy.

To prove that $\overline{\mathcal{W}}(u) < \infty$, we construct in Step 7 a sequence $(u_k)_{k\in\mathbb{N}} \subset W^{2,2}((0,1))$ by using Thm. 4.

1 We first recall the auxiliary function

$$\Psi_2(t) := \int_{-\infty}^{t} \frac{1}{(1+\tau^2)^{5/4}}\,\mathrm{d}\tau, \qquad M := \Psi_2(\infty) = \int_{\mathbb{R}} \frac{1}{(1+\tau^2)^{5/4}}\,\mathrm{d}\tau = \frac{\sqrt{\pi}\,\Gamma\big(\frac{3}{4}\big)}{\Gamma\big(\frac{5}{4}\big)}.$$

where Γ denotes the Gamma function. Following [DG07, Lemma 1], Ψ_2 extends to a continuous, strictly increasing map

$$\Psi_2 \colon \overline{\mathbb{R}} \to [0, M], \qquad \Psi_2^{-1} : [0, M] \to \overline{\mathbb{R}}.$$

We also set the function $\psi \colon \mathbb{R} \to (0, \infty)$ by

$$\psi(\tau) := \frac{1}{(1+\tau^2)^{5/2}}, \qquad \tau \in \mathbb{R}.$$

For $u \in W^{2,2}((0,1))$, we recall the one-dimensional Willmore functional

$$\begin{aligned} \mathcal{W}(u) &:= \int_0^1 \psi(u')\,|u''|^2\,\mathrm{d}x = \int_0^1 \frac{|u''(x)|^2}{(1+u'(x)^2)^{5/2}}\,\mathrm{d}x \\ &= \int_0^1 \kappa_u(x)^2 \sqrt{1+u'(x)^2}\,\mathrm{d}x = \int_{\mathrm{graph}[u]} \kappa_u^2\,\mathrm{d}s \\ &= \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}x} \Psi_2(u'(x)) \right|^2 \mathrm{d}x. \end{aligned}$$

Here $\kappa_u = u''(1+u'^2)^{-3/2}$ denotes the curvature of the graph of u.

2 Following [DMFLM09, pp. 2356 ff.], we now construct the generalized Cantor set $\mathbb{D}_\delta$ and the associated Cantor function f_δ. Let

$$\delta \in \left(0, \frac{1}{2}\right) \tag{C1}$$

with δ be suitably chosen below. Its precise value will be chosen later. Starting from the interval $[0,1]$, we first remove the open middle interval of length $1-2\delta$, namely

$$I_{11} := (\delta, 1-\delta), \quad \text{therefore} \quad [0,1] \setminus I_{11} = [0,\delta] \cup [1-\delta, 1].$$

In the second step, we remove from each of these two intervals its open middle subinterval of length $\delta(1-2\delta)$, namely

$$I_{21} := \big(\delta^2,\ \delta(1-\delta)\big), \quad I_{22} := \big((1-\delta)+\delta^2,\ (1-\delta)+\delta(1-\delta)\big) = (1-\delta+\delta^2,\ 1-\delta^2).$$

We repeat the procedure on $[0,1] \setminus (I_{11} \cup I_{21} \cup I_{22})$ and remove four open intervals $I_{31}, I_{32}, I_{33}, I_{34}$. Iterating this construction, after $j-1$ steps there remain 2^{j-1} closed intervals, each of length δ^{j-1}. In step j, we remove from the middle of each of these intervals an open subinterval of length $\delta^{j-1}(1-2\delta)$. These intervals are denoted by

$$I_{jk}, \qquad k = 1, \ldots, 2^{j-1},$$

ordered from left to right. In particular, $I_{j1} = \big(\delta^j, \delta^{j-1}(1-\delta)\big)$.

We thus define the generalized Cantor set by

$$\mathbb{D}_\delta := \bigcap_{\ell=1}^{\infty} C_\ell \quad \text{with} \quad C_\ell := [0,1] \setminus \bigcup_{j=1}^{\ell} \bigcup_{k=1}^{2^{j-1}} I_{jk}, \ell \in \mathbb{N}. \tag{6}$$

Then C_ℓ consists of 2^ℓ intervals, each of length δ^ℓ. Hence

$$\mathcal{L}^1(C_\ell) = 2^\ell \delta^\ell = (2\delta)^\ell \xrightarrow[\ell\to\infty]{} 0,$$

and therefore $\mathbb{D}_\delta = \bigcap_{\ell=1}^\infty C_\ell$ has Lebesgue measure zero.

We now define the approximating densities and functions by

$$g_\ell := \frac{1}{(2\delta)^\ell} \mathbb{1}_{C_\ell} = \frac{1}{(2\delta)^\ell}\left(1 - \sum_{j=1}^{\ell}\sum_{k=1}^{2^{j-1}} \mathbb{1}_{I_{jk}}\right), \qquad f_\ell(x) := \int_0^x g_\ell(\xi)\,\mathrm{d}\xi.$$

By construction, f_ℓ is affine on each connected component of C_ℓ and constant on every interval I_{jk} with $1 \le j \le \ell$. More precisely, if I_{jk} denotes the k-th removed interval at level j, counted from left to right, then

$$f_\ell \equiv \frac{2k-1}{2^j} \qquad \text{on } I_{jk}. \tag{7}$$

On each of the 2^ℓ components of C_ℓ, the function f_ℓ is linear with slope $(2\delta)^{-\ell}$. Since $g_\ell \ge 0$, each f_ℓ is nondecreasing. Moreover,

$$f_\ell(1) = \int_0^1 g_\ell\,\mathrm{d}x = \frac{1}{(2\delta)^\ell}\mathcal{L}^1(C_\ell) = \frac{1}{(2\delta)^\ell}(2\delta)^\ell = 1.$$

Together with $f_\ell(0) = 0$, this shows that $f_\ell : [0,1] \to [0,1]$.

By Lemma 16, the sequence (f_ℓ) converges uniformly to a continuous function $f_\delta : [0,1] \to [0,1]$ as $\ell \to \infty$, namely the generalized Cantor function associated with $\mathbb{D}_\delta$. Moreover,

$$(f'_\delta)^a = 0 \text{ a.e. in } (0,1), \text{ and } Df_\delta = D^s f_\delta \text{ as Radon measures,}$$

whereas the Cantor part $D^c f_\delta$ is a nonnegative measure supported on $\mathbb{D}_\delta$. In particular, all variation of f_δ is concentrated on $\mathbb{D}_\delta$. The function f_δ is nondecreasing, satisfies

$$f_\delta(0) = 0, \qquad f_\delta(1) = 1,$$

and is constant on every connected component of $[0,1] \setminus \mathbb{D}_\delta$.

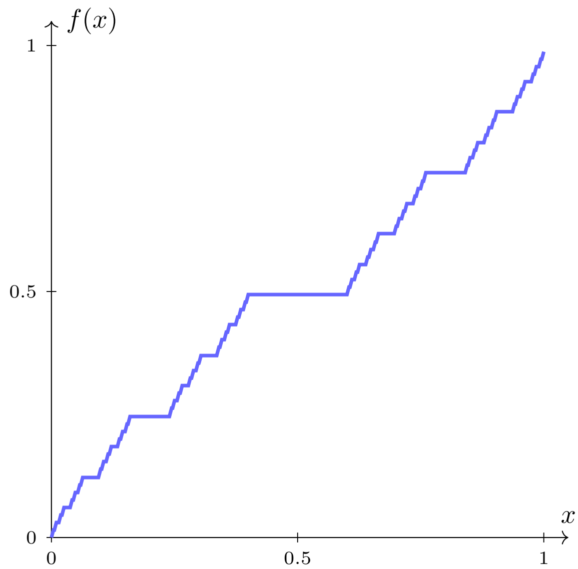


Figure 2: f_δ for $\delta = 0.4$

3 We now construct an auxiliary derivative $w = w_\delta$ for the function U that will later be added to f_δ. The Cantor part of the derivative will still come entirely from f_δ. The role of U is to modify only the absolutely continuous part in such a way that the resulting function has finite relaxed Willmore energy. For the moment, we therefore define only the derivative candidate

$$w_\delta : [0,1] \to [0,\infty],$$

which we require to be continuous, to satisfy

$$w_\delta(x) = +\infty \quad \iff \quad x \in \mathbb{D}_\delta,$$

and to obey

$$\Psi_2 \circ w_\delta \in W^{1,2}((0,1)).$$

Once U is defined as a primitive of w_δ, this condition will yield finite curvature energy for U. The specific singular profile Φ used below and differs from the construction in [DMFLM09]. We fix

$$\beta \in \left(\frac{1}{3}, \frac{1}{2}\right), \tag{C2}$$

where the upper bound $\beta < \frac{1}{2}$ has two roles. First, it implies $\beta < 1$ and therefore guarantees the integrability of the singularities of the profile at the endpoints 0 and 1 of the reference interval. Second, it is responsible for the unboundedness of the curvature near the endpoints of the removed intervals, used in Definition (18). For comparison, see also Remark 10. The lower bound $\beta > \frac{1}{3}$ will be needed later in the local Willmore estimates on the intervals I_{jk} in Step 5.

We define

$$\Phi : [0,1] \to [0,\infty], \qquad \Phi(x) := C_\beta \left(\frac{1}{x^\beta (1-x)^\beta} - 4^\beta \right),$$

where the normalizing constant $C_\beta > 0$ is chosen so that

$$\int_0^1 \Phi(x)\,\mathrm{d}x = 1.$$

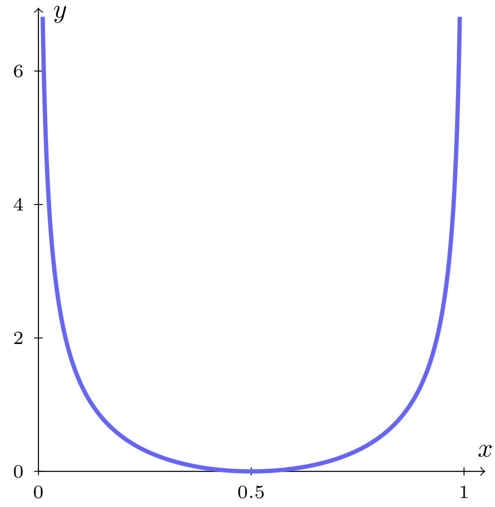
Since $x(1-x) \le \frac{1}{4}$ on $[0,1]$, we have $x^\beta(1-x)^\beta \le 4^{-\beta}$, with equality only at $x = \frac{1}{2}$. Hence $\Phi \ge 0$ and $\Phi(\frac{1}{2}) = 0$, which will later imply monotonicity of its primitive. Moreover, because $\beta < 1$, the singularities at the endpoints are integrable, and

$$\begin{aligned} C_\beta^{-1} &= \int_0^1 \left(\frac{1}{x^\beta (1-x)^\beta} - 4^\beta \right) \mathrm{d}x \\ &= \mathrm{B}(1-\beta, 1-\beta) - 4^\beta = \frac{\Gamma(1-\beta)^2}{\Gamma(2-2\beta)} - 4^\beta > 0, \end{aligned}$$

where B denotes Euler's Beta function [DLMF, (5.12.1)]. Up to a horizontal rescaling and an additive constant, Φ will serve as the basic profile for the absolutely continuous part of the derivative. Its primitive is given for $x \in [0,1]$ by

$$C_\beta^{-1} \int_0^x \Phi(s)\,\mathrm{d}s = \mathrm{B}_x(1-\beta, 1-\beta) - 4^\beta x = \frac{x^{1-\beta}}{1-\beta}\, {}_2F_1(1-\beta, \beta; 2-\beta; x) - 4^\beta x,$$

where B_x is the incomplete Beta function and ${}_2F_1$ denotes the Gauss hypergeometric function [DLMF, (8.17.7)].



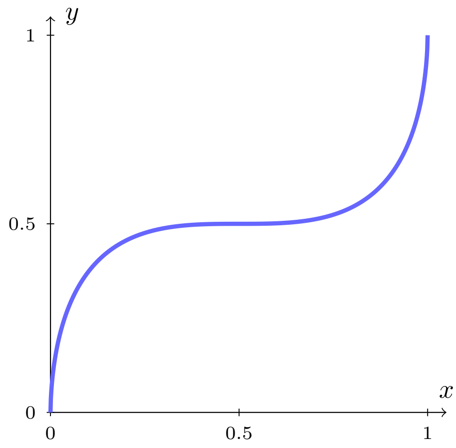


Figure 3: Left: $\Phi(x)$ for $\beta = \frac{2}{5}$, right: its primitive $x \mapsto \int_0^x \Phi(s)\,\mathrm{d}s$ for $\beta = \frac{2}{5}$.

Furthermore, Φ is strictly convex on $(0,1)$. Indeed,

$$\begin{aligned}
\frac{\Phi'(x)}{C_\beta} &= \frac{-\beta}{x^{\beta+1}(1-x)^\beta} + \frac{\beta}{x^\beta(1-x)^{\beta+1}} = \frac{\beta\big[-(1-x)+x\big]}{x^{\beta+1}(1-x)^{\beta+1}} = \frac{\beta\big[2x-1\big]}{x^{\beta+1}(1-x)^{\beta+1}},\\
\frac{\Phi''(x)}{C_\beta} &= \beta\frac{2}{x^{\beta+1}(1-x)^{\beta+1}} - \beta(\beta+1)\frac{(2x-1)(1-2x)}{x^{\beta+2}(1-x)^{\beta+2}} = \beta\frac{2x(1-x)+(1-2x)^2+\beta(1-2x)^2}{x^{\beta+2}(1-x)^{\beta+2}}\\
&= \beta\frac{(1-x)^2+x^2+\beta(1-2x)^2}{x^{\beta+2}(1-x)^{\beta+2}} > 0.
\end{aligned}$$

Hence Φ attains its unique minimum at $x = \frac{1}{2}$, where $\Phi(\frac{1}{2}) = 0$.

Next choose

(C3) $$s > 1 \iff 2^{1-3s} < 2^{-s-1},$$

so that the interval $\big(2^{1-3s},\, 2^{-s-1}\big)$ is nonempty so that later we can find a number δ between 2^{1-3s} and 2^{-s-1}.

Let a_{jk} denote the midpoint of the interval I_{jk}, so that

$$I_{jk} = \left(a_{jk} - \frac{1}{2}\delta^{j-1}(1-2\delta),\, a_{jk} + \frac{1}{2}\delta^{j-1}(1-2\delta)\right).$$

On each interval I_{jk} we place a vertically rescaled copy of Φ, shifted upward by 2^{sj}. More precisely, for $x \in I_{jk}$ we set

$$\Phi_{jk}(x) := 2^{sj} + 2^{sj}\Phi\left(\frac{x - a_{jk}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right),$$

and we define

$$w_\delta(x) := \begin{cases} \Phi_{jk}(x), & \text{if } x \in I_{jk} \text{ for some } j \geq 1,\ k = 1,\dots,2^{j-1},\\ +\infty, & \text{if } x \in \mathbb{D}_\delta. \end{cases}$$

The term 2^{sj} is an additional level-dependent background slope. It will later be crucial in the curvature estimates, since it enters the denominator of the relevant expressions.

By construction, w_δ is continuous as a function with values in $[0,\infty]$. On each interval I_{jk} this is immediate. At the endpoints of I_{jk} it follows from the fact that $\Phi(t)\to+\infty$ as $t\searrow 0$ or $t\nearrow 1$. Finally, if $x\in\mathbb{D}_\delta$ is not an endpoint of any removed interval, then every sequence approaching x from $[0,1]\setminus\mathbb{D}_\delta$ must eventually lie in intervals I_{jk} with $j\to\infty$, and hence $w_\delta\geq 2^{sj}\to+\infty$.

Since $\mathcal{L}^1(\mathbb{D}_\delta)=0$, only the intervals I_{jk} contribute to the integral. Using $|I_{jk}|=\delta^{j-1}(1-2\delta)$ and $\int_0^1\Phi(x)\,\mathrm{d}x=1$, we obtain

$$\begin{aligned}
&\int_{I_{jk}} w_\delta(x)\,\mathrm{d}x=\int_{I_{jk}}\Phi_{jk}(x)\,\mathrm{d}x=2^{sj}(1-2\delta)\delta^{j-1}+2^{sj}(1-2\delta)\delta^{j-1}\int_0^1\Phi(t)\,\mathrm{d}t\\
&\qquad\qquad=2(1-2\delta)\delta^{j-1}2^{sj}.\\
\Rightarrow\quad &\int_0^1 w_\delta(x)\,\mathrm{d}x=\sum_{j=1}^{\infty}\sum_{k=1}^{2^{j-1}}\int_{I_{jk}}w_\delta(x)\,\mathrm{d}x=\sum_{j=1}^{\infty}2^{j-1}2(1-2\delta)\delta^{j-1}2^{sj}\\
&\qquad\qquad=\frac{1-2\delta}{\delta}\sum_{j=1}^{\infty}(2\delta)^j2^{sj}.
\end{aligned}$$

If, in addition, we assume

$$\text{(C4)}\qquad \delta<2^{-s-1}\quad\iff\quad 2^{s+1}\delta<1,$$

then geometric series converges, and

$$\int_0^1 w_\delta(x)\,\mathrm{d}x=\frac{1-2\delta}{\delta}\left(\frac{2^{s+1}\delta}{1-2^{s+1}\delta}\right)=2^{s+1}\frac{1-2\delta}{1-2^{s+1}\delta}<\infty.$$

Hence w_δ is integrable on $(0,1)$. In particular, it represents an element of $L^1((0,1))$, and it takes the value $+\infty$ on the null set $\mathbb{D}_\delta$.

4 Since $w_\delta:[0,1]\to[0,\infty]$ is continuous and $\Psi_2:[0,\infty]\to[0,M]$ is continuous, we have $\Psi_2\circ w_\delta\in C^0([0,1])$. In particular, $0\leq\Psi_2\circ w_\delta\leq M$, and therefore

$$\Psi_2\circ w_\delta\in L^\infty((0,1))\subset L^2((0,1)).$$

For $x\in I_{jk}$, it holds $w_\delta(x)=\Phi_{jk}(x)$, and hence $\Psi_2\circ w_\delta$ is of class C^1 on each interval I_{jk}, with

$$\text{(8)}\qquad\begin{aligned}
(\Psi_2\circ w_\delta)'(x)&=(\Psi_2\circ\Phi_{jk})'(x)=\Psi_2'\big(\Phi_{jk}(x)\big)\cdot\Phi_{jk}'(x)\\
&=\frac{2^{sj}}{(1+\Phi_{jk}(x)^2)^{5/4}}\cdot\Phi'\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}}+\frac{1}{2}\right)\cdot\frac{1}{(1-2\delta)\delta^{j-1}}.
\end{aligned}$$

We now define a candidate weak derivative for $\Psi_2\circ w_\delta$ on all of $(0,1)$ by

$$G_\delta(x):=\begin{cases}(\Psi_2\circ w_\delta)'(x), & x\in[0,1]\setminus\mathbb{D}_\delta,\\ 0, & x\in\mathbb{D}_\delta.\end{cases}$$

In Step 5 we will prove that $G_\delta\in L^2((0,1))$, and hence also $G_\delta\in L^1((0,1))$. Assuming this for the moment, we now show that $\Psi_2\circ w_\delta$ is a pointwise absolutely continuous function and G_δ is indeed the weak derivative of $\Psi_2\circ w_\delta$, i.e.

$$\text{(9)}\qquad \Psi_2\circ w_\delta(x)=M+\int_0^x G_\delta(\xi)\,\mathrm{d}\xi\qquad\text{for all } x\in[0,1],$$

where $M=\Psi_2(\infty)$. Note that $\Psi_2\circ w_\delta(0)=\Psi_2(w_\delta(0))=\Psi_2(\infty)=M$, since $0\in\mathbb{D}_\delta$.

Let $I_{jk} = (\alpha_{jk}, \beta_{jk})$. Since $\alpha_{jk}, \beta_{jk} \in \mathbb{D}_\delta$, we have $\Psi_2 \circ w_\delta(\alpha_{jk}) = \Psi_2 \circ w_\delta(\beta_{jk}) = M$. Therefore,

$$\begin{aligned}\int_{I_{jk}} G_\delta(\xi)\,\mathrm{d}\xi &= \lim_{\varepsilon \searrow 0} \int_{\alpha_{jk}+\varepsilon}^{\beta_{jk}-\varepsilon} (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi \\ &= \lim_{\varepsilon \searrow 0} \big(\Psi_2 \circ w_\delta(\beta_{jk} - \varepsilon) - \Psi_2 \circ w_\delta(\alpha_{jk} + \varepsilon)\big) = M - M = 0.\end{aligned}$$

Now let $x \in [0, 1]$. If $x \in \mathbb{D}_\delta$, then $\Psi_2 \circ w_\delta(x) = M$, and $(0, x) \setminus \mathbb{D}_\delta$ is the disjoint union of all removed intervals contained in $(0, x)$. Hence, since $G_\delta(x) = 0$ for all $x \in \mathbb{D}_\delta$, it follows

$$M + \int_0^x G_\delta(\xi)\,\mathrm{d}\xi = M + \sum_{I_{jk} \subset (0,x)} \int_{I_{jk}} G_\delta(\xi)\,\mathrm{d}\xi = M = \Psi_2 \circ w_\delta(x), \quad \checkmark$$

which proves (9) in this case.

Now suppose that $x \notin \mathbb{D}_\delta$. Then there exists a unique interval $I_{j_0k_0} = (\alpha_{j_0k_0}, \beta_{j_0k_0})$ such that $x \in I_{j_0k_0}$. Since $\alpha_{j_0k_0} \in \mathbb{D}_\delta$, the previous case gives

$$\int_0^{\alpha_{j_0k_0}} G_\delta(\xi)\,\mathrm{d}\xi = 0.$$

This yields (9) also for $x \notin \mathbb{D}_\delta$ since

$$\begin{aligned}\int_0^x G_\delta(\xi)\,\mathrm{d}\xi + M &= \int_0^{\alpha_{j_0k_0}} G_\delta(\xi)\,\mathrm{d}\xi + \int_{\alpha_{j_0k_0}}^x G_\delta(\xi)\,\mathrm{d}\xi + M \\ &= 0 + \lim_{\varepsilon \searrow 0} \int_{\alpha_{j_0k_0}+\varepsilon}^x (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + M \\ &= \lim_{\varepsilon \searrow 0} \big(\Psi_2 \circ w_\delta(x) - \Psi_2 \circ w_\delta(\alpha_{j_0k_0} + \varepsilon)\big) + M = \Psi_2 \circ w_\delta(x). \quad \checkmark\end{aligned}$$

Hence, once Step 5 establishes $G_\delta \in L^2((0, 1))$, we conclude that

$$\Psi_2 \circ w_\delta \in W^{1,2}((0, 1)), \qquad (\Psi_2 \circ w_\delta)' = G_\delta \quad \text{a.e. in } (0, 1).$$

5 We now estimate the local curvature contribution on each removed interval I_{jk}. Let

$$E_{jk} := \int_{I_{jk}} \big|(\Psi_2 \circ w_\delta)'(x)\big|^2\,\mathrm{d}x.$$

In what follows, we write $A \preceq B$ if $A \le CB$ for some constant $C > 0$ independent of j and k. It is important to observe that the following estimates are uniform with respect to k and j. Using the formula (8) from Step 4 and a change of variables we obtain

$$\begin{aligned}E_{jk} &= \frac{\delta^2}{(1-2\delta)^2\delta^{2j}} \int_{I_{jk}} \frac{\left[2^{sj}\Phi'\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right)\right]^2}{\left(1 + \left[2^{sj} + 2^{sj}\Phi\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right)\right]^2\right)^{5/2}}\,\mathrm{d}x \\ &\preceq \frac{\delta}{(1-2\delta)\delta^j} \int_0^1 \frac{\left[2^{sj}\Phi'(y)\right]^2}{\left(1 + \left[2^{sj} + 2^{sj}\Phi(y)\right]^2\right)^{5/2}}\,\mathrm{d}y \overset{2\delta<2^{-s}}{\underset{\text{(C4)}}{\preceq}} \delta^{-j} \int_0^1 \frac{\left[2^{sj}\Phi'(y)\right]^2}{\left(2^{sj} + 2^{sj}\Phi(y)\right)^5}\,\mathrm{d}y\end{aligned}$$

Consequently, since the integrand is invariant under the substitution, it follows that

$$\begin{aligned}E_{jk} &\preceq \delta^{-j} 2^{sj(2-5)} \int_0^1 \frac{y^{-2(\beta+1)}(1-y)^{-2(\beta+1)}}{(1 + y^{-\beta}(1-y)^{-\beta})^5}\,\mathrm{d}y \preceq 2^{-3sj}\delta^{-j} \int_0^1 \frac{y^{-2(\beta+1)}}{(1 + y^{-\beta})^5}\,\mathrm{d}y \\ &\preceq 2^{-3sj}\delta^{-j} \int_0^1 \frac{y^{-2(\beta+1)}}{y^{-5\beta}}\,\mathrm{d}y = 2^{-3sj}\delta^{-j} \int_0^1 y^{-2+3\beta}\,\mathrm{d}y.\end{aligned}$$

Since $\beta > \frac{1}{3}$, the exponent satisfies

$$-2 + 3\beta > -1,$$

and therefore the last integral is finite. We conclude that

$$E_{jk} \preceq \delta^{-j} 2^{-3sj} = \left(\delta^{-1} 2^{-3s}\right)^j.$$

In particular, this estimate is uniform with respect to k. Summing over all removed intervals, we obtain

$$\int_{[0,1]\setminus\mathbb{D}_\delta} \left[(\Psi_2 \circ w_\delta)'(x)\right]^2 \mathrm{d}x = \sum_{j=1}^{\infty} \sum_{k=1}^{2^{j-1}} E_{jk} \preceq \sum_{j=1}^{\infty} \sum_{k=1}^{2^{j-1}} \left(\delta^{-1} 2^{-3s}\right)^j \le \sum_{j=1}^{\infty} \left(\delta^{-1} 2^{1-3s}\right)^j < \infty$$

This series converges provided

$$\text{(C5)} \qquad \delta^{-1} 2^{1-3s} < 1 \quad \Longleftrightarrow \quad \delta > 2^{1-3s}.$$

Together with (C4), this means that we may choose

$$2^{1-3s} < \delta < 2^{-s-1}.$$

Since $s > 1$, this interval is nonempty, so that such a choice of δ is an admissible choice. Thus $(\Psi_2 \circ w_\delta)' \in L^2\big((0,1) \setminus \mathbb{D}_\delta\big)$, and since $\mathbb{D}_\delta$ has Lebesgue measure zero, Step 4 yields

$$\Psi_2 \circ w_\delta \in W^{1,2}((0,1)).$$

By the discussion in Sect. 2, $w_\delta \in C^0([0,1];[0,+\infty])$ with respect to the topology of the extended half-line. The crucial point is the additional term 2^{sj}, stemming from the structure $|u''|^2(1+u'^2)^{-5/2}$ in the definition of Φ_{jk}. It yields the decay factor 2^{-3sj} ensuring convergence of the sum over j.

6 We now fix parameters $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right)$, $s > 1$, and $\delta \in \left(2^{1-3s}, 2^{-s-1}\right)$ such that (C4) and (C5) hold. We then define the function from the statement of this theorem

$$u(x) := U(x) + f_\delta(x), \qquad \text{where} \qquad U(x) := \int_0^x w_\delta(\xi)\,\mathrm{d}\xi,$$

and f_δ is the generalized Cantor function from Step 2. Since $w_\delta \in L^1((0,1))$ by Step 3, the above integral is well defined. Hence

$$U \in W^{1,1}((0,1)) \subset BV((0,1)),$$

so in U is absolutely continuous on $[0,1]$. Since also $f_\delta \in BV((0,1)) \cap C^0([0,1])$, it follows that

$$u \in BV((0,1)) \cap C^0([0,1]) \quad \text{with} \quad Du = w_\delta\,\mathcal{L}^1 + D^c f_\delta.$$

Equivalently, the absolutely continuous density of Du is w_δ, its Cantor part coincides with that of f'_δ, and its jump part vanishes:

$$(u')^a = w_\delta \quad \text{a.e. on } (0,1), \qquad D^c u = D^c f_\delta, \qquad D^j u = 0.$$

In fact, with respect to the chosen continuous representative w_δ, we have the generalized Cantor set

$$\mathbb{D}_\delta = \left\{ x \in [0,1] \;\middle|\; \left|(u')^a(x)\right| = \infty \right\} = \left\{ x \in [0,1] \;\middle|\; (u')^a(x) = +\infty \right\}$$

is closed and, in particular, of measure zero. Since f_δ is continuous, its singular part is purely Cantor. Moreover,

$$|D^c u|((0,1)) = |D^c f_\delta|((0,1)) = f_\delta(1) - f_\delta(0) = 1 > 0.$$

In particular, $u \notin SBV((0,1))$.

It remains to clarify the role of U. Although U provides the absolutely continuous part needed for finite curvature energy, it does not belong to $W^{2,1}((0,1))$. Indeed, on each open interval I_{jk}, the function U is of class C^2, and

$$U''(x) = w_\delta'(x) = \frac{2^{sj}}{(1-2\delta)\delta^{j-1}}\Phi'\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right), \qquad x \in I_{jk}.$$

Hence $U'' \in L^1_{\mathrm{loc}}(I_{jk})$ for every k, j. However, with the same change of variables as in Step 3 we obtain

$$\begin{aligned} 2^{-sj}\int_{I_{jk}} |U''(x)|\,\mathrm{d}x &= \int_0^1 |\Phi'(y)|\,\mathrm{d}y = C_\beta\beta\int_0^1 \frac{|2y-1|}{y^{\beta+1}(1-y)^{\beta+1}}\,\mathrm{d}y \\ &\succeq \int_0^{1/4}\frac{1}{y^{\beta+1}}\,\mathrm{d}y + \int_{3/4}^1 \frac{1}{(1-y)^{\beta+1}}\,\mathrm{d}y \succeq \int_0^{1/4}\frac{1}{y^{4/3}}\,\mathrm{d}y = \infty, \end{aligned}$$

since $\beta > 1/3$. Thus $U'' \notin L^1(I_{jk})$ for every k, j, and therefore

$$U \notin W^{2,1}((0,1)).$$

Nevertheless, Step 5 shows that $\Psi_2 \circ w_\delta \in W^{1,2}((0,1))$. Consequently, the graph of U has finite curvature energy:

$$\int_0^1 \kappa_U(x)^2\sqrt{1+U'(x)^2}\,\mathrm{d}x = \int_{\mathrm{graph}[U]} \kappa_U(x)^2\,\mathrm{d}s_U(x) = \int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Psi_2\big(\underbrace{U'(x)}_{=w_\delta(x)}\big)\Big)^2\,\mathrm{d}x < \infty.$$

where $\kappa_U = U''(1+U'^2)^{-3/2}$ denotes the classical curvature on $(0,1)\setminus\mathbb{D}_\delta$. Since

$$\mathcal{H}^1(\mathrm{graph}(U)) \le 1 + \|U'\|_{L^1((0,1))} < \infty,$$

it follows by Cauchy-Schwarz that $\kappa_U \in L^1(\mathrm{graph}[U])$ as well.

Thus $u = U + f_\delta$ already has the required nontrivial Cantor part, while U provides the absolutely continuous contribution with finite curvature energy. The finiteness of the relaxed Willmore energy of u will be established in the next step.

7 We now prove that

$$\overline{\mathcal{W}}(u) \le \int_0^1 \big|(\Psi_2\circ w_\delta)'(x)\big|^2\,\mathrm{d}x = \mathcal{W}(U) =: \mathcal{W}^a(u).$$

Proof: By the relaxation theorem 4, there is a recovery sequence $q_n \in W^{2,2}((0,1))$ such that $q_n \to u$ in $L^1((0,1))$ and

$$\mathcal{F}_2(q_n) \longrightarrow |Du|((0,1)) + \int_0^1 \big|(\Psi_2\circ w_\delta)'(x)\big|^2\,\mathrm{d}x.$$

The lower semicontinuity of the total variation gives

$$|Du|((0,1)) \le \liminf_{n\to\infty}\int_0^1 |q_n'|\,\mathrm{d}x.$$

Since $\mathcal{F}_p(q_n) = \int_0^1 |q_n'|\,\mathrm{d}x + \mathcal{W}(q_n)$, it follows that

$$\limsup_{n\to\infty} \mathcal{W}(q_n) \le \lim_{n\to\infty} \mathcal{F}_2(q_n) - \liminf_{n\to\infty} \int_0^1 |q_n'|\,\mathrm{d}x \le \int_0^1 \left|(\Psi_2 \circ w_\delta)'(x)\right|^2 \mathrm{d}x.$$

Taking the infimum over all approximating sequences proves the claim. $\square$

Therefore, $\overline{\mathcal{W}}(u) \le \mathcal{W}^a(u) < \infty$. Together with $|D^c u|((0,1)) > 0$, which was shown in Step 6, this proves the theorem. $\square$

6 Corollary (Arbitrarily small relaxed Willmore energy with a nontrivial Cantor part)
For every $\varepsilon > 0$, there exists a continuous and nondecreasing function

$$u_\varepsilon \in BV((0,1)) \setminus SBV((0,1))$$

such that

$$|D^c u_\varepsilon|((0,1)) = 1 \qquad \text{and} \qquad \overline{\mathcal{W}}(u_\varepsilon) < \varepsilon.$$

In particular,

$$\inf\left\{\overline{\mathcal{W}}(v) \,\middle|\, v \in BV((0,1)),\ |D^c v|((0,1)) = 1\right\} = 0.$$

Proof: Let $u = U + f_\delta$ be the function constructed in Thm. 5, and let $w_\delta = U'$ denote the absolutely continuous part of its derivative. On every removed interval of generation j, the construction gives

$$w_\delta \ge 2^{sj} \quad \Rightarrow \quad \forall j \ge 1\colon \quad w_\delta(x) \ge c_0 := 2^s > 0 \qquad \text{for a.e. } x \in (0,1).$$

For $A \ge 1$, define

$$u_A := A\,U + f_\delta \quad \Rightarrow \quad Du_A = Aw_\delta\,\mathcal{L}^1 + D^c f_\delta.$$

Hence $D^c u_A = D^c f_\delta \ne 0$, and therefore $u_A \notin SBV((0,1))$ for every $A \ge 1$.

The blow-up set of the continuous $\overline{\mathbb{R}}$-valued representative of Aw_δ is still $\mathbb{D}_\delta$, and the positive singular measure $D^s u_A = D^c f_\delta$ is concentrated on this set. Moreover, for a.e. $x \in (0,1) \setminus \mathbb{D}_\delta$,

$$\left|\big(\Psi_2 \circ (Aw_\delta)\big)'(x)\right|^2 = \frac{A^2 |w_\delta'(x)|^2}{\big(1 + A^2 w_\delta(x)^2\big)^{5/2}}.$$

Using $A \ge 1$ and $w_\delta \ge c_0$, we further obtain $w_\delta\sqrt{1 + c_0^{-2}} \ge \sqrt{w_\delta^2 + 1}$

$$\frac{1}{w_\delta^5} \le \frac{\big(1 + c_0^{-2}\big)^{5/2}}{(1 + w_\delta^2)^{5/2}} \quad \Rightarrow \quad \frac{A^2 |w_\delta'|^2}{(1 + A^2 w_\delta^2)^{5/2}} \le \frac{|w_\delta'|^2}{A^3 w_\delta^5} \le \frac{\big(1 + c_0^{-2}\big)^{5/2} |w_\delta'|^2}{A^3 (1 + w_\delta^2)^{5/2}}.$$

Therefore, it follows

$$\int_0^1 \frac{A^2 |w_\delta'(x)|^2}{(1 + A^2 w_\delta(x)^2)^{5/2}}\,\mathrm{d}x \le A^{-3}\left(1 + c_0^{-2}\right)^{5/2} \int_0^1 \frac{|w_\delta'(x)|^2}{(1 + w_\delta(x)^2)^{5/2}}\,\mathrm{d}x.$$

The final expression is integrable by the construction in Theorem 5. Repeating the arguments from Steps 4 and 5 gives

$$\Psi_2 \circ (Aw_\delta) \in W^{1,2}((0,1)).$$

Hence $u_A \in X_\psi^2((0,1))$. Step 7 from Thm. 5 therefore yields

$$\begin{aligned}\overline{\mathcal{W}}(u_A) &\le \int_0^1 \frac{A^2 |w_\delta'(x)|^2}{\big(1 + A^2 w_\delta(x)^2\big)^{5/2}}\,\mathrm{d}x \\ &\le \frac{(1 + c_0^{-2})^{5/2}}{A^3} \int_0^1 \frac{|w_\delta'(x)|^2}{\big(1 + w_\delta(x)^2\big)^{5/2}}\,\mathrm{d}x.\end{aligned}$$

Thus there exists a constant $C > 0$, independent of A, such that

$$\overline{\mathcal{W}}(u_A) \leq CA^{-3}.$$

Choose $A > \max\{1, C^{1/3}\varepsilon^{-1/3}\}$ and set $u_\varepsilon := u_A$. Then

$$|D^c u_\varepsilon|((0,1)) = 1 \qquad \text{and} \qquad \overline{\mathcal{W}}(u_\varepsilon) < \varepsilon.$$

□

7 Remark (No positive energy gap for the pure relaxed Willmore term)
Geometrically, the Cantor part is kept fixed, while only the absolutely continuous part of the graph is stretched in the vertical direction. The regular graph becomes steeper and longer, but its bending contribution decreases like A^{-3}.

This observation concerns only the pure relaxed Willmore term $\overline{\mathcal{W}}$. It is not a corresponding small-energy statement for the full Dal Maso-Fonseca-Leoni-Morini functional $\mathcal{F}_2$. Indeed, for smooth functions this functional contains, in addition to the Willmore term, the total variation term. For the rescaled functions $u_A = A\,U + f_\delta$, the total variation satisfies

$$|Du_A|((0,1)) = A\int_0^1 w(x)\,\mathrm{d}x + |D^c f_\delta|((0,1)).$$

Thus the total variation grows linearly in A, although the pure Willmore term decays like A^{-3}. This is precisely why the above argument applies to $\overline{\mathcal{W}}$, but not to the full relaxed functional $\overline{\mathcal{F}}_2$.

This remark rules out one natural strategy. In the parametric Willmore theory, energy thresholds and small-energy regularity results are decisive tools, see, for instance, the small-energy theory of Kuwert and Schätzle [KS01]. For the pure relaxed Willmore term of graphs, no analogous mechanism can exist: there is no positive energy threshold below which Cantor parts are excluded. Consequently, any regularity theory for minimizers of relaxed Willmore-type graph functionals must rely on minimality or on the lower-order terms, and not on smallness of the curvature energy alone. One idea is to consider fixed boundary data with smallness assumptions, as used for elliptic and parabolic Willmore problems, see [Gul24, Gul26].

8 Remark (Cantor part vanishes by rotation)
It is important to note that the appearance of a Cantor part in the derivative is tied to the chosen projection direction. Let $g : [0,1] \to \mathbb{R}$ be continuous and nondecreasing, and rotate its graph clockwise by an angle $\varphi \in (0, \frac{\pi}{2})$. Writing

$$X_\varphi(x) := x\cos\varphi + g(x)\sin\varphi, \qquad Y_\varphi(x) := -x\sin\varphi + g(x)\cos\varphi,$$

we see that X_φ is strictly increasing. Hence the rotated curve can again be written as a graph, say

$$Y_\varphi = v_\varphi(X_\varphi).$$

Moreover, for $x_1 < x_2$, by expoiling the monotonicity of f

$$|Y_\varphi(x_2) - Y_\varphi(x_1)| \leq \sin\varphi\,|x_2 - x_1| + \cos\varphi\,|g(x_2) - g(x_1)| \leq L_\varphi |X_\varphi(x_2) - X_\varphi(x_1)|,$$

where one may take

$$L_\varphi := \tan\varphi + \cot\varphi.$$

Thus v_φ is Lipschitz. By [AFP00, Theorem 3.16, p. 127], $v_\varphi \in W^{1,\infty}$, and therefore its distributional derivative has no singular part.

Applied to the Cantor function f_δ, this shows that the Cantor part of Df_δ is not invariant under changes of projection: it disappears after such a rotation. The same projection effect occurs for the function u constructed in Thm. 5. Since u is continuous and nondecreasing, every clockwise rotation by an angle $\varphi \in (0, \frac{\pi}{2})$ yields a Lipschitz graph, whose distributional derivative has no singular part. In the original projection, however,

$$D^c u = D^c f_\delta \neq 0.$$

The crucial difference from the pure Cantor function is that this Cantor part is concentrated precisely where the absolutely continuous slope blows up, namely on $Z^+[(u')^a]$. Thus the Cantor part is compatible with finite relaxed Willmore energy: it is a projection-induced singular part and creates no additional curvature cost in the approximation constructed above.

Plot with explicit values

We illustrate the function u from Thm. 5 for the parameter choice $s = 2 > 1$, $\delta = \frac{1}{31}$, and $\beta = \frac{2}{5} \in \left(\frac{1}{3}, \frac{1}{2}\right)$. First, we check that all assumptions are satisfied. Indeed, the conditions (C4), (C5) are satisfied:

$$2^{1-3s} = 2^{-5} = \frac{1}{32} < \frac{1}{31} = \delta < \frac{1}{8} = 2^{-s-1}.$$

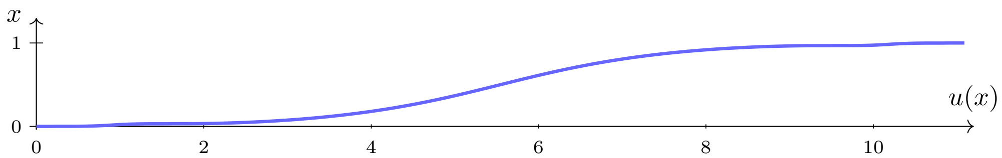


Figure 4: The function u constructed in Thm. 5 with $s = 2$, $\delta = \frac{1}{31}$, and $\beta = \frac{2}{5}$. The axes are interchanged: the value $u(x)$ is plotted on the horizontal axis and x on the vertical axis, so that both axes can be shown with the same scale.

Since u is nondecreasing, its maximum is attained at $x = 1$. We have

$$u(1) = U(1) + f_\delta(1).$$

By the normalization of the Cantor function, $f_\delta(1) = 1$, while

$$u(1) = U(1) + 1 = \int_0^1 w_\delta(x)\,\mathrm{d}x + 1 = 2^{s+1}\frac{1-2\delta}{1-2^{s+1}\delta} + 1 = 11 + \frac{2}{23} \approx 11.09.$$

The graph as a C^1-curve

In this subsection we show that the graph of the function constructed in Thm. 5 can be represented as the image of a regular C^1-curve in $\mathbb{R}^2$, parametrized by arc-length. More precisely, although the derivative of u has a nontrivial Cantor part, the corresponding graph admits a continuous unit tangent field, which becomes vertical precisely on the Cantor set. Since u has no jump part, this graph coincides with the usual graph of u.

As in [AM17a], c_u denotes the Cartesian curve associated with $u \in BV((0,1)) \cap C^0([0,1])$, namely

$$c_u : [0,1] \to \mathbb{R}^2, \qquad c_u(x) := (x, u(x)).$$

Since u has no jump part, c_u is continuous and belongs to $BV((0,1);\mathbb{R}^2) \cap C^0([0,1];\mathbb{R}^2)$. Writing

$$Du = (u')^a\,\mathcal{L}^1 + D^c u,$$

we have, in the sense of vector-valued Radon measures; see [AM17a, Remark 3.1, p. 459],

$$Dc_u = (1, (u')^a)\, \mathcal{L}^1 + (0,1)\, D^c u. \tag{10}$$

Since $\mathcal{L}^1$ and $|D^c u|$ are mutually singular, the total variation of Dc_u splits accordingly, and hence

$$|Dc_u| = \sqrt{1 + |(u')^a|^2}\, \mathcal{L}^1 + |D^c u|. \tag{11}$$

We define the *arc-length function*

$$s : [0,1] \to [0,L], \qquad s(x) := |Dc_u|([0,x]), \qquad \text{where} \quad L := |Dc_u|([0,1]).$$

The measure $\mathcal{L}^1$ is atomless, and $|D^c u|$ is atomless because it is the Cantor part of Du, see, for instance, [AFP00, p. 139]. Therefore $|Dc_u|$ has no atoms, and the distribution function s is continuous on $[0,1]$. Moreover, if $0 \le x < y \le 1$, then by (11)

$$s(y) - s(x) = |Dc_u|((x,y]) \ge \int_x^y \sqrt{1 + |(u')^a(t)|^2}\, \mathrm{d}t \ge \int_x^y 1\, \mathrm{d}t = y - x > 0.$$

Thus s is strictly increasing. Consequently, $s : [0,1] \to [0,L]$ is a homeomorphism, and its inverse $s^{-1} : [0,L] \to [0,1]$ is well defined.

We can now introduce the arc-length parametrization of the graph of the function constructed in Thm. 5 by setting

$$\gamma : [0,L] \to \mathbb{R}^2, \qquad \gamma(\sigma) := c_u(s^{-1}(\sigma)).$$

The following result shows that, in the present construction, this parametrized Cartesian curve is in fact a regular C^1-curve. This is in agreement with the general arc-length parametrization theory for continuous Cartesian curves developed by Acerbi and Mucci, see [AM17a, Prop. 5.5 and Thm. 10.2 and the discussion after formula (10.5)]. In their framework such curves arise naturally in the study of the geometric functionals $\mathcal{F}_p$, in particular in the case $p = 1$. For the reader's convenience, and also to make the special structure of the present one-dimensional construction explicit, we give a direct proof.

9 Theorem
Let $u\colon [0,1] \to \mathbb{R}$ be the function constructed in Thm. 5. In particular, $u \in BV((0,1)) \cap C^0([0,1])$, Du has a nontrivial Cantor part, and $(u')^a = w_\delta$ on $[0,1] \setminus \mathbb{D}_\delta$, as the chosen continuous representative. Let

$$c_u(x) := (x, u(x)), \qquad s(x) := |Dc_u|([0,x]), \qquad L := |Dc_u|([0,1]),$$

and define the arc-length parametrization

$$\gamma : [0,L] \to \mathbb{R}^2, \qquad \gamma(\ell) := c_u(s^{-1}(\ell)).$$

Then the following assertions hold.

(1) *The curve γ is an injective regular C^1-curve parametrized by arc-length. Moreover, for every $x \in [0,1]$,*

$$\gamma(s(x)) = c_u(x), \qquad \gamma_1(s(x)) = x.$$

Its unit tangent field, pulled back to the original parameter x, is given by

$$\gamma'(s(x)) = \begin{cases} \dfrac{(1, w_\delta(x))}{\sqrt{1 + w_\delta(x)^2}}, & \text{if } x \in [0,1] \setminus \mathbb{D}_\delta, \\ (0,1), & \text{if } x \in \mathbb{D}_\delta. \end{cases} \tag{12}$$

2. *The curve γ has square-integrable curvature. More precisely, $\gamma \in W^{2,2}(0,L;\mathbb{R}^2)$. If κ_γ denotes the scalar curvature of the arc-length parametrized curve γ, defined by*

$$\gamma'' = \kappa_\gamma\, \nu_\gamma \qquad \textit{a.e. in } (0,L),$$

where ν_γ is the unit normal obtained by rotating γ' by $\pi/2$, then $\kappa_\gamma \in L^2(0,L)$ and

$$\kappa_\gamma(\ell) = \begin{cases} \dfrac{w_\delta'(s^{-1}(\ell))}{(1+w_\delta(s^{-1}(\ell))^2)^{3/2}}, & \ell \in s\left([0,1]\setminus \mathbb{D}_\delta\right), \\ 0, & \ell \in s(\mathbb{D}_\delta). \end{cases} \qquad \textit{for a.e. } \ell \in (0,L)$$

Consequently,

$$\int_0^L |\kappa_\gamma(\ell)|^2 \,\mathrm{d}\ell = \int_0^1 \frac{|w_\delta'(x)|^2}{\left(1+w_\delta(x)^2\right)^{5/2}}\,\mathrm{d}x = \mathcal{W}^a(u) < \infty. \tag{13}$$

3. *The graph $c_u([0,1])$ is not a one-dimensional C^2-submanifold of $\mathbb{R}^2$. More precisely, for every $x_0 \in \mathbb{D}_\delta$, the set $c_u([0,1])$ does not admit a local regular C^2-parametrization in any neighbourhood of $c_u(x_0)$.*

Proof: **1 C^1-arc-length parametrization.**
Since u is monotonically increasing in the construction of Thm. 5, its Cantor part is a positive measure. Hence $|D^c u| = D^c u$. By the polar decomposition theorem for vector-valued Radon measures [AFP00, Corollary 1.29], there exists a unique $|Dc_u|$-measurable function $\tau \in [L^1([0,1],|Dc_u|)]^2$ such that

$$\tau : [0,1] \to \mathbb{S}^1, \qquad Dc_u = \tau\,|Dc_u|.$$

We now characterize this polar vector explicitly. On the absolutely continuous part we have $(u')^a = w_\delta$. Then (11) gives

$$(1,(u')^a)\mathcal{L}^1 = \tau\sqrt{1+|(u')^a|^2}\mathcal{L}^1 \quad\Rightarrow\quad \tau(x) = \frac{(1,w_\delta(x))}{\sqrt{1+w_\delta(x)^2}} \qquad \mathcal{L}^1\text{-a.e. on } [0,1]\setminus\mathbb{D}_\delta.$$

On the Cantor part, by [AM17a, Remark 3.1, p. 459] the measure Dc_u equals $(0,1)D^c u$, whereas $|Dc_u| = D^c u$. Hence

$$\tau(x) = (0,1) \qquad D^c u\text{-a.e. on } \mathbb{D}_\delta.$$

We now choose a continuous representative of this measure-theoretic tangent. Consider the map

$$F : [0,\infty] \to \mathbb{S}^1, \qquad F(t) := \begin{cases} \dfrac{(1,t)}{\sqrt{1+t^2}}, & t < \infty, \\ (0,1), & t = \infty. \end{cases}$$

Here $[0,\infty]$ is endowed with the usual topology of the extended half-line. Since $w_\delta \colon [0,1] \to [0,\infty]$ is continuous and $w_\delta(x) = +\infty$ precisely on $\mathbb{D}_\delta$, the composition $F \circ w_\delta$ is continuous on $[0,1]$. Moreover, the preceding identification shows that $F(w_\delta) = \tau$ holds $|Dc_u|$-a.e. Thus, replacing τ by this continuous representative, we may write

$$\tau(x) = \begin{cases} \dfrac{(1,w_\delta(x))}{\sqrt{1+w_\delta(x)^2}}, & \text{if } x \in [0,1]\setminus\mathbb{D}_\delta, \\ (0,1), & \text{if } x \in \mathbb{D}_\delta. \end{cases}$$

which later will show (12).

Now, keeping in mind that s is a homeomorphism, we finally define the curve γ by setting

$$\gamma(\ell) := c_u(0) + \int_0^{\ell} \tau(s^{-1}(r))\,\mathrm{d}r, \qquad \ell \in [0, L].$$

The map $\tau \circ s^{-1} : [0, L] \to \mathbb{S}^1$ is continuous. Hence

$$\gamma \in C^1([0, L]; \mathbb{R}^2), \qquad \gamma'(\ell) = \tau(s^{-1}(\ell)) \quad \text{for every } \ell \in [0, L].$$

Since $|\tau| = 1$ everywhere, we obtain $|\gamma'(\ell)| = 1$ for every $\ell \in [0, L]$, and therefore γ is parametrized by arc-length.

It remains to prove that this parametrization coincides with the Cartesian curve c_u after the change of variables s. We first claim that the pushforward measure of $\mathcal{L}^1$ under s^{-1} is

$$(s^{-1})_{\#}\big(\mathcal{L}^1 \llcorner [0, L]\big) = |Dc_u|. \tag{14}$$

Indeed, for every $x \in [0, 1]$,

$$\big((s^{-1})_{\#}(\mathcal{L}^1 \llcorner [0, L])\big)([0, x]) = \mathcal{L}^1\big(s([0, x])\big) = \mathcal{L}^1([0, s(x)]) = s(x) = |Dc_u|([0, x]).$$

Since both sides are finite Borel measures on $[0, 1]$ and coincide on the generating family of intervals $[0, x]$, (14) follows. Now let $x \in [0, 1]$. Using the change-of-variables formula for push-forward measures (14) and the polar decomposition $Dc_u = \tau\,|Dc_u|$, we obtain

$$\begin{aligned}\gamma(s(x)) &= c_u(0) + \int_0^{s(x)} \tau(s^{-1}(r))\,\mathrm{d}r = c_u(0) + \int_{[0,x]} \tau\,\mathrm{d}|Dc_u| \\ &= c_u(0) + Dc_u([0, x]).\end{aligned}$$

Since $c_u \in BV((0, 1); \mathbb{R}^2) \cap C^0([0, 1]; \mathbb{R}^2)$, the one-dimensional structure theorem for BV functions [AFP00, Theorem 3.28] applied componentwise gives, for $0 < y < x < 1$,

$$c_u(x) - c_u(y) = Dc_u((y, x]).$$

Since c_u is continuous, the measure Dc_u has no atoms. Hence

$$Dc_u((y, x]) = Dc_u([y, x]) \quad \text{and} \quad c_u(x) - c_u(0) = Dc_u([0, x]).$$

Consequently, for γ and its first component γ_1 it holds

$$\gamma(s(x)) = c_u(x) \quad \text{and} \quad \gamma_1(s(x)) = x \qquad \text{for every } x \in [0, 1].$$

Equivalently, since s is onto, $\gamma_1(\ell) = s^{-1}(\ell)$ for every $\ell \in [0, L]$. In particular, γ is injective. Moreover, $\gamma([0, L]) = c_u([0, 1])$. Thus γ is an injective regular C^1-curve whose image is precisely the graph of u. Finally, from the definition of γ we obtain

$$\forall\, \ell \in [0, L]\colon \quad \gamma'(\ell) = \tau(s^{-1}(\ell)) \quad \overset{\ell = s(x)}{\Rightarrow} \quad \forall\, x \in [0, 1]\colon \quad \tau(x) = \gamma'(s(x)).$$

Using the explicit representation of τ, this gives (12).

2 Square-integrable curvature.

We first show that the arc-length parametrization constructed above has square-integrable curvature. This should be contrasted with the fact proved below that the same graph is not a one-dimensional C^2-submanifold of $\mathbb{R}^2$. Motivated by the representation of the tangent vector γ'

in (12), where the quotient of the components equals $w_\delta/1$, we define the *tangent angle with respect to the horizontal parameter* by

$$\theta(x) = \begin{cases} \arctan(w_\delta(x)), & \text{if } x \in [0,1] \setminus \mathbb{D}_\delta, \\ \pi/2, & \text{if } x \in \mathbb{D}_\delta, \end{cases} \qquad x \in [0,1].$$

Equivalently, θ is obtained by composing w_δ with the continuous extension of $\arctan$ to $[0,\infty]$, where $\arctan(+\infty) := \pi/2$. Since $w_\delta : [0,1] \to [0,\infty]$ is continuous, we have $\theta \in C^0([0,1])$. Moreover, the unit tangent field found in 1 can be written as

$$\tau(x) = \big(\cos\theta(x), \sin\theta(x)\big), \qquad x \in [0,1].$$

Thus the value $\theta = \pi/2$ on $\mathbb{D}_\delta$ is not an additional choice, but exactly the angle corresponding to the vertical polar vector of Dc_u. On every connected component I_{jk} of $[0,1] \setminus \mathbb{D}_\delta$, the function w_δ is C^1. Therefore, classically on I_{jk}, it follows

$$\theta'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \arctan(w_\delta(x)) = \frac{w_\delta'(x)}{1 + w_\delta(x)^2}.$$

On the absolutely continuous part of $|Dc_u|$ we have $\mathrm{d}|Dc_u| = \sqrt{1 + w_\delta^2}\,\mathrm{d}x$.

$$\Rightarrow \quad \theta'(x)\,\mathrm{d}x = \frac{w_\delta'(x)}{1 + w_\delta(x)^2}\,\mathrm{d}x = \frac{w_\delta'(x)}{\big(1 + w_\delta(x)^2\big)^{3/2}}\,\mathrm{d}|Dc_u|, \qquad \text{on } [0,1] \setminus \mathbb{D}_\delta,$$

We therefore define the *candidate for the derivative of the tangent angle with respect to arc-length* by

$$g(x) := \begin{cases} \dfrac{w_\delta'(x)}{\big(1 + w_\delta(x)^2\big)^{3/2}}, & x \in [0,1] \setminus \mathbb{D}_\delta, \\ 0, & x \in \mathbb{D}_\delta. \end{cases}$$

Then, using the decomposition (11) $|Dc_u| = \sqrt{1 + w_\delta^2}\,\mathcal{L}^1 + D^c u$ and the fact that $g = 0$ on $\mathbb{D}_\delta$, we obtain

$$\begin{aligned} \int_{[0,1]} |g|^2\,\mathrm{d}|Dc_u| &= \int_{[0,1]\setminus\mathbb{D}_\delta} \frac{|w_\delta'(x)|^2}{\big(1 + w_\delta(x)^2\big)^3} \sqrt{1 + w_\delta(x)^2}\,\mathrm{d}x \\ &= \int_0^1 \frac{|w_\delta'(x)|^2}{\big(1 + w_\delta(x)^2\big)^{5/2}}\,\mathrm{d}x < \infty. \end{aligned}$$

Thus, we conclude $g \in L^2([0,1], |Dc_u|)$ and with the push-forward identity (14) $g \circ s^{-1} \in L^2(0,L)$.
Claim. For every $x \in [0,1]$, one has

$$\theta(x) = \frac{\pi}{2} + \int_{[0,x]} g\,\mathrm{d}|Dc_u|. \tag{15}$$

Proof of (15)**:** Write the connected components of $[0,1] \setminus \mathbb{D}_\delta$ as $I_{jk} = (\alpha_{jk}, \beta_{jk})$. Since $w_\delta(x) \to +\infty$ as $x \searrow \alpha_{jk}$ and as $x \nearrow \beta_{jk}$, we have

$$\lim_{x \searrow \alpha_{jk}} \theta(x) = \lim_{x \nearrow \beta_{jk}} \theta(x) = \frac{\pi}{2}.$$

On each I_{jk}, the function w_δ is C^1, and hence on I_{jk} we obtain

$$\theta' = \frac{w_\delta'}{1 + w_\delta^2} \quad \text{on } I_{jk} \quad \Rightarrow \quad g\,\mathrm{d}|Dc_u| = \frac{w_\delta'}{1 + w_\delta^2}\,\mathrm{d}x = \theta'\,\mathrm{d}x \quad \text{on } I_{jk},$$

since on I_{jk} we have $\mathrm{d}|Dc_u| = \sqrt{1+w_\delta^2}\,\mathrm{d}x$. Consequently,

$$\int_{I_{jk}} g\,\mathrm{d}|Dc_u| = \int_{\alpha_{jk}}^{\beta_{jk}} \frac{w_\delta'(x)}{1+w_\delta(x)^2}\,\mathrm{d}x = \lim_{R\nearrow\beta_{jk}} \theta(R) - \lim_{R'\searrow\alpha_{jk}} \theta(R') = 0. \tag{16}$$

Now let $x \in [0,1]$. We distinguish two cases. First assume that $x \in \mathbb{D}_\delta$. Since $g = 0$ on $\mathbb{D}_\delta$, and since the connected components of $[0,1]\setminus\mathbb{D}_\delta$ are precisely the intervals I_{jk}, we obtain

$$\int_{[0,x]} g\,\mathrm{d}|Dc_u| = \sum_{I_{jk}\subset(0,x)} \int_{I_{jk}} g\,\mathrm{d}|Dc_u|.$$

The series is well-defined because $g \in L^1([0,1],|Dc_u|)$, which follows from $g \in L^2([0,1],|Dc_u|)$ and $|Dc_u|([0,1]) < \infty$. By (16), every summand vanishes. Hence

$$\frac{\pi}{2} + \int_{[0,x]} g\,\mathrm{d}|Dc_u| = \frac{\pi}{2} = \theta(x).$$

Now assume that $x \notin \mathbb{D}_\delta$. Then there is a unique interval $I_{j_0k_0} = (\alpha_{j_0k_0}, \beta_{j_0k_0})$ such that $x \in I_{j_0k_0}$. Since $\alpha_{j_0k_0} \in \mathbb{D}_\delta$, the first case gives

$$\begin{aligned}
\int_{[0,x]} g\,\mathrm{d}|Dc_u| &= \int_{[0,\alpha_{j_0k_0}]} g\,\mathrm{d}|Dc_u| + \int_{(\alpha_{j_0k_0},x]} g\,\mathrm{d}|Dc_u| \\
&= 0 + \int_{\alpha_{j_0k_0}}^{x} \frac{w_\delta'(t)}{1+w_\delta(t)^2}\,\mathrm{d}t = \theta(x) - \lim_{R\searrow\alpha_{j_0k_0}} \theta(R) \\
&= \theta(x) - \frac{\pi}{2}.
\end{aligned}$$

This proves (15).

Next, we define the *tangent angle parametrized over length* by

$$\vartheta\colon [0,L] \to [0,\pi/2], \qquad \vartheta := \theta\circ s^{-1}.$$

Since $s^{-1}_{\#}\mathcal{L}^1 = |Dc_u|$, identity (15) gives for all $\ell \in [0,L]$

$$\vartheta(\ell) = \frac{\pi}{2} + \int_0^\ell g(s^{-1}(r))\,\mathrm{d}r \quad\Rightarrow\quad \vartheta \in W^{1,1}((0,L)), \quad \vartheta' = g\circ s^{-1} \quad \text{a.e. in } (0,L).$$

Furthermore, since $g = 0$ on $\mathbb{D}_\delta$ we conclude

$$\begin{aligned}
\int_0^L |\vartheta'(\ell)|^2\,\mathrm{d}\ell &= \int_0^L |g(s^{-1}(\ell))|^2\,\mathrm{d}\ell \overset{(14)}{=} \int_{[0,1]} |g(x)|^2\,\mathrm{d}|Dc_u| \\
&= \int_0^1 \frac{|w_\delta'(x)|^2}{(1+w_\delta(x)^2)^3}\sqrt{1+w_\delta(x)^2}\,\mathrm{d}x = \int_0^1 \frac{|w_\delta'(x)|^2}{(1+w_\delta(x)^2)^{5/2}}\,\mathrm{d}x.
\end{aligned}$$

The last integral is finite by Step 5 of the construction. Hence, we obtain $\vartheta \in W^{1,2}((0,L))$. By construction of the arc-length parametrization, it holds for all $\ell \in [0,L]$

$$\gamma'(\ell) = \tau(s^{-1}(\ell)) = \big(\cos\vartheta(\ell), \sin\vartheta(\ell)\big).$$

The Sobolev chain rule therefore gives $\gamma' \in W^{1,2}((0,L);\mathbb{R}^2) \Rightarrow \gamma \in W^{2,2}((0,L);\mathbb{R}^2)$. Moreover, for the second derivative it holds

$$\gamma''(\ell) = \vartheta'(\ell)\,(-\sin\vartheta(\ell), \cos\vartheta(\ell)) \qquad \text{for a.e. } \ell \in (0,L).$$

Since $|\gamma'| \equiv 1$, the scalar curvature of the arc-length parametrized curve is

$$\kappa_\gamma(\ell) = \vartheta'(\ell) = \begin{cases} \dfrac{w_\delta'(s^{-1}(\ell))}{(1+w_\delta(s^{-1}(\ell))^2)^{3/2}}, & \ell \in s\left([0,1] \setminus \mathbb{D}_\delta\right), \\ 0, & \ell \in s(\mathbb{D}_\delta). \end{cases} \qquad \text{for a.e. } \ell \in (0,L).$$

Therefore, (13) follows from

$$\int_0^L |\kappa_\gamma|^2(\ell)\,\mathrm{d}\ell = \int_0^L |\vartheta'(\ell)|^2\,\mathrm{d}\ell = \int_0^1 \frac{|w_\delta'(x)|^2}{(1+w_\delta(x)^2)^{5/2}}\,\mathrm{d}x = \mathcal{W}^a(u).$$

3 Failure of local C^2-regularity.

We prove that the classical curvature of the smooth graph pieces becomes unbounded along sequences of points converging to Cantor points. Let $I_{jk} = (\alpha_{jk}, \beta_{jk})$ be one of the removed intervals at level j. Its length is $(1-2\delta)\delta^{j-1}$ and its midpoint is denoted by a_{jk}. Define the affine rescaling

$$\rho_{jk}(x) := \frac{x - a_{jk}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}, \qquad x \in I_{jk}.$$

Then ρ_{jk} maps I_{jk} bijectively onto $(0,1)$, with $\rho_{jk}(\alpha_{jk}) = 0$ and $\rho_{jk}(\beta_{jk}) = 1$.

Since f_δ is constant on every removed interval I_{jk}, the function $u = U + f_\delta$ satisfies on I_{jk}

$$u'(x) = U'(x) = w_\delta(x) = 2^{sj} + 2^{sj}\Phi(\rho_{jk}(x)), \quad u''(x) = w_\delta'(x) = \frac{2^{sj}}{(1-2\delta)\delta^{j-1}}\Phi'(\rho_{jk}(x)).$$

Thus u is C^2 on each I_{jk}, and the classical signed curvature of the graph of u at points $x \in I_{jk}$ is given by

$$\kappa_u(x) = \frac{u''(x)}{\left(1+u'(x)^2\right)^{3/2}} = \frac{w_\delta'(x)}{\left(1+w_\delta(x)^2\right)^{3/2}}.$$

This is the standard curvature formula for a Cartesian graph, see, for instance, [AM17a, (2.5)].

We first present two elementary estimates for Φ. Later we shall choose $y = y_j \searrow 0$. Therefore it is enough to consider the range

$$0 < y \leq \frac{1}{4} \qquad \Rightarrow \qquad 1 - y \geq \frac{3}{4}, \quad |2y-1| \geq \frac{1}{2}.$$

Hence there exists a constant $C_1 > 0$, depending only on β, such that

$$0 \leq \Phi(y) = C_\beta\left(\frac{1}{y^\beta(1-y)^\beta} - 4^\beta\right) \leq C_1 y^{-\beta} \qquad \text{for } 0 < y \leq \frac{1}{4}.$$

Moreover, there exists a constant $C_2 > 0$, depending only on β, such that

$$|\Phi'(y)| = C_\beta \beta \frac{|2y-1|}{y^{\beta+1}(1-y)^{\beta+1}} \geq C_2 y^{-\beta-1} \qquad \text{for } 0 < y \leq \frac{1}{4}.$$

We then conclude

$$\left|\kappa_u(\rho_{jk}^{-1}(y))\right| \succeq \frac{2^{sj}}{2^{3sj}(1-2\delta)\delta^{j-1}} \frac{y^{-\beta-1}}{(1+y^{-\beta})^3} \succeq \delta^{-j}2^{-2sj}y^{2\beta-1} \qquad \text{for } 0 < y \leq \frac{1}{4}. \tag{17}$$

We now show that, near every Cantor point, there are removed intervals of arbitrarily high generation. Fix

$$\chi \in \mathbb{D}_\delta.$$

Since $\chi \in \mathbb{D}_\delta$, it is never removed in the Cantor construction. Hence, for every $j \in \mathbb{N}_0$, there exists a closed interval $J_j = J_j(\chi)$ of the j-th generation such that

$$\chi \in J_j, \qquad |J_j| = \delta^j, \qquad \bigcap_{j=0}^{\infty} J_j = \{\chi\}.$$

and the intervals may be chosen nested: $J_{j+1} \subset J_j$. For $j \geq 1$, let $I_j^* = I_{jk_j}$ be the middle open interval removed from J_{j-1} when passing from the $(j-1)$-st to the j-th generation. Then $I_j^* \subset J_{j-1}$ and $|I_j^*| = (1-2\delta)\delta^{j-1}$. Since $\chi \in J_{j-1}$ and $\operatorname{diam}(J_{j-1}) = \delta^{j-1} \to 0$, we have

$$\operatorname{dist}(I_j^*, \chi) \leq \delta^{j-1} \to 0.$$

Consequently, every choice of points $x_j \in I_j^*$ satisfies $x_j \to \chi$ as $j \to \infty$.

Since, by condition (C2), we have $2\beta - 1 < 0$, we define for j sufficiently large

$$y_j := 2^{2sj/(2\beta-1)}. \tag{18}$$

Then $y_j \to 0$, and hence $y_j \in (0, \frac{1}{4})$ for all sufficiently large j. Since $\rho_{jk_j} : I_j^* \to (0,1)$ is affine and bijective, we may choose $x_j \in I_j^*$ such that

$$\rho_{jk_j}(x_j) = y_j \quad \Rightarrow \quad x_j \underset{j\to\infty}{\longrightarrow} \chi \tag{19}$$

since $I_j^* \to \chi$ in the sense that $\operatorname{dist}(I_j^*, \chi) \to 0$. Moreover, by the choice of y_j and by using (17), we therefore obtain

$$|\kappa_u(x_j)| \geq C_4\, \delta^{-j} 2^{-2sj} y_j^{2\beta-1} = C_4\, \delta^{-j} \to +\infty \qquad \text{as } j \to \infty,$$

because $0 < \delta < 1$. Thus the classical curvature of the smooth graph pieces is unbounded along the sequence $x_j \to \chi$.

Since the classical curvature of the smooth graph pieces is unbounded along $p_j := (x_j, u(x_j)) \to p_* := (\chi, u(\chi))$, the graph cannot admit a regular local C^2-parametrization at p_*. Otherwise, the curvature of such a parametrization would be continuous and hence locally bounded, contradicting the invariance of curvature under regular C^2-reparametrization on the smooth graph pieces.

Since $\chi \in \mathbb{D}_\delta$ was arbitrary, the graph fails to be locally C^2 at every point of $c_u(\mathbb{D}_\delta)$, with the one-sided interpretation at its two endpoints. In particular, $c_u([0,1])$ is not a regular embedded C^2-curve with boundary. □

10 Remark (A variant with a C^2-regular curve)
The failure of C^2-regularity in Theorem 9 can be avoided by changing the parameter regime. Choose

$$\beta \in \left(\frac{1}{2}, 1\right), \qquad s > 1, \qquad 2^{-2s} < \delta < 2^{-s-1}.$$

Since $s > 1$, the interval for δ is nonempty and

$$2^{1-3s} < 2^{-2s} < \delta.$$

Thus the summability conditions used in the construction remain satisfied. Moreover, $\beta < 1$ guarantees the integrability of the profile, while $\beta > \frac{1}{3}$ guarantees the finiteness of the local

Willmore integrals. Consequently, repeating the construction of Theorem 5 with these parameters yields a function

$$u \in BV((0,1)) \cap C^0([0,1]), \qquad D^c u \neq 0, \qquad \overline{\mathcal{W}}(u) < \infty.$$

We claim that the graph of u is now a regular embedded C^2-curve. Let I_{jk} be a removed interval of generation j and again write $y = \rho_{jk}(x)$. Then

$$\left|\kappa_u\big(\rho_{jk}^{-1}(y)\big)\right| \leq \frac{\delta}{1-2\delta}\,\delta^{-j}2^{-2sj}\frac{|\Phi'(y)|}{\big(1+\Phi(y)\big)^3}. \tag{20}$$

For $0 < y \leq \frac{1}{4}$, the estimates used in (17) can be reversed in the appropriate places:

$$\begin{aligned}
\Phi(y) &= C_\beta\left(\frac{1}{y^\beta(1-y)^\beta} - 4^\beta\right) \succeq y^{-\beta} && \text{for } 0 < y \leq \frac{1}{4},\\
|\Phi'(y)| &= C_\beta\beta\frac{|2y-1|}{y^{\beta+1}(1-y)^{\beta+1}} \preceq y^{-\beta-1} && \text{for } 0 < y \leq \frac{1}{4}.
\end{aligned}$$

We then conclude

$$\left|\kappa_u(\rho_{jk}^{-1}(y))\right| \preceq \frac{2^{sj}}{2^{3sj}(1-2\delta)\delta^{j-1}}\frac{y^{-\beta-1}}{(1+y^{-\beta})^3} \preceq \delta^{-j}2^{-2sj}y^{2\beta-1} \qquad \text{for } 0 < y \leq \frac{1}{4}.$$

By the symmetry of Φ, the analogous estimate holds near $y = 1$:

$$\frac{|\Phi'(y)|}{\big(1+\Phi(y)\big)^3} \leq C(1-y)^{2\beta-1} \qquad \text{for } \frac{3}{4} \leq y < 1.$$

Since $2\beta - 1 > 0$, we therefore have

$$\kappa_u\big(\rho_{jk}^{-1}(y)\big) \longrightarrow 0$$

as $y \searrow 0$ or $y \nearrow 1$. Thus the curvature on each removed interval extends continuously to its endpoints by assigning the value zero there. In particular, h_β is bounded on $[0,1]$. It follows from (20) that

$$\sup_{x \in I_{jk}} |\kappa_u(x)| \leq C\delta^{-j}2^{-2sj} = C\left(\frac{2^{-2s}}{\delta}\right)^j.$$

Since $\delta > 2^{-2s}$, the right-hand side converges to zero as $j \to \infty$.

Hence the scalar curvature on $[0,1] \setminus \mathbb{D}_\delta$ extends continuously to $[0,1]$. Let $s : [0,1] \to [0,L]$ and $\gamma = c_u \circ s^{-1}$ be the arc-length parametrization from Theorem 9. The extended scalar curvature, pulled back by s^{-1}, is continuous on $[0,L]$. The weak Frenet identity proved there, $\gamma'' = \kappa_\gamma\nu_\gamma$ almost everywhere, therefore has a continuous right-hand side. Hence $\gamma' \in C^1([0,L];\mathbb{R}^2)$ and $\gamma \in C^2([0,L];\mathbb{R}^2)$. Since $|\gamma'| = 1$, the image $c_u([0,1])$ is a regular one-dimensional C^2-submanifold of $\mathbb{R}^2$ with boundary.

11 Remark (Interpretation of Menne's theorem in the present example)
The example in Thm. 9 should not be read as contradicting Menne's second-order rectifiability theorem in [Men13, Thm. 3.6]. On the contrary, it illustrates the precise measure-theoretic meaning of that theorem.

Let V_u be the multiplicity-one integral varifold associated with $\Gamma_u = \gamma([0,L])$. Since $\gamma \in W^{2,2}(0,L;\mathbb{R}^2)$ is arc-length parametrized, V_u has locally bounded first variation. More precisely, in the open set

$$U_0 := \mathbb{R}^2 \setminus \{\gamma(0), \gamma(L)\}$$

its first variation is represented by integration,

$$\delta V_u(X) = -\int_{\Gamma_u} H_{V_u} \cdot X \, d\|V_u\|, \quad H_{V_u}(\gamma(\ell)) = \gamma''(\ell), \quad \int_{\Gamma_u} |H_{V_u}|^2 \, d\|V_u\| = \int_0^L |\kappa_\gamma(\ell)|^2 \, d\ell < \infty.$$

If the endpoints are included in the ambient open set, the first variation has in addition the two usual boundary atoms. In either case $\|\delta V_u\|$ is a Radon measure, and hence Menne's second-order rectifiability theorem applies.

Consequently, there are countably many C^2-curves $\Sigma_i \subset \mathbb{R}^2$ such that

$$\|V_u\| \left(\Gamma_u \setminus \bigcup_{i=1}^{\infty} \Sigma_i \right) = 0.$$

Moreover, on these C^2-pieces the classical curvature agrees $\|V_u\|$-almost everywhere with the generalized mean curvature.

The point is that this conclusion is only a measure-theoretic covering statement. It does not imply that Γ_u itself is a C^2-curve, nor that Γ_u can be decomposed into countably many C^2-subarcs lying inside the graph. In the present construction the Cantor trace has positive $\|V_u\|$-measure, while Γ_u is not locally a regular C^2-curve at any Cantor point. Hence some of the covering curves Σ_i must pass through uncountably many points of the Cantor trace, but they cannot coincide with Γ_u in any neighbourhood of such a point. They may therefore cover Cantor points in the sense of Menne's theorem while leaving the graph between those points.

Thus the example shows that countable C^2-rectifiability of an integral varifold with L^2-mean curvature is much weaker than a global, or even locally compatible, C^2-curve structure of its support.

4 General $p > 1$ version of the Cantor construction

In this section we show that the mechanism behind Thm. 5 is not specific to the Willmore case $p = 2$, but works for every exponent $p > 1$, provided the weight ψ decays like $t^{-\alpha}$ with $\alpha > 2p - 1$. This extends the examples of [DMFLM09, Remark 3.2 (iv)], which cover only exponents p close to 1, to the full range $p > 1$. Since the weight associated with the p-curvature energy has decay exponent $\alpha = 3p - 1 > 2p - 1$, the relaxed L^p-curvature energies $\overline{\mathcal{W}}_p$ are included for every $p > 1$ (Corollary 13). As before, the construction admits a geometric interpretation: the associated curve has an arc-length parametrization of class $C^1 \cap W^{2,p}$, and a variant of the singular profile even yields a C^2-regular image (Remark 14). As a further application, we obtain a surface of revolution with finite Willmore energy whose profile function carries a nontrivial Cantor part (Corollary 15).

12 Theorem (Cantor parts in X^p_ψ for every $p > 1$)
Let $p > 1$, and let $\psi : \mathbb{R} \to (0, +\infty)$ be a bounded Borel function satisfying (3). *Assume moreover that there exist constants $C > 0$ and*

$$\alpha > 2p - 1 \quad \text{such that} \quad \psi(t) \le \frac{C}{t^\alpha} \qquad \text{for all } t \ge 1.$$

Then, for every bounded open interval $I = (a, b)$, there exists a function

$$u \in X^p_\psi(I)$$

such that its distributional derivative has a nontrivial Cantor part.

Proof: According to the definition in [DMFLM09], for $p>1$ the class X^p_ψ is described in terms of the absolutely continuous density $(u')^a = \frac{dDu}{d\mathcal{L}^1}$ by requiring

$$\Psi_p\big((u')^a\big) \in W^{1,p},$$

where Ψ_p is defined in (4). Moreover, the singular part $D^s u$ has to be concentrated on the set where the chosen continuous representative of $(u')^a$ takes the values $\pm\infty$. In the construction below only the value $+\infty$ occurs.

It is enough to consider $I=(0,1)$. The construction on an arbitrary bounded open interval $I=(a,b)$ is obtained by applying the same argument after an affine change of variables in the domain. All estimates remain unchanged up to constants depending only on $|I|$.

In this proof we only highlight the changes that have to be made in the proof of Thm. 5. In Step 1, we replace the definitions of M and Ψ_2 by

$$M_p := \int_{-\infty}^{+\infty} \psi(t)^{1/p}\,\mathrm{d}t < +\infty, \qquad \Psi_p(t) := \int_{-\infty}^{t} \psi(s)^{1/p}\,\mathrm{d}s.$$

The finiteness of M_p is part of assumption (3). The additional decay estimate is used only to control the positive tail and to obtain the quantitative bounds required for the Cantor-profile construction. Indeed, since $\alpha > 2p-1 > p$, one has

$$\int_1^\infty \psi(t)^{1/p}\,\mathrm{d}t \le C^{1/p}\int_1^\infty t^{-\alpha/p}\,\mathrm{d}t < \infty.$$

Then Ψ_p extends continuously to $[-\infty,+\infty]$, with $\Psi_p(-\infty)=0$ and $\Psi_p(+\infty)=M_p$. The Cantor construction itself in Step 2 remains unchanged. In Step 3, we replace the condition on β in (C2) by

(C2p) $$\beta \in \left(\frac{p-1}{\alpha-p}, 1\right).$$

This interval is nonempty precisely because $\alpha > 2p-1$. The definition of function Φ also remains completely unchanged. The condition (C3) on $s>1$ is replaced by

(C3p) $$s > \frac{p}{\alpha-2p+1} \quad\Longleftrightarrow\quad 2^{\frac{1-s(\alpha-p)}{p-1}} < 2^{-s-1},$$

which is later important for the choice of δ. The definitions of Φ_{jk} and w_δ are not altered and the proof that $w_\delta \in L^1(0,1)$ remains the same:

$$\int_0^1 w_\delta(x)\,\mathrm{d}x = 2^{s+1}\frac{1-2\delta}{1-2^{s+1}\delta} < \infty$$

provided the unchanged condition (C4) is valid:

(C4) $$\delta < 2^{-s-1} \quad\Longleftrightarrow\quad 2^{s+1}\delta < 1.$$

Moreover, w_δ is continuous as a function with values in $[0,+\infty]$. At endpoints of removed intervals this follows from the divergence of Φ at 0 and 1. If $\chi \in \mathbb{D}_\delta$ is not such an endpoint and $x_n \to \chi$ with $x_n \notin \mathbb{D}_\delta$, then $x_n \in I_{j_n k_n}$ with $j_n \to \infty$. Hence

$$w_\delta(x_n) \ge 2^{s j_n} \to +\infty.$$

Since $w_\delta = +\infty$ on $\mathbb{D}_\delta$, this proves continuity at χ.

In Steps ④ and ⑤, we argue as before, replacing Ψ_2 by Ψ_p throughout. On each connected component of $(0,1)\setminus\mathbb{D}_\delta$, the function $\Psi_p\circ w_\delta$ is smooth. We therefore define the natural candidate for its weak derivative on $(0,1)$ by

$$G_\delta(x) := \begin{cases} (\Psi_p\circ w_\delta)'(x), & x\in[0,1]\setminus\mathbb{D}_\delta,\\ 0, & x\in\mathbb{D}_\delta. \end{cases}$$

Here we have to prove that

$$\Psi_p\circ w_\delta \in W^{1,p}((0,1)), \quad (\Psi_p\circ w_\delta)' = G_\delta \quad \text{a.e. in } (0,1).$$

Since w_δ is continuous with values in $[0,+\infty]$ and Ψ_p is continuous on $[0,+\infty]$, we have

$$\Psi_p\circ w_\delta \in C^0([0,1]) \subset L^p((0,1)).$$

Furthermore, ψ is bounded and Borel measurable. Therefore, Ψ_p is locally absolutely continuous on $\mathbb{R}$, with

$$\Psi_p'(t) = \psi(t)^{1/p} \qquad \text{for a.e. } t\in\mathbb{R}.$$

Moreover, $w_\delta\in C^1_{\mathrm{loc}}(I_{jk})$. Hence $\Psi_p\circ w_\delta\in W^{1,1}_{\mathrm{loc}}(I_{jk})$ is locally absolutely continuous on I_{jk}, and on each interval I_{jk} the one-dimensional chain rule gives

$$\begin{aligned}(\Psi_p\circ w_\delta)'(x) &= (\Psi_p\circ\Phi_{jk})'(x) = \Psi_p'\big(\Phi_{jk}(x)\big)\cdot\Phi_{jk}'(x)\\ &= \frac{2^{sj}}{(1-2\delta)\delta^{j-1}}\cdot\psi^{1/p}\big(\Phi_{jk}(x)\big)\cdot\Phi'\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}}+\frac{1}{2}\right)\end{aligned}$$

instead of (8). Since $\Phi_{jk}(x)\ge 2^{sj}\ge 1$, the decay assumption on ψ gives

$$\psi\big(\Phi_{jk}(x)\big) \le C\big(\Phi_{jk}(x)\big)^{-\alpha}.$$

Using the explicit form of w_δ on I_{jk}, the decay assumption on ψ, and the change of variables $y=\rho_{jk}(x)$, we obtain

$$\begin{aligned}\int_{I_{jk}}\big|(\Psi_p\circ w_\delta)'(x)\big|^p\,\mathrm{d}x &\preceq \frac{2^{sj(p-\alpha)}}{(1-2\delta)^p\delta^{(j-1)p}}\cdot\int_{I_{jk}}\frac{\left|\Phi'\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}}+\frac12\right)\right|^p}{\left|1+\Phi\left(\frac{x-a_{jk}}{(1-2\delta)\delta^{j-1}}+\frac12\right)\right|^\alpha}\,\mathrm{d}x\\ &\preceq \frac{2^{sj(p-\alpha)}}{(1-2\delta)^{p-1}\delta^{(j-1)(p-1)}}\cdot\int_0^1\frac{|\Phi'(y)|^p}{|1+\Phi(y)|^\alpha}\,\mathrm{d}y\\ &\underset{\text{(C4)}}{\overset{2\delta<2^{-s}}{\preceq}} \frac{2^{sj(p-\alpha)}}{\delta^{j(p-1)}}\int_0^1 y^{-p(\beta+1)}y^{\alpha\beta}\,\mathrm{d}y.\end{aligned}$$

The last integral is integrable if and only if $-p(\beta+1)+\alpha\beta>-1$. This is exactly our choice of β in (C2p). Consequently,

$$\int_{I_{jk}}\big|(\Psi_p\circ w_\delta)'(x)\big|^p\,\mathrm{d}x \preceq (2^{s(p-\alpha)}\delta^{1-p})^j.$$

Summing over all removed intervals, we obtain

$$\int_{[0,1]\setminus\mathbb{D}_\delta}\big|(\Psi_p\circ w_\delta)'(x)\big|^p\,\mathrm{d}x = \sum_{j=1}^\infty\sum_{k=1}^{2^{j-1}}\int_{I_{jk}}\big|(\Psi_p\circ w_\delta)'(x)\big|^p\,\mathrm{d}x \preceq \sum_{j=1}^\infty\Big(2^{1+s(p-\alpha)}\delta^{1-p}\Big)^j.$$

Therefore, the last geometric series converges provided that we replace (C5) by

$$\text{(C5p)} \qquad 2^{1+s(p-\alpha)}\delta^{1-p} < 1 \quad \iff \quad \delta > 2^{\frac{1-s(\alpha-p)}{p-1}}.$$

By the choice of s in (C3p), this condition is compatible with the upper bound $\delta < 2^{-s-1}$. Hence we may choose

$$2^{\frac{1-s(\alpha-p)}{p-1}} < \delta < 2^{-s-1}.$$

Thus, $(\Psi_p \circ w_\delta)' \in L^p([0,1] \setminus \mathbb{D}_\delta)$.

It remains to characterize G_δ as the weak derivative of $\Psi_p \circ w_\delta$. Since $\Psi_p \circ w_\delta$ is locally absolutely continuous on every removed interval I_{jk} and tends to M_p at both endpoints of each such interval, the same argument as in Step *4* of Thm. 5 gives

$$(\Psi_p \circ w_\delta)(x) = M_p + \int_0^x G_\delta(\xi)\,\mathrm{d}\xi \qquad \text{for all } x \in [0,1].$$

Together with $G_\delta \in L^p((0,1))$ similar to Step *5* of Thm. 5, this yields

$$\Psi_p \circ w_\delta \in W^{1,p}((0,1)), \qquad (\Psi_p \circ w_\delta)' = G_\delta \quad \text{a.e.}$$

As in Step *6* of the proof of Thm. 5, we define for all $x \in [0,1]$

$$u(x) := U(x) + f_\delta(x) \quad \text{with} \quad U(x) := \int_0^x w_\delta(t)\,\mathrm{d}t$$

where f_δ is the Cantor function associated with $\mathbb{D}_\delta$. Since $w_\delta \in L^1((0,1))$, we have $U \in W^{1,1}((0,1))$. Moreover, f_δ is continuous and nondecreasing. Hence, it follows that $u \in BV((0,1)) \cap C^0([0,1])$. In particular,

$$(u')^a = w_\delta \qquad \mathcal{L}^1\text{-a.e.,} \qquad \text{and} \qquad D^s u = D^c u = D^c f_\delta.$$

All pointwise statements about the blow-up set of $(u')^a$ are understood with respect to the chosen continuous representative w_δ. Since $w_\delta(x) = +\infty \iff x \in \mathbb{D}_\delta$, the chosen continuous representative of $(u')^a$ satisfies

$$Z^+[(u')^a] = \mathbb{D}_\delta \cap (0,1), \qquad Z^-[(u')^a] = \emptyset.$$

The singular measure $D^s u$ is positive and supported on $\mathbb{D}_\delta$. Hence its positive part is concentrated on $Z^+[(u')^a]$, while its negative part is zero and therefore trivially concentrated on $Z^-[(u')^a]$.

Together with $\Psi_p\big((u')^a\big) = \Psi_p \circ w_\delta \in W^{1,p}((0,1))$ it follows that $u \in X^p_\psi(0,1)$. Additionally, Du has a nontrivial Cantor part. The analogue of Step *7* of Thm. 5 is not needed here, since the corresponding structural result is already contained in [DMFLM09, Thm. 3.4.]. This concludes the proof. □

13 Corollary (Cantor parts for L^p-curvature energies)
Let $p > 1$. Then there exists a function $u \in BV((a,b)) \cap C^0([a,b])$ such that

$$D^c u \neq 0 \qquad \textit{and} \qquad \overline{\mathcal{W}}_p(u) < \infty.$$

In particular, finite relaxed L^p-curvature energy does not imply $u \in SBV((a,b))$.

Proof: For a classical u it holds

$$\mathcal{W}_p(u) = \int_a^b \psi_p(u')|u''|^p\,\mathrm{d}x \quad \text{with} \quad \psi_p(t) := (1+t^2)^{\frac{1-3p}{2}}.$$

The function ψ_p is bounded, positive, and satisfies the structural condition (3). Moreover,

$$\psi_p(t) \leq Ct^{-(3p-1)} \qquad \text{for } t \geq 1.$$

Therefore the decay exponent is $\alpha = 3p - 1 > 2p - 1$.

Thm. 12, applied with $\psi = \psi_p$, yields a function $u \in X^p_{\psi_p}((a,b))$ with nontrivial Cantor part $D^c u \neq 0$. By the relaxation theorem for $\mathcal{F}_p$, the membership $u \in X^p_{\psi_p}((a,b))$ implies $\overline{\mathcal{F}}_p(u) < \infty$. Since, for smooth functions, $\mathcal{W}_p(u) \leq \mathcal{F}_p(u)$, the same recovery sequence also gives $\overline{\mathcal{W}}_p(u) < \infty$. This proves the claim. □

14 Remark (Geometric interpretation)
For the specific curvature weight $\psi_p(t) = (1+t^2)^{(1-3p)/2}$, the function constructed in Corollary 13 admits the same geometric interpretation as in Theorem 9. The Cartesian curve $c_u : [0,1] \to \mathbb{R}^2$, associated with the function u constructed above admits an arc-length parametrization $\gamma : [0, L] \to \mathbb{R}^2$ such that $\gamma \in C^1([0,L];\mathbb{R}^2) \cap W^{2,p}((0,L);\mathbb{R}^2)$. Moreover, the scalar curvature of the arc-length parametrized curve belongs to $L^p((0,L))$:

$$\overline{\mathcal{W}}_p(u) \leq \int_0^L |\kappa_\gamma(\ell)|^p \,\mathrm{d}\ell < \infty.$$

Furthermore, as in Remark 10, one may obtain a C^2-regular image by a different choice of the singular profile. Indeed, the decay exponent is $\alpha = 3p-1$. The condition (C2p) in Thm. 12 requires $\beta \in \left(\frac{p-1}{\alpha-p}, 1\right)$. Since now

$$\frac{p-1}{\alpha-p} = \frac{p-1}{2p-1} < \frac{1}{2},$$

we may choose instead

$$\beta \in \left(\frac{1}{2}, 1\right).$$

If, in addition, the choice of δ is strengthened to

$$2^{-2s} < \delta < 2^{-s-1}, \qquad s > 1,$$

then the previous admissibility condition remains satisfied, because

$$2^{\frac{1-s(\alpha-p)}{p-1}} = 2^{-2s-\frac{s-1}{p-1}} < 2^{-2s}.$$

With these choices, the curvature on the smooth graph pieces extends continuously across the Cantor set by setting it equal to zero there. Consequently the graph $c_u([0,1])$ is a one-dimensional C^2-submanifold of $\mathbb{R}^2$ with boundary.

15 Corollary (Cantor parts for Willmore surfaces of revolution)
There exists $r \in BV((a,b)) \cap C^0([a,b])$, $r \geq r_0 > 0$, with $D^c r \neq 0$, such that the surface of revolution generated by the arc-length parametrized graph of r

$$\Sigma_r := \big\{(x, r(x)\cos\varphi, r(x)\sin\varphi) : x \in (a,b),\ \varphi \in [0,2\pi)\big\},$$

has finite Willmore energy.

Proof: Let u be the function obtained from the construction in the case $p = 2$, using the C^2-variant described in Remark 10, so that the graph $c_u([a,b])$ is a one-dimensional C^2-submanifold of $\mathbb{R}^2$. Moreover, $D^c u \neq 0$. Choose $r_0 > 0$ and set $r := r_0 + u > 0$. Since adding a constant does not change the distributional derivative, we have $D^c r = D^c u \neq 0$. Also, $c_r([a,b])$ is just the vertical translate of $c_u([a,b])$, and hence it is again a one-dimensional C^2-submanifold of $\mathbb{R}^2$.

The arc-length parametrization of $c_r([a,b])$ is denoted by

$$\gamma : [0,L] \to \mathbb{R}^2, \qquad \gamma(\ell) = (x(\ell), \rho(\ell)).$$

Since the generating curve is C^2, the map

$$F : [0,L] \times [0,2\pi] \to \mathbb{R}^3, \qquad F(\ell,\varphi) := \big(x(\ell), \rho(\ell)\cos\varphi, \rho(\ell)\sin\varphi\big)$$

is C^2. Therefore $|\partial_\ell F| = 1,\ |F_\varphi| = \rho \geq r_0,\ F_\ell \cdot F_\varphi = 0$, F is a regular C^2-parametrization of the surface of revolution. The principal curvatures are, up to the choice of orientation,

$$k_1(\ell,\varphi) = \kappa_\gamma(\ell), \qquad k_2(\ell,\varphi) = \frac{x'(\ell)}{\rho(\ell)}.$$

Indeed, k_1 is the curvature of the meridian curve, while k_2 is the rotational curvature. Therefore

$$\int_{\Sigma_r} k_1^2 \,\mathrm{d}A = 2\pi \int_0^L \rho(\ell)\, |\kappa_\gamma(\ell)|^2 \,\mathrm{d}\ell \leq 2\pi \|\rho\|_{L^\infty((0,L))} \int_0^L |\kappa_\gamma(\ell)|^2 \,\mathrm{d}\ell < \infty.$$

Here we used that ρ is continuous on the compact interval $[0,L]$ and that $\kappa_\gamma \in L^2((0,L))$.

For the rotational principal curvature we obtain

$$\int_{\Sigma_r} k_2^2 \,\mathrm{d}A = 2\pi \int_0^L \frac{x'(\ell)^2}{\rho(\ell)} \,\mathrm{d}\ell \leq \frac{2\pi}{r_0} \int_0^L x'(\ell)^2 \,\mathrm{d}\ell \leq \frac{2\pi L}{r_0} < \infty.$$

Since the mean curvature H is either $k_1 + k_2$ or $\frac{1}{2}(k_1 + k_2)$, depending on convention, we have in either case $|H|^2 \leq C\,(k_1^2 + k_2^2)$ with a universal constant $C > 0$. Hence $\mathcal{W}(\Sigma_r) < \infty$. □

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5 Appendix

16 Lemma

Let $\mathbb{D}_\delta$ and $(f_\ell)_{\ell \in \mathbb{N}}$ be defined as in Step 2 of Thm. 5. Then the following assertions hold.

There exists a function $f_\delta \in C^0([0,1])$ with $f_\delta(0) = 0, f_\delta(1) = 1$ such that

$$f_\ell \to f_\delta \qquad \textit{uniformly on } [0,1].$$

and moreover

$$Df_\delta = D^c f_\delta, \qquad |D^c f_\delta|((0,1)) = 1.$$

where $D^c f_\delta$ is a nonnegative Cantor measure supported on $\mathbb{D}_\delta$.

Proof: Let $m \geq \ell$. By (7), the functions f_m and f_ℓ agree on every interval I_{jk} with $j \leq \ell$. Let $J_{i\ell} = [a_{i\ell}, b_{i\ell}]$ be a connected component of C_ℓ. Then f_m and f_ℓ have the same endpoint values on $J_{i\ell}$, and

$$f_\ell(b_{i\ell}) - f_\ell(a_{i\ell}) = 2^{-\ell}.$$

Since both functions are nondecreasing, it follows that, for every $x \in J_{i\ell}$,

$$\big|f_m(x) - f_\ell(x)\big| \leq 2^{-\ell}.$$

Together with the equality on the removed intervals of levels $j \leq \ell$, this gives

$$\|f_m - f_\ell\|_{C^0([0,1])} \leq 2^{-\ell}.$$

Hence (f_ℓ) is Cauchy in $C([0,1])$ and converges uniformly to some $f_\delta \in C([0,1])$. Since every f_ℓ is nondecreasing and satisfies $f_\ell(0) = 0$, $f_\ell(1) = 1$, the same is true for f_δ.

It remains to characterize the distributional derivative of f_δ. Each f_ℓ is nondecreasing, and therefore the uniform limit f_δ is nondecreasing as well. Hence $f_\delta \in BV((0,1))$, and Df_δ is a finite nonnegative Radon measure.

On every removed interval I_{jk}, the functions f_ℓ are eventually constant. Passing to the uniform limit, f_δ is constant on each I_{jk}. Therefore

$$f'_\delta = 0 \qquad \text{a.e. on } [0,1] \setminus \mathbb{D}_\delta.$$

Since $\mathcal{L}^1(\mathbb{D}_\delta) = 0$, this implies $D^a f_\delta = 0$. Furthermore, f_δ is continuous, and hence it has no jump part: $D^j f_\delta = 0$. Since

$$|Df_\delta|((0,1)) = f_\delta(1) - f_\delta(0) = 1,$$

the derivative is nontrivial. Therefore the whole derivative is its Cantor part:

$$Df_\delta = D^c f_\delta.$$

Finally, since f_δ is constant on every connected component of $[0,1] \setminus \mathbb{D}_\delta$, the measure $D^c f_\delta$ is supported on $\mathbb{D}_\delta$. $\square$

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