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Introduction

We are concerned here with the flow of n-dimensional hypersurfaces $M_0 \subset \mathbf{R}^{n+1}$ by their mean curvature vector. That is, we look for a family of hypersurfaces $\{ M_t \}$ which represent a smooth flow of the surface M_0 with time, such that at time t, each point p_t on the surface M_t is moving in the direction of the unit normal to M_t, with a speed equal to $H(p_t)$, the mean curvature of the surface M_t at the point p_t.

Mean curvature flow has a very real physical realization. It is a well known physical phenomenon that if a piece of metal is heated up to such a high temperature that it liquefies, and then allowed to cool so that it becomes solid again, then the metal will form grains during the cooling stage (see diagram 0.1). These grains are irregular in size and shape and so make the metal irregular and more prone to break under stress. If we heat the metal up again, the grain boundaries begin to move, and it is known that these boundaries evolve in such a way that after some time the grains become reasonably uniform in size and shape, thus getting rid of a number of the structural problems that the metal had before. It has been shown that the way these boundaries move is very well approximated by mean curvature flow. (See diagram 0.2.)

Mean curvature flow can also be used as a tool for solving certain existence problems in the field of Partial Differential Equations (P.D.E.). The method by which we do this gives us a good idea as to what the resulting solution actually looks like. This is in contrast to standard P.D.E. existence and uniqueness theorems (see for example [11]), where often no real information as to what the solution looks like is made available.

We present this method here:
Take an initial surface M_0 and evolve it via mean curvature flow. If $M_t \to M_\infty$ as $t \to \infty$ (for some hypersurface M_∞) then M_∞ will have mean curvature 0 (if the flow is smooth) and hence M_∞ will be a minimal surface. Thus evolving M_0, under boundary conditions that we choose, may lead us to a minimal surface M_∞ with certain desired boundary conditions. This property of mean curvature flow is well explained by A. Stone in Chapter 1 of [10].

In the first chapter, we see that mean curvature flow is interesting from a geometrical point of view, in that it reduces area as quickly as is possible via a smooth flow. (See Theorem 1.5 of Chapter 1 for a more precise description of this property.)

Summary of contents

We are in particular interested in the mean curvature flow of surfaces which are rotationally symmetric. Our interest in the flow of such surfaces is largely a result of the paper by G. Huisken [2] where singularities occurring under the flow are investigated using rotationally symmetric surfaces. A rotationally symmetric surface M_0 is a surface obtained by rotating the graph of some function $y_0 : \mathbf{R} \to \mathbf{R}$ around the x-axis (see diagram 0.3). y_0 is known as the "generator" function of the surface M_0.

We concern ourselves mainly with the existence of solutions M_t to the flow, given initial data M_0, where M_0 is a rotationally symmetric surface (Chapter 4). We are also interested in "pinching" of such surfaces M_0. We say that a barbell like surface M_0 "pinches" off at time $T < \infty$, if, as t approaches T, the neck of the barbell pinches off and we are left with two sphere-like shapes on either side of the neck (see diagram 0.4).

If M_0 is a rotationally symmetric surface, then by symmetry of the surface M_0, and uniqueness of the flow for rotationally symmetric surfaces, M_t stays rotationally symmetric while solutions M_t exist. In our main theorem (Chapter 4, Theorem 4.3) we prove existence of solutions M_t for some short time interval $t \in [0, T)$, whenever: the initial data M_0 is a rotationally symmetric surface generated by some function y_0, and M_0 is smooth ($C^{2,\alpha}(\Omega)$), and M_0 has height strictly bounded away from 0 (ie. $\inf_{\mathbf{R}} y_0 > 0$). We also show there, that if $[0, T)$ is the maximal time interval for which we have such solutions M_t, and $T < \infty$, then $\inf_{\mathbf{R}} y(\cdot, t) \to 0$ as $t \to T$ (where $y(\cdot, t)$ is the generator of the surface M_t), and so the evolving surface M_t pinches off as $t \to T$.

This existence result is much better than we can usually hope to get using the parabolic theory, since we start off with no assumptions whatsoever on the growth of the initial surface near ∞. Usually some sort of growth assumption near ∞ is needed in order to obtain existence via parabolic theory.

Chapter 1 is concerned with formally introducing mean curvature flow and presenting some of its well known properties. In particular: convex surfaces shrink to a point in finite time under the flow, surfaces which are initially disjoint under the flow stay disjoint under the flow, and mean curvature flow reduces area as quickly as is possible via a smooth flow (as mentioned above).

In Chapter 2 we define what a rotationally symmetric surface of dimension n is, and calculate the evolution equation for rotationally symmetric surfaces evolving under the flow:

$$y_t = \frac{y_{xx}}{(1 + (y_x)^2)} - \frac{(n-1)}{y}$$

where $y(\cdot, t)$ is the generator of the surface M_t.

An interior height estimate and an interior gradient estimate for rotationally symmetric surfaces evolving under the flow is derived in Chapter 3.

In Chapter 4 we prove existence of solutions M_t to the flow when the initial data M_0 is rotationally symmetric (as explained above). To do this we use the interior height and gradient estimates of Chapter 3.

Pinching is examined for general surfaces in Chapter 5. We present some geometrical criteria there with the property that any initial surface M_0 satisfying these criteria will pinch off in finite time under the flow, and M_t will not have shrunk to a point when it does so.

A certain class of surfaces known as "self-similar" surfaces are defined in Chapter 6, and we point out there some of the interesting properties such surfaces have while evolving under the flow.

In Chapter 7 we show that if an initial surface M_0 is rotationally symmetric, and M_0 has polynomial growth of order p, then M_t has growth less than or equal to polynomial growth of order p.

The evolution equation for the gradient $\sqrt{1 + (y_x)^2}$ of a rotationally symmetric surface evolving under the flow is calculated in Chapter 8. We also obtain there the gradient estimate: $f \leq \frac{C}{y}$ for some constant $C < \infty$ independent of t, whenever $\sup_{M_0} fy$ is bounded initially.

Notation

$\vec{\iota_1}, \ldots \vec{\iota_{n+1}}$: The standard euclidean basis in $\mathbf{R}^{n+1}$.

x: The position vector of $\mathbf{R}^{n+1}$. Given $p \in \mathbf{R}^{n+1}$, $x(p) = (x^1(p), \ldots, x^{n+1}(p)) = \sum_j \langle p, \vec{\iota_j} \rangle \vec{\iota_j}$

Ω: An open set of an n-dimensional hyperplane sitting in $\mathbf{R}^{n+1}$

D_i: Differentiation in Ω with respect to the hyperplane in which it sits.

$M = M^n$: a base Riemannian manifold of dimension n.

$M_0 = M_0^n$: an initial n -dimensional hypersurface sitting in $\mathbf{R}^{n+1}$.

F_0: a smooth immersion $F_0 : M \rightarrow \mathbf{R}^{n+1}$, describing the initial surface M_0 by $F_0(M) = M_0$.

$F = F(\cdot, t)$: F is a smooth immersion $F(\cdot, t) : M_0 \rightarrow \mathbf{R}^{n+1}$ describing the surface M_t at time t under the flow, by $F(M, t) = M_t$.

M_t: The surface at time t under the flow with initial data M_0.

$\mathbf{S}^n$: The n-dimensional unit sphere with centre 0 sitting in $\mathbf{R}^{n+1}$.

$\vec{\nu} = \vec{\nu}(\cdot, t)$: The outward unit normal to the surface M_t.

$H = H(\cdot, t)$: The mean curvature of the surface M_t.

$\vec{H} = \vec{H}(\cdot, t)$: The mean curvature vector of the surface M_t. $\vec{H}$ is defined by $\vec{H} = -H \cdot \vec{\nu}$.

$h_{ij} = A = A(\cdot, t)$: The second fundamental form of the surface M_t. We define $h_{ij} = \langle \hat{\nabla}_{\vec{e_i}} \vec{\nu}, \vec{e_j} \rangle$ for a choice of unit normal $\vec{\nu}$, where $\vec{e_1}, \ldots, \vec{e_n}$ is an orthonormal basis.

g_{ij}: The first fundamental form. $g_{ij} = \langle \vec{e_i}, \vec{e_j} \rangle$.

g: The determinant of the first fundamental form. $g = det(g_{ij})$.

π: $\pi(Y_p)$ is the orthogonal projection of the vector $Y_p \in \mathbf{R}^{n+1}$ onto the tangent space of M_t at p.

$\hat{\nabla}$: gradient of the surface M_t (inherited from the space $\mathbf{R}^{n+1}$).

∇: tangential gradient of the surface M_t ($\nabla = \pi \circ \hat{\nabla}$).

$\triangle$: Laplace- Beltrami operator on the surface M_t.

Chapter 1

Formalization of mean curvature flow

1.1 Introduction

In this chapter we present a formal definition of what it means for a surface to be evolving under mean curvature flow. We also derive a number of important properties that surfaces evolving under mean curvature flow have. In particular convex surfaces shrink to a point in finite time, surfaces stay disjoint under the flow, and the flow reduces area at a rate which is faster than the rate at which any other flow reduces area.

1.2 Formal introduction to mean curvature flow

Definition 1.1 *Let $F_0 : M^n \to \mathbf{R}^{n+1}$ be a smooth immersion with $F_0(M) = M_0$. If $\exists$ smooth immersions $F(\cdot, t) : M^n \to \mathbf{R}^{n+1}$such that*

$$\begin{aligned} F(\cdot, 0) &= F_0, & (1.1) \\ \textit{and } \frac{\partial}{\partial t} F &= \vec{H} & (1.2) \end{aligned}$$

then we say that M_0 evolves under mean curvature flow, and that the surface at time t under this evolution is $M_t = F(M, t)r$.

$\vec{H}$ is the mean curvature vector on the surface M_t,

$$\vec{H}(\cdot, t) = \vec{H} = -H(\cdot, t) \cdot \vec{\nu}(\cdot, t) = -H \cdot \vec{\nu},$$

where $\vec{\nu}$ is a choice of unit normal for the surface M_t, and H is the mean curvature of the surface M_t. In particular, given our choice of unit normal $\vec{\nu}$, we give H the sign so that

$$\triangle x = \vec{H}$$

is true. (That this formula is true up to the sign of H is shown in Appendix 3, Lemma C.2.)

We shall refer to 1.2 as the **evolution equation** or the **flow equation** or **the flow problem.** Note that 1.2 is equivalent up to tangential diffeomorphisms to the equation

$$(\frac{\partial}{\partial t}F)^{\perp} = \vec{H} \tag{1.3}$$

i.e. the surfaces we get under 1.2 and 1.3 are the same at each time.

Given some initial surface M_0, it is not immediately clear that a smooth flow exists. We break this existence problem up into two cases:

Problems:

i) Can we find solutions M_t for $t \in [0,T)$ for some maximal time interval $[0,T)$, for some $T < \infty$ (**short time existence**)?

ii) Can we find solutions M_t for $t \in [0,\infty)$ (**longtime existence**)?

1.3 Discussion of short time existence

Here we would like to discuss the first of the above problems — short time existence. A question that immediately arises when considering this problem is "does a short time existence solution imply a long time existence solution?" The answer is a categorical no. (See Example 1 below.) Here we consider an initial surface M_0 which can be written as a graph and show that we do have short time existence. We consider this example initially because it powerfully illustrates the link between the flow problem and the theory of parabolic equations.

Let Ω be an open domain in $\mathbf{R}^n$. Let u_0 be a $C^{2,\alpha}(\Omega)$ function $u_0 : \Omega \to \mathbf{R}$ (for some $\alpha > 0$), and let $M_0 = graph(u_0)$. So here $M = \mathbf{R}^n$, and $F_0(x) = (x, u_0(x))$.

Problem: Can we find a $C^{2,\alpha'}(\Omega \times [0,T))$ function u, such that $u : \Omega \times [0,T) \to \mathbf{R}$ satisfies

$$\begin{aligned} u(\cdot,0) &= u_0 && (1.4)\\ \text{and } u_t &= \sqrt{1+(Du)^2}D_i(\frac{D_i u}{\sqrt{1+|Du|^2}}) && (1.5)\end{aligned}$$

From Appendix 2 we see that if we can find such a u, and if we let $F(x) = (x, u(x))$, then F satisfies equation 1.3. Hence we have solutions $M_t = graph(u(\cdot,t))$ to the flow problem for some short time interval $[0,T)$.

Remark: From Appendix 2, we see that the mean curvature of a hypersurface written as a graph u (note: all hypersurfaces can locally be written as a graph relative to a suitable choice of co-ordinate axes) depends on the second spatial derivatives of u. So in order to obtain a smooth solution to the flow we need u to be at least $C^{2,\alpha}(\Omega \times [0,T))$.

Equation 1.5 is a quasi-linear parabolic equation (i.e. it is of the form $u_t = Qu$, where Q is a quasi-linear elliptic partial differential operator). Let us impose the Dirichlet boundary conditions

$$u(\cdot,t) = b(\cdot,t) \tag{1.6}$$

on the boundary $\partial\Omega$, where b is some function $\in C^{2,\alpha}(\partial\Omega \times [0,T))$, along with our initial condition $u(\cdot,0) = u_0$ on Ω. Then by the theory of parabolic equations, (e.g. see [8]), 1.5 has a unique solution for some short time interval [0,T].

So this takes care of short time existence for a surface which is a graph, flowing under Dirichlet boundary conditions.

Parabolic theory can also be used to show [3] that:

Theorem 1.1 *Let M_0 be a smooth compact (without boundary) initial surface. Then for initial data M_0 we have short time existence (and uniqueness) of a solution to the flow problem.*

Example 1.

Let $S^n_{R_0}(z)$ be an n-dimensional sphere of radius R_0, with centre z, sitting in $\mathbf{R}^{n+1}$. The surface at time t under mean curvature flow is $M_t = S^n_{R(t)}(z)$, where $R(t) = \sqrt{R_0^2 - 2nt}$. So M_0 shrinks to a point at $T = \frac{R_0^2}{2n}$ under the mean curvature flow. (See diagram 1.1.)

It was shown by Huisken [1] ($n \geq 2$), and Gauge/Hamilton [9] ($n = 1$) that the behaviour of a convex surface under the flow is the same as it is for the sphere. That is,

Theorem 1.2 *If M_0 is convex, compact, smooth (without boundary) then M_0 shrinks to a point in finite time under mean curvature flow and it becomes sphere like as it does so.*

(See diagram 1.2.)

This might lead one to conjecture that all smooth compact closed surfaces will shrink to a point in finite time. This is **not** the case as will be shown shortly.

For the case $n = 1$, (i.e. M_0 is a curve in the plane), non-embedded smooth closed curves M_0 can develop singularities before shrinking to a point. (See diagram 1.3.)

But Grayson [6] shows that:

Theorem 1.3 *Embedded smooth, closed curves in the plane become convex in finite time and then by Theorem 1.2 shrink to a point in finite time.*

For $n \geq 2$ $\exists$ embedded compact surfaces (without boundary) M_0 sitting in $\mathbf{R}^{n+1}$ such that M_0 develops a singularity under mean curvature flow at a finite time T and at time T the surface M_0 has not shrunk to a point. This fact is shown in Chapter 5.

1.4 Area reducing property of the flow

Now we wish to discuss the area of a surface which is evolving under the flow. Clearly for convex surfaces evolving under the flow, the area is being reduced at some rate, since after finite time a convex surface will shrink to a point under the flow (as is shown in Theorem 1.2). We show here that mean curvature flow reduces area for any smooth initial data M_0.

Lemma 1.4 *Mean curvature flow reduces area.*

proof: This can be easily seen by noting that locally

$$|M_t| = Area(M_t) = \int_{M_t} 1 du_t = \int_\Omega \sqrt{g} dx$$

(where here we have used a local co-ordinate chart $(x, U \subset M_t)$, $x : U \to \Omega \subset \mathbf{R}^n$ to describe the surface M_t (locally), and we have written the metric g_{ij} in terms of this co-ordinate chart) so that

$$\begin{aligned}
\frac{\partial}{\partial t}|M_t| &= \frac{\partial}{\partial t}\int_\Omega \sqrt{g} dx \\
&= \int_\Omega \frac{\partial}{\partial t}\sqrt{g} dx \\
&= \int_\Omega \frac{1}{2\sqrt{g}}\frac{\partial}{\partial t} det(g_{ij}) dx \\
&= \int_\Omega \frac{1}{2}\sqrt{g} g^{ij}\frac{\partial}{\partial t} g_{ij} dx \\
&= \int_\Omega \frac{1}{2}\sqrt{g} g^{ij}(-2Hh_{ij}) dx
\end{aligned}$$

(since in [1] Lemma 3.2, it is shown that $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$)

$$\begin{aligned}
&= -\int_\Omega \sqrt{g} H^2 dx \\
&= -\int_{M_t} H^2 du_t \\
&\leq 0. \quad \blacksquare
\end{aligned}$$

So we have calculated the rate at which the area of a surface reduces as it evolves under mean curvature flow. In fact mean curvature flow reduces area as quickly as is possible. That is,

Theorem 1.5 *Given an initial surface M_0, and a function $f \in L^2(M_t)$ such that $\|f\|_{L^2(M_t)} = \|H\|_{L^2(M_t)}$, and such that $f \neq H$, then the flow given by*

$$\frac{\partial}{\partial t}\vec{F} = -f \ldots \nu \tag{1.7}$$

has the property that it reduces area at a rate which is strictly less than the rate at which mean curvature flow reduces area. i.e. if M_t^* are the surfaces we get under the flow given by 1.7, then $\frac{\partial}{\partial t}|M_t^*|_{t=0} \geq \frac{\partial}{\partial t}|M_t|_{t=0}$ with strict inequality if $f \neq H$.

proof:

$$\begin{aligned}\frac{\partial}{\partial t}|M_t^*|_{|t=0} &= -\int_{M_0^*} fH du_t \\ &= -\int_{M_0} fH du_t\end{aligned}$$

since by an almost identical calculation to [1] Lemma 3.2 we have that $\frac{\partial}{\partial t}g_{ij} = -2fh_{ij}$ for a surface under evolution by 1.7.

But

$$\begin{aligned}\langle f, H\rangle &= \int_{M_0} fH du_t \\ &\leq (\int_{M_0} f^2)^{1/2}(\int_{M_0} H^2)^{1/2} \\ &= \|f\|_2\|H\|_2 \\ &= \|H\|_2^2\end{aligned}$$

From analysis, this inequality is strict **unless** $f = \alpha H$ for some constant α. But

$$\begin{aligned}f = \alpha H &\Rightarrow \|f\|_2 = \alpha\|H\|_2 \\ &\Rightarrow \alpha = 1\end{aligned}$$

So the inequality is strict unless $f = H$. ∎

1.5 Disjoint surfaces under the flow

Here we show that two initially disjoint surfaces evolving under the flow stay disjoint.

Theorem 1.6 *If M_0 and N_0 are two initially disjoint surfaces, then they will stay disjoint under the flow* **or** *they will touch at some first time and they will do so on the boundary of at least one of the surfaces. (If a surface is non-compact, then the boundary includes ∞.)*

proof: Assume the surfaces touch at some interior point $z \in \mathbf{R}^{n+1}$, at some first time T. Now at time T we can write a small portion of the surface M_T containing z in it's interior, as the graph of a function $u(\cdot, T)$ defined on some hyperplane Ω. Assuming z is not on the boundary of either surface, then we can write a small portion of N_T containing z in it's interior, as the graph of a function $v(\cdot, T)$ defined on the same hyperplane Ω. Then for a small time interval $t \in [T-\epsilon, T]$, the portion of the surfaces M_t, and N_t sitting above Ω can be written as the graph of functions $u(\cdot, t) : \Omega \to \mathbf{R}$ and $v(\cdot, t) : \Omega \to \mathbf{R}$ (respectively). Without loss of generality $u < v$ on $[T-\epsilon, T)$. Now as shown earlier in this chapter the equation for flow of a surface written as a graph is:

$$g_t = Qg = \sqrt{1+(Dg)^2} D_i\left(\frac{D_i g}{\sqrt{1+|Dg|^2}}\right)$$

So both u and v satisfy this equation.
Now $Qu = a^{ij}(x, Du) D_{ij} u$ where

$$a^{ij}(x, Du) = \frac{\delta_{ij}(1+|Du|^2) - D_i u D_j u}{(1+|Du|^2)}.$$

So

$$\begin{aligned} Qu - Qv &= a^{ij}(x, Du) D_{ij}(u-v) + [a^{ij}(x, Du) - a^{ij}(x, Dv)] D_{ij} v \\ &= (u-v)_t \end{aligned}$$

Setting $w = u - v$, $a^{ij}(x) = a^{ij}(x, Du)$, and $[a^{ij}(x, Du) - a^{ij}(x, Du)] D_{ij} v = b^i(x) D_i w$, and $Lw = a^{ij}(x) D_{ij} w + b^i D_i w$, we see that $Lw - w_t = 0$, and $w < 0$ on $[T-\epsilon, T)$ on the whole of Ω. The existence of the smooth functions b^i can be seen by use of the mean value theorem for functions going from $\mathbf{R}^n$ to $\mathbf{R}$:

$$\begin{aligned} a^{ij}(x, Du) - a^{ij}(x, Dv) &= \int_0^1 \frac{\partial}{\partial t}[a^{ij}(x, tDu + (1-t)Dv)] dt \\ &= \int_0^1 \frac{\partial a^{ij}(x, tDu + (1-t)Dv)}{\partial p_k} D_k(u-v) dt \\ &= D_k(u-v) \int_0^1 \frac{\partial a^{ij}(x, tDu + (1-t)Dv)}{\partial p_k} dt \\ &= D_k(u-v) a_k^{ij} \end{aligned}$$

so $b^k = a_k^{ij} D_{ij} v$.
Invoking the parabolic strong maximum principle, we see that w cannot achieve it's maximum in the interior of $\overline{\Omega} \times [0, T]$. So since $w \leq 0$ on $[T - \epsilon, T]$ then if $w(x) = 0$ at time T then x must be a maximum of w on $\overline{\Omega} \times [0, T]$, and so $x \in \partial\overline{\Omega}$. This contradicts the fact that z is in the interior of $u(\Omega, T)$. ■

1.6 Barriers

We now discuss a class of natural barriers for surfaces evolving under the flow. If M_0 is an initial surface disjoint from a surface $\overline{M_0}$, and $\overline{M_0}$ has everywhere ≥ 0 (≤ 0) mean curvature, and if the mean curvature vectors of $\overline{M_0}$ point towards M_0 then $\overline{M_0}$ will serve as a barrier for the surface M_0.

Theorem 1.7 *If M_0 and $\overline{M_0}$ are two initially disjoint surfaces, and $\overline{M_0}$ has ≥ 0(≤ 0) mean curvature, and the mean curvature vectors of $\overline{M_0}$ point towards M_0, then M_0 will stay disjoint from $\overline{M_0}$ under the flow, or the surfaces will touch at some first time and they will do so on the boundary of at least one of the surfaces. (For non-compact surfaces the boundary includes ∞.)*

proof: The proof is essentially the same as for the previous theorem. The only change that needs to be made is: we let $v(\cdot, t) = v_0$ where v_0 denotes a portion of $\overline{M_0}$ where the surfaces are assumed to touch at time T. Then we get the parabolic equation:

$$Lw - w_t \geq 0,$$

and $w \leq 0$ on $[T - \epsilon, T]$, and so the strong maximum principle is still valid. ■

1.7 The monotonicity formula

It is shown by Ecker/Huisken [5] (in Section 1 of this chapter) that if M_0 is an initial surface which does not rise too steeply (e.g. has polynomial growth), and f is function $f(\vec{x}, t)$ defined on M_t which does not rise to steeply (e.g. f has polynomial growth), then we have the monotonicity formula:

$$\frac{\partial}{\partial t} \int_{M_t} f \rho du_t = \int_{M_t} (\frac{\partial}{\partial t} f - \triangle f) \rho du_t - \int_{M_t} f \rho |\vec{H} + \frac{1}{2\tau} (\vec{x} - \vec{x_0})^{\perp}|^2 du_t,$$

where $\tau = t_0 - t$, and ρ is the "backward heat kernel":

$$\rho(\vec{x}, t) = (4\pi\tau)^{-n/2} \exp(\frac{-|\vec{x_0} - \vec{x}|^2}{4\tau}), t_0 > t,$$

for a fixed point $(\vec{x_0}, t_0)$. From this Ecker/Huisken then obtain the monotonicity formula:

Theorem 1.8 *Suppose the function $f = f(\vec{x}, t)$ satisfies the inequality*

$$(\frac{\partial}{\partial t} - \triangle)f \leq \vec{a} \cdot \nabla f$$

for some vector field $\vec{a}$. If $a_0 = \sup_{M\times[0,t_1]} |\vec{a}| < \infty$ for some $t_1 > 0$, then

$$\sup_{M_t} f \leq \sup_{M_0} f$$

$\forall t \in [0, t_1]$.

The surfaces and functions we are going to consider will have (assumed or explicit) polynomial growth.

Chapter 2

Rotationally symmetric surfaces

2.1 Introduction

In this chapter we introduce rotationally symmetric surfaces. We derive the equation

$$y_t = \frac{y_{xx}}{(1+(y_x)^2)} - \frac{(n-1)}{y}$$

for flow of a rotationally symmetric surface, (where $y(\cdot, t)$ is the generator of the surface M_t) and a number of other evolution equations for interesting functions defined on the surface M_t. (e.g. we derive the evolution equation for the curvature $A = h_{ij}$ where h_{ij} is expressed with respect to a particular co-ordinate frame $\{ \vec{e_i} \}$ chosen by us.) In the last part of this chapter we show that for a particular class of initial surfaces M_0, we can derive a time independent gradient estimate for the surfaces M_t obtained under mean curvature flow.

2.2 Formal introduction to rotationally symmetric surfaces

Definition 2.1 *A surface M_0 is said to be rotationally symmetric if*

$$M_0 = f(\mathbf{R} \times \mathbf{S}^{n-1})$$

$$\text{where } f(x_1, s_1, \ldots, s_n) = \begin{pmatrix} x \\ y_0(x)s_1 \\ \vdots \\ y_0(x)s_n \end{pmatrix}$$

for some $y_0 : \mathbf{R} \to \mathbf{R}$.

y_0 is to be thought of as the graph which generates the surface. e.g. for $n = 2$ we just rotate the graph of y_0 around the x-axis. (See diagram 0.3.) The mean curvature of a rotationally symmetric surface is

$$H = \frac{(n-1)}{y\sqrt{1+(y_x)^2}} - \frac{y_{xx}}{(1+(y_x)^2)^{3/2}} \tag{2.1}$$

(as shown in appendix 1), and the outward unit normal is given by

$$\vec{\nu} = \begin{pmatrix} \frac{-y_x}{\sqrt{1+(y_x)^2}} \\ \frac{s_1}{\sqrt{1+(y_x)^2}} \\ \vdots \\ \frac{s_n}{\sqrt{1+(y_x)^2}} \end{pmatrix} \tag{2.2}$$

(also shown in appendix 1).

2.3 Mean curvature flow of rotationally symmetric surfaces

In this section we wish to examine the behaviour of rotationally symmetric surfaces evolving under mean curvature flow. In particular we will derive the evolution equation stated at the beginning of this chapter.

By symmetry, a rotationally symmetric surface will stay rotationally symmetric under mean curvature flow.

In Chapter 2 we noted that the evolution equation

$$(\frac{\partial}{\partial t}F)^{\perp} = \vec{H} \tag{2.3}$$

is equivalent up to tangential diffeomorphisms to the mean curvature flow equation

$$\frac{\partial}{\partial t}F = \vec{H} \tag{2.4}$$

If our initial surface $M = M_0$ is rotationally symmetric, and F is a solution to 2.3 which preserves the x_1 co-ordinate, then we have

$$F(x_1, 0, \ldots, 0, y_0(x_1), t) = (x_1, 0, \ldots, 0, y(x_1, t))$$

for some function. $y : M_0 \times \mathbf{R} \to \mathbf{R}$, with $y(\cdot, 0) = y_0$.
$y(\cdot, t)$generates the surface M_t at time t. Note that for points $p = (x_1, 0, \ldots, 0, y, t)$, $x_2 = 0, \ldots, x_n = 0$ is preserved under the flow by symmetry, and uniqueness of the flow for rotationally symmetric surfaces. Hence, $\frac{\partial}{\partial t} F = (0, 0, \ldots, 0, y_t(x_1, t))$. Hence, by 2.2, we have

$$\langle \frac{\partial}{\partial t} F, \vec{\nu} \rangle = \frac{y_t}{\sqrt{1 + (y_x)^2)}} \tag{2.5}$$

But by 2.3 we have

$$\langle \frac{\partial}{\partial t} F, \vec{\nu} \rangle = -H \tag{2.6}$$

Hence, combining 2.5, 2.6, and 2.1, we see that

$$\frac{y_t}{\sqrt{1 + (y_x)^2)}} = \frac{y_{xx}}{(1 + (y_x)^2)^{3/2}} - \frac{(n-1)}{y\sqrt{1 + (y_x)^2}}$$

and hence we have the evolution equation for the flow of a rotationally symmetric surface:

$$y_t = \frac{y_{xx}}{(1 + (y_x)^2)} - \frac{(n-1)}{y}$$

Existence of the flow for rotationally symmetric surfaces will be shown in Chapter 4.

2.4 Evolution equations

Now we wish to derive certain evolution equations for rotationally symmetric surfaces (of dimension n) evolving under mean curvature flow. We do this by generalizing [1], where in Chapter 5 (Lemma 5.1), evolution equations are derived for rotationally symmetric surfaces of dimension $n = 2$ evolving under the flow. Let $y_0 : \mathbf{R} \to \mathbf{R}$ be a smooth positive function. Let M_0 be a rotationally symmetric surface of dimension $n \geq 2$ generated by the function y_0. Let $\vec{\imath}_1, \ldots, \vec{\imath}_{n+1}$ be the standard basis in $\mathbf{R}^{n+1}$ and let $\vec{e_1}, \ldots, \vec{e_n}$ be the local orthonormal frame on M_0 such that

$$\begin{aligned} \langle \vec{e_j}, \vec{\imath}_1 \rangle &= 0 \ \forall j = 2, \ldots, n \\ \langle \vec{e_1}, \vec{\imath}_1 \rangle &> 0. \end{aligned}$$

e.g. along the top ridge,

$$\begin{aligned} \vec{e_1} &= \frac{1}{\sqrt{1 + (y_x)^2}} (1, 0, \ldots, y_x) \\ \vec{e_j} &= \vec{\imath}_j \ \forall j = 2, \ldots, n. \end{aligned}$$

(the top ridge of a rotationally symmetric surface M_0 generated by a function y_0 is the set of points $\{p \in M_0 : p = (x_1, 0, \ldots, 0, y_0(x_1))\ \}$). (See diagram 2.1.) Clearly the vectors $\vec{e_1}, \ldots, \vec{e_n}$ are orthonormal and there are n of them. They are also in the tangent space, as can be seen by a quick examination of the definition of what a rotationally symmetric surface is. So $\vec{e_1}, \ldots, \vec{e_n}$ is a well defined local orthonormal frame. Introduce the notation

$$\begin{aligned} p &= \langle \vec{e_1}, \vec{\iota_1} \rangle y^{-1}, \\ q &= \langle \vec{\nu}, \vec{\iota_1} \rangle y^{-1}. \end{aligned}$$

Then $\langle \vec{e_1}, \vec{\iota_1} \rangle = \frac{1}{\sqrt{1+(y_x)^2}}$ and from 2.2 we see that $\langle \vec{\nu}, \vec{\iota_1} \rangle = \frac{-y_x}{\sqrt{1+(y_x)^2}}$. So $p^2 + q^2 = y^{-2}$. Also $\nabla_i y = -\delta_{i1} q y$. If f is a function defined on a rotationally symmetric surface M_0 such that $f(p) = f(x_1, y_0 s1, \ldots, y_0 s_n)$ is a constant for fixed values of x_1, then clearly $\nabla_i f = \delta_{i1} \nabla_i f$, for the local orthonormal frame $\{\ \vec{e_i}\ \}$ we have chosen. So we need only calculate $\nabla_1 f = \vec{e_1}(f)$ for such functions.

The second fundamental form $A = h_{ij}$ is defined by $h_{ij} = \langle \nabla_{\vec{e_i}} \vec{\nu}, \vec{e_j} \rangle$. Using our choice of $\{\ \vec{e_i}\ \}$, and the calculation of $\vec{\nu}$ for rotationally symmetric surfaces (2.2), we calculate,

$$\begin{aligned} k = h_{11} = \langle \nabla_{\vec{e_1}} \vec{\nu}, \vec{e_1} \rangle &= -y_{xx}(1 + (y_x)^2)^{-3/2} \\ h_{ii} &= p \text{ for } 2 \leq i \leq n, \\ h_{ij} &= 0 \text{ for } i \neq j, \end{aligned}$$

where here we have introduced the notation $k = h_{11}$. Now evolve M_0 by mean curvature flow. As we have stated before, M_t stays rotationally symmetric. The position vector $F = \vec{x}$ of the hypersurface satisfies

$$\frac{\partial}{\partial t} \vec{x} = \triangle \vec{x} = \vec{H} = -H \cdot \vec{\nu} \tag{2.7}$$

as is shown in Appendix 3 Lemma C.2.

Define the function

$$y = (|\vec{x}|^2 - |\langle \vec{x}, \vec{\iota_1} \rangle|^2)^{1/2} \tag{2.8}$$

At time $t = 0$ it agrees with y_0.

note: y as we have defined it here is **not** the same as the y defined by 2.3. $y(\vec{p}, t)$ as we have defined it here measures the height of a point $F(\vec{p}, t)$ above the $x1$-axis at time t under the flow for a **fixed** p: i.e. we take a point $F_0(\vec{p})$ on the original surface and follow it through time, as the surface flows, and calculate its height above the x_1-axis at time t. $y(x)$ as defined by 2.3 measures the height of the surface at time t above x on the x^1-axis. The function y defined by 2.3 will be known as the **generator function** , and the y defined by 2.8 will be known as the **height function** . It should be clear by the context which y we are talking about.

Lemma 2.1 *Let y be the height function. Under mean curvature flow we get the following evolution equations.*

$$\begin{array}{ll} i) & \frac{\partial}{\partial t}\langle \vec{x}, \vec{\iota}_1\rangle = \triangle \langle \vec{x}, \vec{\iota}_1\rangle \\ ii) & \frac{\partial}{\partial t} y = \triangle y - (n-1)y^{-1}. \\ iii) & \frac{\partial}{\partial t} q = \triangle q + |A|^2 q + q((n-1)p^2 + (n-3)q^2 - 2kp \\ iv) & \frac{\partial}{\partial t} p = \triangle p + |A|^2 p + 2q^2(k-p). \\ v) & \frac{\partial}{\partial t} k = \triangle k + |A|^2 k - 2(n-1)q^2(k-p). \\ vi) & \frac{\partial}{\partial t} H = \triangle H + H|A|^2. \end{array}$$

proof:
i) is immediate from 2.7.
ii):

$$\begin{aligned} \frac{\partial}{\partial t} y &= \frac{\partial}{\partial t}(|\vec{x}|^2 - |\langle \vec{x}, \vec{\iota}_1\rangle|^2)^{1/2} \\ &= y^{-1}(\langle \vec{x}, \frac{\partial}{\partial t}\vec{x}\rangle - \langle \vec{x}, \vec{\iota}_1\rangle\langle \frac{\partial}{\partial t}\vec{x}, \vec{\iota}_1\rangle) \\ &= y^{-1}(\langle \vec{x}, \vec{H}\rangle - \langle \vec{x}, \vec{\iota}_1\rangle\langle \vec{H}, \vec{\iota}_1\rangle) \end{aligned}$$

Let $\alpha 1, \ldots, \vec{a_n}$ be a Riemannian orthonormal frame on M_t. Then

$$\begin{aligned} \triangle y &= \vec{a_i}(\vec{a_i}(y)) \\ &= \vec{a_i}(\vec{a_i}((|\vec{x}|^2 - |\langle \vec{x}, \vec{\iota}_1\rangle|^2)^{1/2})) \\ &= \vec{a_i}(y^{-1}(\langle \vec{x}, \hat{\nabla}_{\vec{a_i}}\vec{x}\rangle - \langle \vec{x}, \vec{\iota}_1\rangle\langle \hat{\nabla}_{\vec{a_i}}\vec{x}, \vec{\iota}_1\rangle)) \end{aligned}$$

using $\hat{\nabla}_{\vec{a_i}}\vec{x} = \vec{a_i}$ (from Appendix 3)

$$= \vec{a_i}(y^{-1}(\langle \vec{x}, \vec{a_i}\rangle - \langle \vec{x}, \vec{\iota}_1\rangle\langle \vec{a_i}, \vec{\iota}_1\rangle))$$

using $\hat{\nabla}_{\vec{a_i}}\vec{a_i} = \vec{H}$ and $\hat{\nabla}_{\vec{a_i}}\vec{x} = \vec{a_i}$ (both from Appendix 3)

$$
\begin{aligned}
&= -y^{-1}|\nabla y|^2 + y^{-1}(\langle \vec{a_i}, \vec{a_i}\rangle + \langle \vec{x}, \vec{H}\rangle - \langle \vec{a_i}, \vec{\iota}_1\rangle^2 - \langle \vec{x}, \vec{\iota}_1\rangle\langle \vec{H}, \vec{\iota}_1\rangle) \\
&= -y^{-1}|\nabla y|^2 + y^{-1}(n + \langle \vec{x}, \vec{H}\rangle - \langle \vec{a_i}, \vec{\iota}_1\rangle^2 - \langle \vec{x}, \vec{\iota}_1\rangle\langle \vec{H}, \vec{\iota}_1\rangle) \\
&= \frac{\partial}{\partial t} y + y^{-1}(-|\nabla y|^2 + n - \langle \vec{a_i}, \vec{\iota}_1\rangle^2)
\end{aligned}
$$

Now $\langle b_i, \vec{\iota}_1\rangle^2$ is independent of any orthonormal frame $\{\ b_i\ \}$ we care to choose (this is so because for any orthonormal frame $\{\ b_i\ \}$, $\langle b_i, \vec{\iota}_1\rangle b_i$ is the projection of $\vec{\iota}_1$ onto the tangent space, and so $\langle b_i, \vec{\iota}_1\rangle^2$is the length of this projected vector). Hence $\langle \vec{a_i}, \vec{\iota}_1\rangle^2 = \langle \vec{e_i}, \vec{\iota}_1\rangle^2$, where $\vec{e_i}$ is our specially chosen orthonormal frame. Hence using

$$
\begin{aligned}
\langle \vec{e_j}, \vec{\iota}_1\rangle &= 0 \forall j = 2, \ldots, n, \\
\langle \vec{e_1}, \vec{\iota}_1\rangle^2 &= y^2 p^2 \\
&= 1 - y^2 q^2 \\
&= 1 - |\nabla y|^2,
\end{aligned}
$$

we see that

$$\frac{\partial}{\partial t} y = \triangle y - (n-1) y^{-1}.$$

iii): To derive iii) we infer from [1] Lemma 3.3 that

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle \vec{\nu}, \vec{\iota}_1\rangle &= \langle \nabla H, \vec{\iota}_1\rangle \\
&= \triangle\langle \vec{\nu}, \vec{\iota}_1\rangle + |A|^2\langle \vec{\nu}, \vec{\iota}_1\rangle
\end{aligned}
$$

The last identity is shown in appendix 3 Lemma C.8 (the identity is shown using the Codazzi equation). Combining this with equation ii) we get

$$\frac{\partial}{\partial t} q = \triangle q + 2y^{-2}\nabla_i y \nabla_i\langle \vec{\nu}, \vec{\iota}_1\rangle - 2y^{-3}\langle \vec{\nu}, \vec{\iota}_1\rangle|\nabla y|^2 + (n-1)y^{-3}\langle \vec{\nu}, \vec{\iota}_1\rangle + |A|^2 q.$$

The result then follows from the identities $\nabla_i y = -\delta_{i1} q y$, $\nabla_1\langle \vec{\nu}, \vec{\iota}_1\rangle = kpy$, and $y^{-2} = p^2 + q^2$.

iv):

$$\frac{\partial}{\partial t} p = \frac{\partial}{\partial t}(y^{-2} - q^2)^{1/2}$$

(using ii) and iii))

$$
\begin{aligned}
&= \triangle p - 3y^{-4}p^{-1}|\nabla y|^2 + p^{-1}|\nabla q|^2 + p^{-1}|\nabla p|^2 \\
&+ (n-1)p^{-1}y^{-4} - p^{-1}q^2|A|^2 \\
&- p^{-1}q^2((n-1)p^2 + (n-3)q^2 - 2kp)
\end{aligned}
$$

Now substituting in

$$\begin{aligned}
\nabla_i p &= \delta_{i1} q(p-k) \\
\nabla_i q &= \delta_{i1}(q^2+kp) \\
|A|^2 &= k^2+(n-1)p^2 \\
y^{-4} &= p^4+2p^2q^2+q^4 \\
\nabla_i y &= -\delta_{i1} q y,
\end{aligned}$$

and collecting like terms, we get

$$= \triangle p + p|A|^2 + 2q^2(k-p).$$

vi):

$$\frac{\partial}{\partial t} H = \triangle H + H|A|^2$$

was proved in [1] (Lemma 3.5).
v): Using $H = k + (n-1)p$, we have

$$\frac{\partial}{\partial t} k = \frac{\partial}{\partial t} H - (n-1)\frac{\partial}{\partial t} p$$

then from iv) and vi) and $H = k + (n-1)p$,

$$= \triangle k + k|A|^2 - 2(n-1)q^2(k-p). \quad \blacksquare \tag{2.9}$$

2.5 Gradient estimates

Now assume that M_0 is a periodic surface, and that M_0 has positive mean curvature. Then applying the strong parabolic maximum principle to the evolution equation vi) for H (and using the fact that the surface stays periodic under the flow, and hence any maximum is an interior maximum), we see that $H \geq$ the minimum of its initial values. So H stays strictly greater than 0. From the previous section, we see that $y_x = -\frac{q}{p}$. We wish to show that y_x stays uniformly bounded. Also note that since M_0 lies beneath an infinite cylinder (see Chapter 6 on self-similar solutions), and since surfaces stay disjoint under the flow (by Chapter 1, Theorem 1.6), we see that M_0 must develop a singularity in finite time, since in finite time the cylinder collapses to the x_1-axis (see Chapter 6 on self similar solutions).

Lemma 2.2 *If $n = 2$, then $\exists$ a constant C_1 depending only on M_0 such that*

$$|y_x| = |\frac{q}{p}| \leq C_1 \tag{2.10}$$

independent of time.

proof: We calculate from [1] Lemma 5.2 the evolution equation for a general surface under the flow)

$$\frac{\partial}{\partial t} \leq \triangle(\frac{|A|^2}{H^2}) + \frac{2}{H}\nabla_i H \nabla_i(\frac{|A|^2}{H^2}) \tag{2.11}$$

Hence, by the strong parabolic maximum principle, $\frac{|A|^2}{H^2} \leq$ the maximum of it's initial values. Hence $|A|^2 \leq (C_2)^2 H^2$ where C_2 depends only on the initial surface (i.e. it is independent of time). Furthermore we calculate from the previous lemma that

$$\frac{\partial}{\partial t}(\frac{q}{H}) = \triangle(\frac{q}{H}) + \frac{2}{H}\nabla_i H \nabla_i(\frac{q}{H}) + \frac{q}{H}((n-1)p^2 + (n-3)q^2 - 2kp)$$

Assume $n = 2$. Then in view of 2.11, the last term is negative if $\frac{q}{H} \geq 2C_2$ and it is positive if $\frac{q}{H} \leq -2C_2$. Thus (by the maximum principle again)

$$|q| \leq C_3 H, \tag{2.12}$$

with a constant C_3 depending only on C_2 and the maximum of $\frac{q}{H}$ at time $t = 0$. Note this argument relies on the fact that for $n = 2$, and hence $(n-3) = -1 < 0$. We cannot use the same argument for $n \geq 3$. We will comment on this at the end of the proof. So keep assuming $n = 2$, and finally consider the evolution equation

$$\frac{\partial}{\partial t}\frac{k}{p} = \triangle\frac{k}{p} + \frac{2}{p}\nabla_i p \nabla_i \frac{k}{p} + 2\frac{q^2}{p^2}(p^2 - k^2),$$

which also follows from the previous lemma. Now the last term on the right hand side of the above equation is negative if $\frac{k}{p} \leq 1$, hence (by the maximum principle)

$$\frac{k}{p} \leq \max(1, \max_{M_0}(\frac{k}{p})) \tag{2.13}$$

So combining 2.12 and 2.13 we get

$$|q| \leq C_3 H = C_3(p+k) \leq C_4 p$$

as desired. ∎

Huisken then shows [2] that the upper blow-up estimate then follows from this lemma (for $n = 2$):

$$\max_{M_t} |A|^2 \leq C_5 \frac{1}{(T-t)} \forall t < T.$$

(See [2] for a discussion of the blow up estimates.) (Also see Chapter 6 on self similar solutions where it is noted that curvature blows up as $t \to T$, and where this blow up rate is discussed.)

Remark .

Using the maximum principle of Chapter 1, we can prove Lemma 2.2 under the weaker assumptions that

a) M_0 is an entire surface with polynomial growth,

b) $\inf_{M_0} H > 0$ on M_0,

c) the quantities $\frac{|A|^2}{H^2}$ $\frac{|q|}{H}$, $\frac{k}{p}$ are bounded on M_0.

Can we expect a gradient estimate for $n > 2$? Well, for $n > 2$ notice that we have $H = (n-1)p + k$, and so now, e.g. for n=3, we have twice as much curvature pulling the surface downwards. Perhaps this means that some sort of cusp like singularity could occur as the singularity occurs at time T on the x_1-axis. This an area that has not been investigated fully. In Chapter 8 on gradient estimates, the evolution equation for the gradient is calculated. It is shown there that for entire rotationally symmetric surfaces M_0 which initially have $C_0 = \sup M_0 |y_x| y < \infty$, then we have the gradient estimate $|y_x| \leq \frac{C_0}{y}$ while the solution exists. This is true for all $n \geq 2$. It seems possible that some sort of interior gradient estimate could be calculated (similar to that calculated by Ecker/Huisken in [4] where the local properties of surfaces evolving under the flow are investigated) using this global gradient estimate, by modifying the calculation by which it was derived.

Chapter 3

Interior estimates

3.1 Introduction

Here we wish to prove a couple of interior estimates for rotationally symmetric surfaces evolving under mean curvature flow. The first theorem is an interior height estimate, derived by placing a ball above the initial surface and using the fact that surfaces stay disjoint under the flow.

The next theorem is an interior gradient estimate for rotationally symmetric surfaces, derived using the interior gradient estimate for graphs calculated by Ecker/Huisken in [4]. The interior gradient estimate derived here should be compared with the gradient estimate calculated in Chapter 8, Lemma 8.4, and the gradient estimate of Chapter 2, Lemma 2.2.

Both of these estimates are crucial to the main existence heorem 4.3 of Chapter 4.

3.2 Interior height estimates

Theorem 3.1 *Let M_0 be an entire rotationally symmetric surface generated by the function $y_0 : \mathbf{R} \to \mathbf{R}$.*

Let

$$h = \sup_{[a,b]} y_0$$

Then for $\beta > 0, \beta < \frac{|a-b|}{2}$, we have

$$\sup_{[a+\beta,b-\beta]\times[0,\frac{\beta^2}{2n}]} y \leq h + \beta$$

where y is the generator function of the surface M_t.

proof: Choose some $\epsilon > 0$ such that $\beta > \epsilon$. Choose some $x_1 \in [a+\beta, b-\beta]$. Place an n-dimensional sphere $S^n_{R_0}$ = of radius $R_0 = \beta - \epsilon$ and centre $(x_1, 0, \ldots, 0, h+\beta)$ above the point $p = (x_1, 0, \ldots, 0, y_0(x_1))$ on the surface M_0 = rotationally symmetric surface generated by y_0. Then note that the sphere S_{R_0} is initially disjoint from the rotationally symmetric surface M_0 generated by y_0. By Theorem 1.6 of Chapter 1, surfaces stay disjoint under the flow. Hence M_t and $B_{R(t)}$ remain disjoint under the flow, while solutions to both flow problems exist. The sphere collapses at time $T = \frac{R_0^2}{2n}$, so while $t \leq \frac{R_0^2}{2n}$ we have $y(x_1) \leq h + \beta$ (since the centre of the sphere has height $h+\beta$). Letting $\epsilon \to 0$, we see that $y(x_1) \leq h + \beta$ for $t \leq \frac{\beta^2}{2n}$. Since x_1 was an arbitrary point $x_1 \in [a+\beta, b-\beta]$ we are done. ■

In Chapter 4 we need an interior height estimate of a slightly different form. The required height estimate is an easy corollary of the above theorem:

Corollary 3.2 *Let y_0 be a rotationally symmetric surface with boundary generated by the function $y_0|_{[\hat{a},\hat{b}]}$, where $\hat{a} \leq a$, $\hat{b} \geq b$. Assume that the surface evolves under mean curvature flow in such a way that points p on the boundary on the left satisfy the condition $\langle p, \iota_1 \rangle \leq a$ for all time, and the right hand boundary satisfies $\langle p, \iota_1 \rangle \geq b$ for all time. Let*

$$h = \sup_{[a,b]} y_0$$

Then for $\beta > 0, \beta < \frac{|a-b|}{2}$, we have

$$\sup_{[a+\beta, b-\beta] \times [0, \frac{\beta^2}{2n}]} y \leq h + \beta$$

where y is the generator function of the surface M_t.

proof: The proof is the same as the proof for the above theorem. ■

Now we prove our interior gradient estimate for rotationally symmetric surfaces evolving under the flow.

3.3 Interior gradient estimates

Theorem 3.3 *Assume $y > \sigma$ on the x_1 - interval $[a,b]$ and the time interval $[0,T]$, for some $\sigma > 0$. Then we have the interior gradient estimate:*

$$\begin{aligned} |y_x(x_1,t)| &\leq C_1(n)(n + \frac{2}{\sigma} \sup_{[x_1-R_0, x_1+R_0]} |(y_0)_x| \cdot \sup_{[x_1-R_0, x_1+R_0]} (y_0)) \\ &\quad \exp[C_2(n) R_0^{-2} \sup_{[x_0-R_0-\sqrt{T2n}, x_0+R_0+\sqrt{T2n}] \times [0,T]} (y_0^2 + T2n)] \end{aligned}$$

where $R_0 = \frac{\sigma}{2}$.

proof: Choose an n-dimensional ball $B^n_{R_0}(x_0)$ of small radius R_0 sitting in the $(x_1,\ldots,x_n)$-plane, with centre $\vec{x_0} = (x_1, 0, \ldots, 0)$ (remember the rotationally symmetric surface M_0 is an n-dimensional surface sitting in $\mathbf{R}^{n+1}$). If we choose $R_0 = \frac{\sigma}{2}$, then for any given $t \in [0,T]$ the portion of the surface M_t which lies above B^n covers the whole of B^n (when projected onto B^n) and hence can be written as the graph of a function $u(\cdot,t) : B^n \to \mathbf{R}$. u is defined by $u(x_1,\ldots,x_n) = \sqrt{y(x_1)^2 - x_2^2 - \ldots - x_n^2}$.
Hence

$$D_1 u(x_1,\ldots,x_n) = \frac{y(x_1)y_x(x_1)}{\sqrt{y(x_1)^2 - (x_2)^2 - \ldots - (x_n)^2}}$$

and

$$D_i u(x_1,\ldots,x_n) = \frac{-x_i}{\sqrt{y(x_1)^2 - x_2^2 - \ldots - x_n^2}} \text{ for } 2 \le i \le n.$$

In particular

$$D_1 u(x_1, 0, \ldots, 0) = y_x(x_1),$$

and

$$D_i u(x_1, 0, \ldots, 0) = 0.$$

Now the gradient estimate for graphs derived by Ecker/Huisken in [4] (Theorem 2.3) is

$$\begin{aligned}\sqrt{1 + |Du(x_0,t)|^2} \;\le\; & C_1(n) \sup_{B_{R_0}(x_0)} \sqrt{1+|Du_0|^2} \\ & \exp[C_2(n) R_0^{-2} (\sup_{B_{R_0}(x_0)\times[0,T]} (u(x,t) - u(x_0,t)))^2]\end{aligned}$$

Put

$$c_0 = C_1(n) \sup_{B_{R_0}(x_0)} \sqrt{1+|Du_0|^2}.$$

Then since $\vec{x_0} = (x_1, 0, \ldots, 0)$ we have the estimate

$$|y_x(x_1,t)| \le c_0 \exp[C_2(n) R_0^{-2} (\sup_{B_{R_0}(x_0)\times[0,T]} (u(x,t) - u(x_0,t)))^2]$$

Remembering that $R_0 = \frac{\sigma}{2}$, we have

$$\begin{aligned}\frac{1}{\sqrt{y^2 - (x_2)^2 - \cdots - (x_n)^2}} &\le \frac{1}{\sigma^2 - (\frac{\sigma}{2})^2} \\ &\le \frac{2}{\sigma}\end{aligned}$$

Hence

$$\sup_{B_{R_0}} |D_i u_0| \le 1$$

for $i = 2, \ldots, n$, and

$$\sup_{B_{R_0}} |D_1 u_0| \le \frac{2}{\sigma} \sup_{[x_1 - R_0, x_1 + R_0]} |(y_0)_x| \cdot \sup_{[x_1 - R_0, x_1 + R_0]} (y_0)$$

Hence

$$c_0 \le C_1(n)(n + \frac{2}{\sigma} \sup_{[x_1 - R_0, x_1 + R_0]} |(y_0)_x| \cdot \sup_{[x_1 - R_0, x_1 + R_0]} (y_0)) \tag{3.1}$$

Now

$$\begin{aligned}(\sup_{B_{R_0}(x_0)\times[0,T]} (u(x,t) - u(x_0,t)))^2 &\le 2 \sup_{[x_0 - R_0, x_0 + R_0]\times[0,T]} y(\cdot,t)^2 \\ &\le \sup_{[x_0 - R_0 - \sqrt{T2n}, x_0 + R_0 + \sqrt{T2n}]\times[0,T]} y_0^2 + T2n\end{aligned}$$

by the interior height estimate of the previous theorem. Hence

$$\begin{aligned}|y_x(x_1,t)| &\le C_1(n)(n + \frac{2}{\sigma} \sup_{[x_1 - R_0, x_1 + R_0]} |(y_0)_x| \cdot \sup_{[x_1 - R_0, x_1 + R_0]} (y_0)) \\ &\quad \exp[C_2(n) R_0^{-2} \sup_{[x_0 - R_0 - \sqrt{T2n}, x_0 + R_0 + \sqrt{T2n}]\times[0,T]} (y_0^2 + T2n)]\end{aligned}$$

as required.

Chapter 4

Short time existence

4.1 Introduction

In this chapter we derive the main result of this essay. That is: if M_0 is a rotationally symmetric surface generated by a function $y_0 : \mathbf{R} \to \mathbf{R}$, then under the assumption that $\inf_{\mathbf{R}} y_0$ is bounded away from 0, we have solutions M_t generated by some $y(\cdot, t)$ for as long as $\inf_{\mathbf{R}} y(\cdot, t)$ is bounded away from 0. This is a better result than that which can usually be obtained by parabolic theory, since here we make no assumptions about the growth near ∞ (usually such existence results obtained via parabolic theory do require some sort of growth assumption near ∞). The catenoid barriers used here were a suggestion of Huisken.

4.2 Short time existence

Let M_0 be an entire rotationally symmetric surface generated by a positive function $y_0 : \mathbf{R} \to \mathbf{R}$. We wish to show that there exist solutions M_t to the flow problem for some maximal short time interval $[0, T)$.

Assume that $y_0 \in C^{2,\alpha}(\mathbf{R})$ and that $\inf_{\mathbf{R}} y_0 > \delta$, for some $\delta > 0$, and some $\alpha > 0$.

Firstly let us restrict our attention to the x_1 -interval $[-m, m]$. Remember the parabolic evolution equation for a rotationally symmetric surface is

$$y_t = \frac{y_{xx}}{1 + (y_x)^2} - \frac{(n-1)}{y} \tag{4.1}$$

(as derived in Chapter 2 on rotationally symmetric surfaces). We look for a solution $y^m(x, t)$ to this equation, defined on the x_1-interval $[-m, m]$ and some short time interval

$[0, T^m]$, for some $T^m > 0$. Then M_t^m = rotationally symmetric surface generated by $y^m(\cdot, t)$ is a solution to the flow on $[-m, m] \times [0, T^m]$

Theorem 4.1 **Assume** that $y_0 \in C^{2,\alpha}(\mathbf{R})$, and that $\min_{[-m,m]} y_0 > \delta$, for some $\delta > 0$, some $\alpha > 0$.
Then equation 4.1 has a unique solution $y = y^m \in C^{2,\alpha'}([-m, m] \times [0, T^m])$ for some $0 < \alpha' < \alpha$, and some $T^m > 0$, satisfying the boundary conditions

$$y^m(m, t) = y_0^m(m), \tag{4.2}$$

$$y^m(-m, t) = y_0^m(-m) \tag{4.3}$$

and initial conditions

$$y(\cdot, 0) = y_0 \text{ on } [-m, m] \tag{4.4}$$

.

proof: The proof follows from the Theory of parabolic equations. e.g. See [10] Theorem 2.3.0. Modifying the proof there to suite our particular evolution equation 4.1 we are done. ∎
In fact

Proposition 4.2 *We can choose a T such that each solution y^m of the above theorem is defined on some time interval $[0, T]$, for some $T > 0$ where T is independent of m*

proof: Initially assume $n = 2$. We will prove this theorem by constructing barrier surfaces B_0,C_0 and D_{m_0} as follows:
First let $[0, T^m)$ be the maximal time interval for which we have a solution y^m to 4.1 ,satisfying the specified boundary (4.2,4.3) and initial conditions (4.4).

Let $g^{l,\lambda} : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$g^{l,\lambda}(x) = \lambda \cosh((\frac{x - l}{\lambda}))$$

for some $l \in \mathbf{R}, \lambda > 0$. Let $G^{l,\lambda}$ be the rotationally symmetric surface generated by the function $g^{l,\lambda}$. $H = 0$ everywhere on $G^{l,\lambda}$ as can be seen by appendix 4. Now if we choose λ very small, the surface will become steep very quickly as we move away from the base-point $x_1 = l$, and it's height at the base point $x_1 = l$ will be very small: the height at the base point l is $h = \lambda \cosh((n - 1)(\frac{l-l}{\lambda})) = \lambda \cosh(0) = \lambda$.
To construct the barrier B_0, we are going to cut off a portion of the surface $G^{l,\lambda}$ for some $l \in \mathbf{R}$, some $\lambda > 0$.
Likewise we will construct the barrier C_0 by cutting off a portion of the surface $G^{l',\lambda'}$ for

some $l' \in \mathbf{R}$, some $\lambda' > 0$.
Our third barrier D_{r_0} will be the infinite cylinder of radius $r_0 = \frac{\delta}{2}$.
Now we go ahead and construct these barriers rigorously.
Let D_{r_0} be the 2 dimensional infinite cylinder of radius $r_0 = \frac{\delta}{2}$. (i.e. the 2-dimensional rotationally symmetric surface generated by the constant function $y = \frac{\delta}{2}$.) Evolving D_{r_0} under the flow, we see that at time t the surface is $D_{r(t)}$, where $D_{r(t)}$ is the infinite cylinder of radius $r(t) = \sqrt{r_0^2 - 2t}$ ($D_{r(t)}$ collapses to the x_1 axis at time $t = \frac{r_0^2}{2)}$). Now initially M_0^m (= the rotationally symmetric surface generated by $y_0|_{[-m,m]}$) is disjoint from the surface D_{r_0}, since by assumption $inf_{\mathbf{R}} y_0 = \delta > 0$, and we chose $r_0 = \frac{\delta}{2}$. Thus, since surfaces stay disjoint under the flow (by Theorem 1.6 of Chapter 1), we see that the surface M_t^m (= rotationally symmetric surface generated by $y^m(\cdot, t)$) will stay disjoint from $D_{r(t)}$ under the flow, while both solutions M_t^m and $D_{r(t)}$ exist. (Note: they cannot touch at the boundary since M_t^m has fixed boundary under the flow, and $D_{r(t)}$ is a shrinking towards the x_1 axis.)

Let $t_1 = \frac{3(r_0)^2}{8}$. If $T^m \geq t_1 \ \forall m$ then we are done. So assume $t_1 > T^m$ for some m. Fix this m. Let $\sigma = \frac{r_0}{2}$. Then on the time interval $[0, t_1]$ we have

$$y^m \geq \sigma, \tag{4.5}$$

since at time $t_1 = \frac{3(r_0)^2}{8}$, we have

$$r(t_1) = \sqrt{(r_0)^2 - \frac{3(r_0)^2}{4}} = \frac{r_0}{2},$$

and $y^m(\cdot, t) \geq r(t_1)$ on the interval $[0, t_1]$. (The last statement is true since
a) the surfaces M_t^m and $D_{r(t)}$ stay disjoint under the flow, and hence
b) $y^m(\cdot, t) \geq r(t)$, and hence
c) $y^m(\cdot, t) \geq r(t_1)$ for $t \leq t_1$ since $r(t)$ is strictly decreasing.)

Now we construct B_0 and C_0.
First choose λ so that $0 < \lambda < \sigma$. Let $x^{\lambda,\sigma}$ be the unique point in $\mathbf{R}^-$ such that

$$g^{\lambda,0}(x^{\lambda,\sigma}) = \sigma$$

Such an $x^{\lambda,\sigma}$ exists since

$$g^{\lambda,0}(x) = \lambda \cosh(\frac{x}{\lambda})$$

grows exponentially as we go from $0 \to -\infty$, and at 0,

$$g^{\lambda,0}(0) = \lambda < \sigma$$

The derivative of g is

$$g_x^{\lambda,0}(x) = \sinh(\frac{x}{\lambda})$$

Hence, choosing $x \leq x^{\lambda,\sigma}$ (remember $x^{\lambda,\sigma} \leq 0$) we have

$$\begin{aligned}|g_x^{\lambda,0}(x)| &= |\sinh(\frac{x}{\lambda})| \\ &= \sqrt{\cosh^2(\frac{x}{\lambda}) - 1} \\ &\geq \sqrt{\cosh^2(\frac{x^{\lambda,\sigma}}{\lambda}) - 1} \\ &= \sqrt{\frac{\sigma^2}{\lambda^2} - 1}\end{aligned}$$

Hence choosing $\lambda > 0$ small enough we have

$$|g_x^{\lambda,0}(x)| \geq \sup_{[-m,m]} |(y_0)_x| \tag{4.6}$$

$\forall x \leq x^{\lambda,\sigma}$. Now let $x^{\lambda,y_0(-m)}$ be the point on the x_1 axis such that

$$g^{\lambda,0}(x^{\lambda,y_0(-m)}) = y_0(-m).$$

Such an $x^{\lambda,y_0(-m)}$ exists since

$$g^{\lambda,0}(0) = \lambda < \sigma < \inf_{\mathbf{R}} y_0$$

and $g^{\lambda,0}$ grows exponentially as we go from $0 \to -\infty$. Clearly $x^{\lambda,y_0(-m)} < x^{\lambda,\sigma}$ Now we are in a position to choose l. Choose

$$l = -x^{\lambda,y_0(-m)} - m$$

Then $\frac{(-m-l)}{\lambda} = \frac{x^{\lambda,y_0(-m)}}{\lambda}$, and so

$$g^{\lambda,l}(-m) = \lambda\cosh(\frac{(-m-l)}{\lambda}) \tag{4.7}$$

$$= \lambda\cosh(\frac{x^{\lambda,y_0(-m)}}{\lambda}) \tag{4.8}$$

$$= g^{\lambda,0}(x^{\lambda,y_0(-m)}) \tag{4.9}$$

$$= y_0(-m) \tag{4.10}$$

Now we choose our barrier B_0 to be the rotationally symmetric surface generated by the function $g^{\lambda,l}$ on the interval $[x^{\lambda,\sigma}, -\infty)$. Note that

a) this generator function $g^{\lambda,l}|_{[x^{\lambda,\delta},-\infty)}$ has gradient everywhere negative and less than $-\sup_{[-m,m]}|(y_0)_x|$ by 4.6, and
b) it has mean curvature 0 everywhere, and
c) it is disjoint from the rotationally symmetric surface M_0^m generated by $y_0|[-m,m]$, except at the left hand side boundary of M_0^m where the two surfaces meet.
Hence by Theorem 1.6 of Chapter 1, the surface M_0^m stays disjoint from B_0 on the time interval $[0,T^m)$ under the flow, since M_0^m has fixed boundary under the flow and $\inf_{[-m,m]} y^m > \sigma$ on $[0,T^m)$. Using the barrier B_0 we see that $|y_x^m(-m)|$ is bounded by $|g_x^{\lambda,r}(-m)| \leq k(m) < \infty$ on any time interval that a solution y^m exists.

Similarly we construct a barrier $G^{\lambda',r'}$ so that $|y_x^m(m)|$ is bounded by $|g_x^{\lambda',r'}(m)| \leq k(m) < \infty$ on any time interval that a solution y^m exists. (See diagram 4.1.)

Now, by the flow equation 4.1, we see that y_x^msatisfies the equation

$$y_{xt}^m = y_{tx}^m = \frac{(y_x^m)_{xx}}{1+(y_x^m)^2} - \frac{2(y_{xx}^m)^2 y_x^m}{(1+(y_x^m)^2)^2} + \frac{y_x^m}{(y^m)^2} \tag{4.11}$$

$\Rightarrow$ (letting $\psi = y_x^m e^{\frac{-t}{\sigma^2}}$)

$$\psi_t = a\cdot\psi_{xx} + b\cdot\psi_x + c\cdot\psi \tag{4.12}$$

where

$$a = \frac{1}{1+(y_x^m)^2},$$
$$b = \frac{-2y_{xx}^m y_x^m}{(1+(y_x^m)^2)^2},$$
$$c = \frac{1}{(y^m)^2} - \frac{1}{(\sigma)^2} \leq 0$$

(since $\frac{1}{(y^m)^2} - \frac{1}{\sigma^2} \leq 0$ on $[0,T^m)$ by 4.5). Now equation 4.12 is a uniformly parabolic equation, and $c \leq 0$, so we can apply the parabolic maximum principle to it to obtain the inequality

$$\begin{aligned} \sup_{[-m,m]\times[0,T^m)} |y_x| &\leq e^{\frac{T^m}{\sigma^2}} \sup_P |g| && (4.13)\\ &\leq C(T)\sup_P |y_x^m| && (4.14)\\ &\leq C(T)k(m) && (4.15)\end{aligned}$$

(where P is the parabolic boundary: $P = \{-n\}\times[0,T^m)\cup\{n\}\times[0,T^m)\cup[-m,m]\times\{0\}$)
The last inequality follows since at m and $-m$ the barriers B_0 and C_0 stop the gradient of y getting worse than k(m), and also, $k(m)$ satisfies $k(m) > \sup_{[-m,m]}|y_x|$ by construction. So the equation 4.1 is strictly parabolic on $[-m,m]\times[0,T^m)$, so by the theory of parabolic

equations, we see that y^m can be extended to the time interval $[0, T^m + \sigma)$ for some $\sigma > 0$. (e.g. See [8].) But $[0, T^m)$ maximal, so we have a contradiction. Hence $T^m \geq t_1 \; \forall m$.
For $n > 2$ there is strong numerical evidence that a rotationally symmetric catenoid like surface exists, which becomes vertical if we move a finite distance from the origin in the x_1 direction. By rescaling this surface (in the same way that we rescaled our surfaces $G^{\lambda,l}$, by varying λ) we can use it as the barrier B_0 (likewise for C_0). The same argument then follows through for $n > 2$.

Having obtained a common time interval for which the surfaces exist, we have done most of the hard work.

Now we show that there is a solution y to 4.1, defined on the whole of $\mathbf{R}$ for some short maximal time interval $[0, T)$, which satisfies the initial condition $y(\cdot, 0) = y_0$. We do this via the Arzela-Ascoli theorem. This result is the main result of this essay.

Theorem 4.3 $\exists$ *a solution* y *to 4.1 (defined on the whole of* $\mathbf{R}$*) for some maximal short time interval* $[0, S)$*, and* $\inf_{\mathbf{R}} y(\cdot, t) \to 0$ *as* $t \to S$ *if* $S < \infty$*.*

proof: Let y^m be the y^m of Theorem 4.1. Extend each y^m in a $C^{2,\alpha}$ way to the whole of $\mathbf{R}$.
Let k be a fixed integer, $k > 8\sqrt{T2n}$, where T is the T of the above theorem. We will be concerned with the compact interval [-k,k].

Let $d = \frac{k}{8}$. From Chapter 3 Theorem 3.2 we see that

$$\sup_{[-k+d, k-d] \times [0, \frac{d^2}{2n}]} y^m \leq C_0(\sup_{[-k,k]} y_0, d).$$

Note: $\frac{d^2}{2n} \geq T$, by choice of d, and stipulation on k. Also from that chapter, we have the interior gradient estimate:

$$|y^m_x(x_1, t)| \leq C_1(n, \sigma, R_0 = \frac{\sigma}{2}, T, \sup_{[k-R_0, k+R_0]} |(y_0)_x)|, \sup_{[k-R_0, k+R_0]} (y_0))$$

on the interval $[-k, k] \times [0, T]$. Then using the evolution equation for y^m:

$$y^m_t = \frac{y^m_{xx}}{1 + (y^m_x)^2} - \frac{(n-1)}{y^m},$$

and remembering that $y^m \geq \sigma = \frac{r_0}{2} = \frac{\delta}{4}$ on the time interval $[0, T]$, we see that y^m is strictly parabolic on the interval $[-k, k]$, with ellipticity constant $\lambda = \min_{[-k,k] \times [0,T)} \frac{1}{1+(y^m_x)^2}$ $\geq \frac{1}{1+C_1^2} > 0$ independent of m and $t \in [0, T]$. Hence by the standard induction argument on i in $C^{i,\alpha}$ we have that y^m is $C^\infty([-k, k] \times [0, T])$. Now since we have a uniform height estimate on $[-k+d, k-d]$ for each y^m independent of m, we can use parabolic theory (e.g. see [8]) to obtain the interior estimates:

$$\|y^m\|_{C^{2,\alpha}([-k+2d,k-2d]\times[0,T])} \leq C_2(C_0, \sigma, n, \epsilon, \lambda, T)$$

The Schauder theory (see [8]) then gives us

$$\|y^m\|_{C^{j,\alpha}([-k+2d,k-2d]\times[0,T])} \leq C(C_0, \sigma, n, \epsilon, \lambda, T, j)$$

Hence y^m is a bounded set (for example) in $C^4([-k+2d, m-2d] \times [0,T])$. Hence, by the Arzela-Ascoli theorem (e.g. see [11] Lemma 6.36), $\exists$ a subsequence $y^{m\prime}$ of y^m such that $y^{m\prime}$ converges in $C^3([-k+2d, k-2d] \times [0,T])$ to some function y say. Then since the evolution equation for $y^{m\prime}$ contains terms of at most order 2, we see, letting $m\prime \to \infty$, that y satisfies the same evolution equation as y^m on $[-k+2d, k-2d] \times [0,T]$. Clearly y is $C^\infty([-k+2d, k-2d] \times [0,T])$. Now let us consider the interval $[-2k, 2k]$. Repeating this argument, only using $y^{m\prime}$ in place of y^m, we obtain a subsequence $y^{m\prime\prime}$ which converges to some y in $C^3([-2k+2d, 2k-2d] \times [0,T])$. y agrees with the previous y on $[-k+2d, k-2d] \times [0,T])$, and satisfies the evolution equation 4.1.

Continue in this way out to ∞. Now take a diagonal subsequence $\vec{a_i}$ given by: $\alpha 1 = y^{1\prime}$, $\alpha 2 = y^{2\prime\prime}$, ... Then clearly the function $y = \lim_{m\to\infty} \vec{a_m}$ satisfies the flow equation and satisfies the initial conditions $y(\cdot, 0) = y_0$. The flow lives for at least $[0,T]$. Once the flow has begun it is analytic, and so there is always only one possible extension to the flow. Hence the flow lives for some maximal time interval $[0,S)$. Note that we can always extend the flow for some short time interval past a time t if $\inf_{\mathbf{R}} y(\cdot,t) > 0$, by just repeating the argument we have used here. So the flow lives always while it is bounded away from 0. So we are done. ∎

Chapter 5

Pinching

5.1 Introduction

We say a barbell like surface "pinches" off under mean curvature flow, if, at some finite time, the surface pinches off somewhere along it's neck, leaving a sphere-like shape on either side of it's neck when it does so. (See diagram 0.4.) In this chapter we present geometrical criteria that ensure that an initial surface satisfying these criteria will pinch off in finite time under the flow.

First we construct a barrier surface (suggested by Ecker) which we know will pinch off in finite time (under mean curvature flow). Then we see that a barbell-like surface which can be placed inside this barrier surface will also pinch off in finite time as long as it has large enough bubbles on either side of its neck.

5.2 Pinching criteria

First we construct our barrier surface M_0^*. Let M_0^* be the rotationally symmetric surface given by:

$M_0^* = \{\vec{x} \in \mathbf{R}^{n+1} : (x_2)^2 + \cdots + (x_{n+1})^2 = \epsilon + (x_1)^2(n-1-\beta)\}$ for some $(n-1) \geq \beta > 0$.
Then $\epsilon = |\vec{x}|^2 - (n-\beta)(x_1)^2$ on M_0^*.
Now

$$\begin{aligned}&(\frac{\partial}{\partial t} - \triangle)(|\vec{x}|^2 - (n-\beta)(\vec{x}_1)^2 + 2\beta t)\\ &= (\frac{\partial}{\partial t} - \triangle)(|\vec{x}|^2) - (\frac{\partial}{\partial t} - \triangle)(n-\beta)(\vec{x}_1)^2 + (\frac{\partial}{\partial t} - \triangle)2\beta t)\end{aligned}$$

(see appendix 3: $(\frac{\partial}{\partial t} - \triangle)(|\vec{x}|^2) = -2n$ and $(\frac{\partial}{\partial t} - \triangle)\vec{x} = 0$)

$$= -2n - (n-\beta)(2x_1(\frac{\partial}{\partial t} - \triangle)x_1 - 2|\nabla x_1|^2) + 2\beta$$
$$= -2n + 2(n-\beta)|\nabla x_1|^2 + 2\beta \tag{5.1}$$

Also $|\nabla x_1|^2 \leq 1$ (from appendix 3), so $5.1 \leq 0$.
Hence, (by the monotonicity formula of Chapter 1), on M_t^* we have

$$\begin{aligned} |\vec{x}|^2 - (n-\beta)|x_1|^2 + 2\beta t &\leq sup_{M_0^*}(|\vec{x}|^2 - (n-\beta)|x_1|^2 \\ &= \epsilon \end{aligned}$$

In particular at $x_2 = \cdots = x_{n+1} = 0$, we have $x_1^2 \leq \epsilon - 2\beta t$, so that $(n-\beta)x_1^2 \leq 0$ at $t = \frac{\epsilon}{2\beta}$, and hence the surface M_0^* must have pinched off by $t = \frac{\epsilon}{2\beta}$. (Note: In the chapter on short time existence, it is shown that the only way a rotationally symmetric surface can develop a singularity under the flow is if it pinches off. i.e. $inf_{\mathbf{R}} \to 0$ as $t \to T$, where T is the blow up time.) Now let M_0 be an initial closed surface, and let V_t = volume of solid contained in M_t, V_t^* = volume of solid contained in M_t^*. Assume $M_0 \subset int(V_0^*)$. Then if we can put a ball B_{R_0} of radius R_0 on either side of the neck of M_0^* such that each $B_{R_0} \subset int(V_0)$ and such that $\frac{R_0^2}{2} > \frac{\epsilon}{2\beta}$, then M_0 will pinch off in finite time under the flow. This can be seen as follows: First note that surfaces stay disjoint under the flow (see Chapter 1 Theorem 1.6). So each

$$B_{R(t)} \subset int(V_t) \subset int(V_t^*) \tag{5.2}$$

But M_t^* pinches off at a finite time $T \leq \frac{\epsilon}{2\beta}$. So M_t must develop a singularity at a time $S \leq T \leq \frac{\epsilon}{2\beta}$ by 5.2. But the balls blow up at a time $\frac{R_0^2}{2n} > \frac{\epsilon}{2\beta}$, so a portion of each ball must still be alive at time S. i.e. M_t has **not** shrunk to a point at time S. If the surface M_0 is also

a) {smooth, embedded, simply connected } (e.g. a rotationally symmetric surface)

then it seems almost certain that the singularity that occurs at time S is due to "pinching". i.e. I am conjecturing that generally, for initial surfaces M_0 satisfying condition (a), the only singularities that can occur under the flow are due to pinching **or** the surface shrinking to a point, and since we have excluded the last possibility for the M_0 we are presently considering, M_0 must have pinched off at time S.

Grayson [7] uses a similar argument: Choose $r > 0$, $l > \frac{\pi r}{2}$. Let ∂D be the 2-dimensional rotationally symmetric surface generated by

$$f(x) = \begin{cases} r\cosh(\frac{(|x|-l)}{r}) & , if |x| > l \\ r & , otherwise \end{cases}$$

after mollifying at the point x = l. Then any smooth surface $M_0 \subset D$ (such that M_0 has bounded height - i.e. $|x_3|$ is bounded) which contains a ball of radius R_0 (such that $\frac{R_0^2}{4} > \frac{2lr^2}{2l-\pi r}$) on either side of the neck will develop a singularity in finite time and it will not have shrunk to a point at that time

Chapter 6

Self-similar solutions

6.1 Introduction

In this chapter we wish to examine what sort of similarity a surface might maintain under the flow. We examine a number of the different ways a solution M_t to the flow might be "self - similar" and present examples of such "self-similar" flows were appropriate. In the last part of the chapter we define a very special kind of "self-similar" flow and present there some interesting properties and examples of such flows.

6.2 Self-similar flows in general

Let M_0 be an initial surface. Assume for the moment that M_0 is compact. Let us consider the two cases of short time existence and long time existence separately.

short time existence:

M_t exists for some short maximal time interval $[0, T)$. It is shown in [2] that the curvature blows up as $t \to T$. i.e.

$$\max_{M_t} |A|^2 \to \infty$$

as $t \to T$. So how fast is $|A|^2$ blowing up? Well in [2] it is shown that $U(t) = \max_{M_t} |A|^2$ is Lipschitz continuous, and

$$U(t) \geq \frac{1}{2(T-t)} \tag{6.1}$$

So we have a lower bound for this blow-up rate. As we mentioned before at the end of Chapter 2, in [2] it is shown that for $n = 2$, periodic rotationally symmetric surfaces satisfy the blow up rate

$$U(t) \leq \frac{C_0}{2(T-t)}$$

So in this case we have an upper and lower bound for the blow up rate. This blow up rate is also the blow up rate for convex surfaces. Now, in order to see what sort of singularity occurs at $t = T$, it is first necessary to locate a point in $\mathbf{R}^{n+1}$ where a singularity occurs at $t = T$ (there could be many such points) - call such a point a blow up point. Then we translate the surface so that, without loss of generality, $0 \in \mathbf{R}^{n+1}$ is this blow up point. Then at each time t we wish to magnify the surface (around this blow up point 0) in a way that will ensure that the resulting magnified surfaces converge to some limiting hypersurface as $t \to T$. Well we now make this method rigorous. An obvious candidate for a blow up point would be:

Definition 6.1 *We say that $\vec{x_0} \in \mathbf{R}^{n+1}$ is a blow-up point, if there is $p \in M^n$ such that $\vec{F}(p,t) \to \vec{x_0}$ as $t \to T$ and $|A|(p,t)$ becomes unbounded as $t \to T$.*

But do such $\vec{x_0}$ exist? Well, note that if we assume the blow up rate of the curvature also satisfies the upper bound

$$U(t) = \max_{M_t} |A|^2 \leq \frac{C_0}{2(T-t)} \tag{6.2}$$

(from now on assume that this blow up estimate is satisfied for the solutions M_t we are considering) then

$$\begin{aligned} |\vec{F}(p,t) - \vec{F}(p,s)| &\leq \int_s^t |\vec{H}(p,\tau)| d\tau \\ &\leq C[(T-s)^{1/2} - (T-t)^{1/2}] \forall p \in M^n, 0 \leq s < t < T \end{aligned}$$

Thus $\vec{F}(\cdot,t)$ converges uniformly as $t \to T$, and so such $\vec{x_0}$ do exist (else U(t) would be bounded by a constant independent of t, contradicting the lower bound 6.1) and so our definition is a good one.

Now we will rescale our surfaces so that the resulting surfaces have the property that $|A|^2$ doesn't blow up, and this natural choice of rescaling turns out to be the right one. i.e. we define the rescaled immersions

$$\tilde{\vec{F}}(p,s) = (2(T-t))^{-1/2} \vec{F}(p,t),$$

where $s(t) = -\frac{1}{2}\log(T-t)$ is a new time parameter. The resulting rescaled surfaces $\tilde{M}_s = \tilde{\vec{F}}(\cdot,s)(M^n)$ are then defined for $-\frac{1}{2}\log T \leq s < \infty$ and satisfy the equation

$$\frac{\partial}{\partial s}\tilde{\vec{F}}(p,s) = \tilde{\vec{H}}(p,s) + \tilde{\vec{F}}(p,s),$$

where $\tilde{\vec{H}}$ is the mean curvature vector of $\tilde{M}_s$. (This choice gives $|\tilde{A}(\cdot,t)|^2 \leq C_0$, independent of time t, where $\tilde{A}(\cdot,t)$ is the curvature on $\tilde{M}_t$.)

Huisken then shows that for a given sequence $s_j \to \infty$, $\exists$ a sub-sequence s_{jk} such that $\tilde{M}_{s_{jk}}$ converges smoothly to an immersed non-empty limiting surface $\tilde{M}_\infty$, and that each limiting hypersurface $\tilde{M}_\infty$ obtained in this way satisfies the equation

$$\tilde{\vec{H}} = \langle \vec{X}, \vec{\nu} \rangle \tag{6.3}$$

Natural open questions are the uniqueness of $\tilde{M}_\infty$, and the number of solutions to 6.3. Using the fact that periodic surfaces with positive mean curvature satisfy the blow up rate 6.2, we may use the argument sketched here to obtain a limiting hypersurface $\tilde{M}_\infty$. Huisken then shows that an entire rotationally symmetric surface satisfying 6.3 must be an infinite cylinder of radius 1 (see end of this chapter where the infinite cylinder of radius r is introduced). It is also shown in [2] that if M is a compact (without boundary) surface with positive mean curvature, which satisfies 6.3 then M is a sphere of radius $\sqrt{n}$.

Long time Existence We have solutions M_t for $t \in [0, \infty)$. There are a number of things to look for as $t \to \infty$. Firstly does M_t converge to a limiting hypersurface M_∞ as $t \to \infty$. If not, can we follow the procedure given for the short time existence case to show that rescaled surfaces $\tilde{M}_t$ converge to some $\tilde{M}_\infty$ as $t \to \infty$? If not let us look for other classes of self-similar solutions -
i) **translating self similar solutions**: There are symmetrical surfaces which move via translation under the flow. e.g. see diagram 6.1.
ii) **rotating self similar solutions** There is a family of curves M_0 sitting in the plane such that M_t is a rotation of M_0 ([3]). e.g. see diagram 6.2

6.3 A special self-similar flow

Here we will consider the particular class of self-similar solutions given by the following definition.

Definition 6.2 *Given an initial surface $M_n^0 \subset \mathbf{R}^{n+1}$, we say that M_0 has a "self-similar solution" to the mean curvature flow problem, or that M_0 is "self-similar",* **if** *the surfaces M_t we obtain under the flow satisfy $M_t = c(t)M_0$, for some $c : [0, T) \to \mathbf{R}$.*

e.g. $M_0^n = S_{R_0}^n(0)$ - the n-dim sphere sitting in $\mathbf{R}^{n+1}$ with centre 0. Then M_0^n has a self-similar solution as shown in Example 1.1, Chapter 1 with

$$\begin{aligned} c(t) &= \sqrt{1 - \frac{2n}{R_0^2}t} \\ &= \sqrt{1 - kt} \end{aligned}$$

where $k = \frac{2n}{R_0^2} = \frac{2H(p_0)}{\langle p_0, \vec{\nu}(p_0)\rangle_{\mathbf{R}^{n+1}}}$, where $H(p_0)$ is the mean curvature of **any** point p_0 on the initial surface M_0, and $\vec{\nu}(p_0)$ is the outward unit normal of the initial surface M_0 at this point p_0.
Without loss of generality assume that $H(p_0) \neq 0$ for some p_0 on the original surface M_0. For if $H = 0$ everywhere on the original surface then the solution is trivial: $M_t = M_0$.

Lemma 6.1 If *$H(p_0) \neq 0$ for some $p_0 \in M_0$, and $M_0^n \subset \mathbf{R}^{n+1}$ has a "self-similar solution" under the flow,* **then**
i) $\exists p_0 \in M_0$ such that $\langle p_0, \vec{\nu}(p_0)\rangle \neq 0$, **and**
ii) $k = \frac{2H(p_0)}{\langle p_0, \vec{\nu}(p_0)\rangle_{\mathbf{R}^{n+1}}}$ is a constant $\forall p_0 \in M_0$ such that $\langle p_0, \vec{\nu}(p_0)\rangle \neq 0$ (i.e. k is independent of such p_0) **and**
iii) $c(t) = \sqrt{1 - kt}$ **and**
iv) $H(p_0) = 0 \forall p_0$ such that $\langle p_0, \vec{\nu}(p_0)\rangle = 0$

We will need a preliminary lemma to prove the above lemma:

Lemma 6.2 *If M is a hypersurface, and $\tilde{M}$ is the hypersurface defined by*

$$\tilde{M} = cM$$

for some $c > 0$, then

$$\tilde{H}(\tilde{p}) = \frac{1}{c} H(p)$$

, where $\tilde{H}$ denotes the mean curvature on the hypersurface $\tilde{M}$, and $\tilde{p} = cp$.

proof: Let (x, U) be a local co-ordinate chart on M. Then $(\tilde{x}, \tilde{U})$ is a co-ordinate chart for $\tilde{M}$, where

$$\tilde{x}(\tilde{p}) = x(p)$$

,

$$\tilde{U} = cU$$

Calculate g^{ij}:

$$\tilde{g}_{ij}(\tilde{p}) = \langle \frac{\partial}{\partial \tilde{x}^i}(\tilde{p}), \frac{\partial}{\partial \tilde{x}^j}(\tilde{p})\rangle = c^2 g_{ij}(p)$$

Hence

$$\tilde{g}^{ij}(\tilde{p}) = \frac{1}{c^2} g^{ij}(p)$$

Calculate $\tilde{h}_{ij}$:

$$\tilde{h}_{ij}(\tilde{p}) = \langle \frac{\partial \tilde{\nu}}{\partial \tilde{x}^i}(\tilde{p}), \frac{\partial}{\partial \tilde{x}^j}(\tilde{p})\rangle$$

Then using the fact that

$$\tilde{\nu}(\tilde{p}) = \nu(p)$$

, we see that

$$\tilde{h}_{ij}(\tilde{p}) = ch_{ij}(p)$$

Hence

$$\begin{aligned} \tilde{H}(\tilde{p}) &= \tilde{g}^{ij}\tilde{h}_{ij} \\ &= \frac{1}{c}g^{ij}h_{ij} \\ &= \frac{1}{c}H \quad \blacksquare \end{aligned}$$

Now we prove the first lemma stated.

proof: Let p_0 be an arbitrary point in M_0. Then follow p_0 through time under the flow given by $M_t = c(t)M_0$- i.e. put $p(t) = c(t)p_0$. By our definition of what a self-similar solution is we see that:

$$\langle p'(t), \vec{\nu}(p(t))\rangle = -H(p(t))$$

But $p'(t) = c'(t)p_0$, and $\vec{\nu}(p(t)) = \vec{\nu}(p_0)$, So

$$\begin{aligned} \langle p'(t), \vec{\nu}(p(t))\rangle &= \langle c'(t)p_0, \vec{\nu}(p_0)\rangle \\ &= c'(t)\langle p_0, \vec{\nu}(p_0)\rangle \\ \Rightarrow c'(t)\langle p_0, \vec{\nu}(p_0)\rangle &= -H(p(t)) \\ &= -\frac{H(p_0)}{c(t)} \qquad (6.4) \\ &\quad \text{(by the above lemma)} \end{aligned}$$

So since by assumption $\exists p_0$ such that $H(p_0) \neq 0$, then by 6.4

$$\langle p_0, \vec{\nu}(p_0)\rangle \neq 0$$

for this p_0. Hence we have proved i).

Still using this p_0, we see from 6.4 that we have

$$c'(t) = \frac{-H(p_0)}{\langle p_0, \vec{\nu}(p_0)\rangle c(t)}$$

By the theory of O.D.E 's, this equation together with the initial condition $c(0) = 1$ has

a unique solution. Put $c(t) = \sqrt{1 - \frac{2H(p_0)t}{\langle p_0, \vec{\nu}(p_0)\rangle}}$.
Then

$$\begin{aligned} c'(t) &= \frac{1}{2}(\frac{-2H(p_0)}{\langle p_0, \vec{\nu}(p_0)\rangle})\frac{1}{\sqrt{1 - \frac{2H(p_0)t}{\langle p_0,\ vec\nu(p_0)\rangle}}} \\ &= \frac{-H(p_0)}{\langle p_0, \vec{\nu}(p_0)\rangle c(t)} \end{aligned}$$

as required, and $c(0) = 1$ as required. Note

$$c^{'}(0) = \frac{-H(p_0)}{\langle p_0, \vec{\nu}(p_0)} \neq 0 \tag{6.5}$$

since $H(p_0) \neq 0$ for our special choice of p_0.
Now let $\hat{p_0}$ be any point on M_0 with $\langle \hat{p_0}, \vec{\nu}(\hat{p_0})\rangle \neq 0$. Then by 6.4, and following the exact reasoning above for p_0, we see that

$$c(t) = \sqrt{1 - \frac{2H(\hat{p_0})t}{\langle \hat{p_0}, \vec{\nu}(\hat{p_0})\rangle}}$$

Since $\hat{p_0}$ was an arbitrary point on M_0 satisfying $\langle \hat{p_0}, \vec{\nu}(\hat{p_0})\rangle \neq 0$, we have proved ii) and iii).
To prove iv) note that $c^{'}(0) \neq 0$ (by 6.5) Hence by 6.4 we have that
$H(\hat{p_0}) = 0 \forall \hat{p_0}$ such that $\langle \hat{p_0}, \nu(\hat{p_0})\rangle = 0$. Hence we are done. ∎
Remark1:**If** M_0 is a surface with properties i), ii) and iv),
then $M_t = \sqrt{1-kt}M_0$ is a solution to the flow problem, since

$$\begin{aligned} \langle \frac{\partial}{\partial t}p(t), \vec{\nu}(p)\rangle &= \langle \frac{\partial}{\partial t}(\sqrt{1-kt}p_0), \vec{\nu}(p_0)\rangle \\ &= \langle -\frac{1}{2}\frac{kp_0}{c(t)}, \vec{\nu}(p_0)\rangle \\ &= -H(p(t)) \end{aligned}$$

where the last identity follows by considering the two cases

$$\text{i) } \langle p_0, \vec{\nu}(p_0)\rangle = 0$$

and

$$\text{ii) } \langle p_0, \vec{\nu}(p_0)\rangle \neq 0$$

(and using the fact that $H(p_0) = \frac{H(p(t))}{c(t)}$). Hence the problem of finding non-trivial self-similar solutions is a Differential Geometry one:

M_0 self similar and $\exists p_0 \in M_0$ such that $H(p_0) \neq 0 \Longleftrightarrow M_0$ satisfies conditions i), ii), and iv) and $\exists p_0 \in M_0$ such that $H(p_0) \neq 0$

Remark 2 Surfaces stay disjoint under the flow (Theorem 1.6, Chapter 1). For this reason general self-similar solutions are particularly important, as we can use them as barriers for other surfaces under the flow.

e.g. if M_0 has a self-similar solution, then often we know explicitly how the surface behaves under the flow (e.g. $S^n_{R_0}$) and we can calculate explicitly a blow up time T for the surface. Then if $\overline{M_0}$ is a surface such that the volume of $\overline{M_0}$ is contained in the volume of M_0 then $\overline{M_0}$ must blow up at or before the time that M_0 blows up at under the flow (i.e. at or before time t = T).

Here we present a number of self-similar flows:

Example 1: C_{R_0} The infinite cylinder of radius R_0 dimension n.

Let M_0 be the rotationally symmetric surface generated by the function $y = R_0$, for some constant $R_0 > 0$. Then the surface at time t under the flow is the infinite cylinder of radius $R(t) = \sqrt{1 - 2(n-1)t}$. (See diagram 6.3.)

Example 2: $S^n_{R_0}(z)$ The n-dimensional sphere of radius R_0, centre z. Then the surface at time t under the flow is $S^n_{R(t)}(z)$, where $R(t) = \sqrt{R_0^2 - 2nt}$. (See diagram 1.1.)

Example 3: The catenoid surface C_0 generated by the surface $y = \cosh$. Then the surface at time t under the flow is C_0. (See diagram 6.4 and appendix 4.)

Example 4: Angenent [12] shows that there is torus T_0 which has a self similar solution. (See diagram 6.5.)

Chapter 7

A-priori height estimates

7.1 Introduction

Here we aim to prove *a-priori* height estimates for rotationally symmetric surfaces evolving under mean curvature flow analogous to those proved for entire graphs by Ecker/Huisken in [5]. In particular we wish to show that if the initial surface M_0 is generated by the positive function y_0, and $(y_0)^2 \leq c(c_0 + (x_1)^2)^p$, then $y^2 \leq c(c_0 + x_1^2 + \alpha t)^p$ (where y is the generator of the surface M_t).

7.2 An *a-priori* height estimate

For convenience sake we first prove a Lemma.

Lemma 7.1 Let $\psi = c_0 + (x_1)^2 + \alpha t$
Then $(\frac{\partial}{\partial t} - \triangle)\psi = -2|\nabla x_1|^2 + \alpha$

proof:

$$\begin{aligned} (\frac{\partial}{\partial t} - \triangle)\psi &= 2x_1((\frac{\partial}{\partial t} - \triangle)x_1) - 2|\nabla x_1|^2 + \alpha && (7.1) \\ & \quad ((\tfrac{\partial}{\partial t} - \triangle)x = 0, \text{ from appendix 3 Lemma C.5}) && (7.2) \\ &= -2|\nabla x_1|^2 + \alpha \quad \blacksquare && (7.3) \end{aligned}$$

Now we prove the desired height estimate.

Theorem 7.2 Assume

$$(y_0)^2 \leq c(c_0 + (x_1)^2)^p$$

for some $p \geq 0$. **Then**

$$y^2 \leq c(c_0 + (x_1)^2 + 2t)^p$$

proof: Let $\alpha = 2$. By Lemma C.4 of appendix 3 we have that $|\nabla x_1|^2 \leq 1$. Hence by the lemma above,

$$-(\frac{\partial}{\partial t} - \triangle)\psi = 2|\nabla x_1|^2 - 2 \leq 0 \tag{7.4}$$

by assumption $\frac{(y_0)^2}{\psi^p} \leq c$.

Now

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \triangle)(y^2\psi^{-p}) \\
&= y^2(\frac{\partial}{\partial t} - \triangle)\psi^{-p} + \psi^{-p}(\frac{\partial}{\partial t} - \triangle)(y^2) - 2\nabla_i(y^2)\nabla_i(\psi^{-p}) \\
&= y^2(-p\psi^{-p-1}(\frac{\partial}{\partial t} - \triangle)\psi - p(p+1)\psi^{-p-2}|\nabla\psi|^2) \\
&+\psi^{-p}(2y(\frac{\partial}{\partial t} - \triangle)y - 2|\nabla y|^2) + 4yp\psi^{-p-1}\nabla_i y\nabla_i\psi \\
&\text{(then by 7.4 above, and the evolution equation for y calculated in Chapter 2)} \\
&\leq -p(p+1)y^2\psi^{-p-2}|\nabla\psi|^2 - 2(n-1)\psi^{-p} \\
&-2\psi^{-p}|\nabla y|^2 + 4py\psi^{-p-1}\nabla_i y\nabla_i\psi
\end{aligned}$$

We estimate the last term by Young's inequality:

$$\begin{aligned}
&4py\psi^{-p-1}\nabla_i y\nabla_i\psi \\
&\leq p^2y^2\psi^{-p-2}|\nabla\psi|^2 + 4\psi^{-p}|\nabla y|^2 \\
&\leq p(p+1)y^2\psi^{-p-2}|\nabla\psi|^2 + 2\psi^{-p}|\nabla y|^2 + 2(n-1)\psi^{-p} \\
&\text{(since } |\nabla y| = |\tfrac{y_x}{\sqrt{1+(y_x)^2}}| \leq 1)
\end{aligned}$$

and hence we obtain

$$(\frac{\partial}{\partial t} - \triangle)(y^2\psi^{-p}) \leq 0$$

Then using the monotonicity formula of Chapter 1 we see that $y^2\psi^{-p} \leq c \; \forall$ time, and so we are done.

Chapter 8

Evolution of the gradient for a rotationally symmetric surface

8.1 Introduction

Let M_0 be an entire rotationally symmetric surface generated by a positive function y_0. Here we wish to derive estimates for the gradient $y(\cdot,t)_x$ of the rotationally symmetric surface M_t (where $y(\cdot,t)$ is the generator function of M_t, and M_t is the surface at time t under the flow (for initial data M_0)).
To do this we must calculate the evolution equation for y_x under the flow.
First we prove a number of lemmas about the Laplacian $\triangle$ of a function on a Riemannian manifold, which will be useful in this calculation.

8.2 Preliminary lemmas

Lemma 8.1 *Let g_1, g_2 be two functions defined on a Riemannian manifold M. Then for a local orthonormal frame $\{\ \vec{e_i}\ \}$ we have*

$$\triangle(g_1 g_2) = g_1 \triangle g_2 + g_2 \triangle g_1 + 2\nabla_i g_1 \cdot \nabla_i g_2$$

proof: Choose a local orthonormal **Riemannian** frame $\{\ \vec{a_i}\ \}$ on M (so $\nabla_{\vec{a_i}}\vec{a_i} = 0$, $\langle \vec{a_i}, \vec{a_j} \rangle = \delta_{ij}$). Then

$$\triangle(g_1 g_2) = \vec{a_i}(\vec{a_i}(g_1 g_2)) - (\nabla_{\vec{a_i}}\vec{a_i})(g_1 g_2)$$

(by definition)

$$= \vec{a_i}(\vec{a_i}(g_1 g_2))$$

$$= \vec{a_i}(g_1\vec{a_i}(g_2) + g_2\vec{a_i}(g_1))$$
$$= g_1\vec{a_i}(\vec{a_i}g_2) + \vec{a_i}g_1 \cdot \vec{a_i}g_2$$
$$+g_2\vec{a_i}(\vec{a_i}g_1) + \vec{a_i}g_2 \cdot \vec{a_i}g_1$$
$$= g_1\triangle g_2 + 2\nabla_i g_1 \cdot \nabla_i g_2 + g_2\triangle g_1$$

(Note: the term $2\nabla_i g_1 \cdot \nabla_i g_2$ is independent of any orthonormal frame $\{ei\}$ we care to choose, since for an orthonormal frame $\{ei\}$ we have

$$\begin{aligned} \nabla_i g_1 \nabla_i g_2 &= \vec{e_i}(g_1)\vec{e_i}(g_2) \\ &= \langle dg_1, dg_2 \rangle \end{aligned}$$

) and so we are done. ∎

Lemma 8.2 *Let g be a function defined on a Riemannian manifold M. If $g \neq 0$ locally, then for a local orthonormal frame $\{\ \vec{e_i}\ \}$ we have*

$$\triangle g^{-1} = -g^{-2}\triangle g + 2g^{-3}|\nabla g|^2$$

proof: Choose a local orthonormal **Riemannian** frame $\{\ \vec{a_i}\ \}$ on M (so $\nabla_{\vec{a_i}}\vec{a_i} = 0$, $\langle \vec{a_i}, \vec{a_j} \rangle = \delta_{ij}$).
Then

$$\triangle g^{-1} = \vec{a_i}(\vec{a_i}(g^{-1})) - \nabla_{\vec{a_i}}\vec{a_i}(g^{-1})$$
$$= \vec{a_i}(\vec{a_i}(g^{-1}))$$
$$= \vec{a_i}(-g^{-2}\vec{a_i}g)$$
$$= 2g^{-3}\vec{a_i}g \cdot \vec{a_i}g - g^{-2}\vec{a_i}(\vec{a_i}g)$$
$$= 2g^{-3}|\nabla g|^2 - g^{-2}\triangle g$$

and so we are done. ∎

So now we are in a position to calculate the evolution equation of y_x.

8.3 Evolution of the gradient

In fact we will calculate the evolution equation of the function

$$f = \sqrt{1 + (y_x)^2}$$

We choose to concentrate on the function f rather than y_x, since f is a more geometrical quantity. This can be seen as follows:
Consider the top ridge of a rotationally symmetric surface M. The top ridge of a rotationally symmetric surface M is the set of points $\{ \; p \in M : p = (x_1, 0, \ldots, y(x_1)) \; \}$, where y is the generator of the surface M. We see by appendix 1 (where $\vec{\nu}$ is calculated for rotationally symmetric surfaces in terms of the generator y and the derivatives of y) that $f =< \nu, \iota_1 >$ on this top ridge. Hence f is a geometrical measure of the slope of the surface M. Given a point p on the surface M, we may rotate the surface so that p lies on the top ridge, so the quantity $f = \sqrt{1 + (y_x)^2}$ is just as geometric no matter where we are on the rotationally symmetric surface M.
So now we will calculate.

We use the orthonormal frame $\{\vec{e_i}\}$ calculated in Chapter 2. We use the notation of that chapter here. Note then that

$$f = \sqrt{1 + (y_x)^2} = \frac{1}{py}$$

where p is defined in Chapter 2.

Lemma 8.3 *The function $f = \sqrt{1 + (y_x)^2}$ satisfies the following evolution equation under mean curvature flow:*

$$(\frac{\partial}{\partial t} - \triangle) f = -fk^2 - 2k^2 (y_x)^2 f + (n-1)(y_x)^2 f^{-1} y^{-2}$$

where k is as defined in Chapter 2.

proof: From the lemmas above we calculate.

$$\begin{aligned}
\triangle f &= \triangle(\frac{1}{py}) \\
&= p^{-1}\triangle(y^{-1}) + y^{-1}\triangle(p^{-1}) + 2\nabla_i(p^{-1})\nabla_i(y^{-1}) \\
&= p^{-1}(2y^{-3}|\nabla y|^2 - y^{-2}\triangle y) + y^{-1}(2p^{-3}|\nabla p|^2 - p^{-2}\triangle p) \\
&\quad +2\nabla_i(p^{-1})\nabla_i(y^{-1}) \\
&= 2y^{-3}|\nabla y|^2 p^{-1} - y^{-2}p^{-1}\triangle y + 2p^{-3}y^{-1}|\nabla p|^2 - p^{-2}y^{-1}\triangle p \\
&\quad +2p^{-2}y^{-2}\nabla_i p \nabla_i y
\end{aligned}$$

Now

$$\frac{\partial}{\partial t} f = \frac{\partial}{\partial t}(\frac{1}{py}) = -\frac{1}{yp^2}\frac{\partial}{\partial t}p - \frac{1}{y^2 p}\frac{\partial}{\partial t}y$$

(using the evolution equations for p and y calculated in Chapter 2)

$$
\begin{aligned}
&= -y^{-1}p^{-2}(\triangle p + |A|^2 p + 2q^2(k-p)) \\
&-y^{-2}p^{-1}(\triangle y - (n-1)y^{-1}) \\
&= -y^{-1}p^{-2}\triangle p - y^{-1}p^{-1}|A|^2 \\
&-2y^{-1}p^{-2}q^2(k-p) - y^{-2}p^{-1}\triangle y \\
&+(n-1)y^{-3}p^{-1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \triangle)f &= -y^{-1}p^{-1}|A|^2 - 2y^{-1}p^{-2}q^2(k-p) \\
&+(n-1)y^{-3}p^{-1} - 2y^{-3}|\nabla y|^2 p^{-1} \\
&-2p^{-3}y^{-1}|\nabla p|^2 - 2p^{-2}y^{-2}\nabla_i p \nabla_i y
\end{aligned}
$$

Then using

$$q^2 = y^{-2} - p^{-2},$$

and

$$|A|^2 = k^2 + (n-1)p^2$$

(see Chapter 2 where these two facts are shown) we see that

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \triangle)f = -y^{-1}p^{-1}k^2 - (n-1)y^{-1}p - 2y^{-3}p^{-2}k + 2y^{-3}p^{-1} \\
&+2y^{-1}k - 2y^{-1}p + (n-1)y^{-3}p^{-1} \\
&-2y^{-3}|\nabla y|^2 p^{-1} - 2p^{-3}y^{-1}|\nabla p|^2 - 2p^{-2}y^{-2}\nabla_i p \nabla_i y
\end{aligned}
$$

Now substituting in

$$p = f^{-1}y^{-1}$$

we see that

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \triangle)f &= -fk^2 - 2f^2y^{-1}k + 2y^{-1}k \\
&-2|\nabla y|^2 f y^{-2} - 2f^3y^2|\nabla p|^2 - 2f^2\nabla_i p \nabla_i y \\
&+(n+1)y^{-2}f - (n+1)f^{-1}y^{-2} \qquad (8.1)
\end{aligned}
$$

Now we explicitly calculate the gradient terms in the above formula.

$$
\begin{aligned}
\nabla_i y &= \delta_{i1} y_x f^{-1} \Rightarrow \\
-2|\nabla y|^2 f y^{-2} &= -2(y_x)^2 f^{-1} y^{-2} \qquad (8.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_1 p &= \nabla_1(y^{-1}f^{-1}) \\
&= -f^{-1}y^{-2}\nabla_1 y - y^{-1}f^{-2}\nabla_1 f \\
&= -f^{-1}y^{-2}y_x f^{-1} - y^{-1}f^{-2}y_x y_{xx} f^{-2} \\
&= -f^{-2}y^{-2}y_x + y^{-1}f^{-1}ky_x
\end{aligned}
$$

(Note here we have calculated $\nabla_i f = -\delta_{i1}ky_x f$.)

$$
\begin{aligned}
\Rightarrow \quad & |\nabla p|^2 \\
& = f^{-4}y^{-4}(y_x)^2 - 2f^{-2}y^{-2}y_x y^{-1}f^{-1}ky_x \\
& +y^{-2}f^{-2}k^2(y_x)^2 \\
& = f^{-4}y^{-4}(y_x)^2 - 2f^{-3}y^{-3}(y_x)^2 k \\
& +y^{-2}f^{-2}k^2(y_x)^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow -2f^3y^2|\nabla p|^2 &= -2(y_x)^2 f^{-1}y^{-2} + 4(y_x)^2 ky^{-1} \\
& -2k^2(y_x)^2 f \qquad (8.3)
\end{aligned}
$$

and hence also,

$$
\begin{aligned}
& -2f^2\nabla_i p \nabla_i y \\
& = -2f^2(y^{-1}ky_x f^{-1} - f^{-2}y^{-2}y_x)(y_x f^{-1}) \\
& = (-2fy^{-1}ky_x + 2y^{-2}y_x)(y_x f^{-1}) \\
& = -2y^{-1}k(y_x)^2 + 2y^{-2}(y_x)^2 f^{-1} \qquad (8.4)
\end{aligned}
$$

So substituting in 8.2, 8.3, and 8.4 into 8.1 we have that

$$
(\frac{\partial}{\partial t} - \triangle)f = -fk^2 - 2k^2(y_x)^2 f + (n-1)(y_x)^2 f^{-1}y^{-2} \qquad (8.5)
$$

this is the evolution equation for the gradient. ■

Note that as n increases, this evolution equation gets worse - i.e. the positive term $(n-1)(y_x)^2 f^{-1}y^{-2}$ increases. This is due to the fact that for e.g. n =3 we have twice as much downwards curvature (p) acting on the surface under the flow, and so the gradient should change at a faster rate. We conjecture that this term could (for certain initial rotationally symmetric data M_0) cause the gradient to blow up (approach ∞) in finite time for $n > 2$. Geometrically this suggestion as backed up by the existence of rotationally symmetric catenoid surfaces for $n > 2$, which become vertical a finite distance from the

origin along the x-axis (see the note at the end of appendix 4 where the catenoid surfaces of dimension n¿2 are discussed). In Chapter 2 it was shown that, for n = 2, if a) M_0 is an entire surface with polynomial growth,
b) $\inf_{M_0} H > 0$ on M_0,
c) the quantities $\frac{|A|^2}{H^2}$ $\frac{|q|}{H}$, $\frac{k}{p}$ are bounded on M_0, then we have a time independent gradient estimate. It was also pointed out there, that the argument we used to obtain this gradient estimate breaks down for $n > 2$. Perhaps this term $(n-1)(y_x)^2 f^{-1} y^{-2}$ is just too big for $n > 2$, and as a result we will not obtain a time independent gradient estimate under the assumptions a),b),c).

8.4 A height dependent gradient estimate

We now wish to obtain a gradient estimate depending on the height of the surface above the x_1-axis, independent of n, for rotationally symmetric surfaces.

Lemma 8.4 If $C_0 = \sup_{M_0} yf < \infty$**,**
then

$$yf < C_0$$

$\forall$ *time t that the flow exists. Hence we have the estimate*

$$f < \frac{C_0}{y}$$

$\forall$ *time t that the flow exists.*

proof: We calculate the evolution equation for yf and then use the monotonicity formula (see appendix 5)

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \triangle)(fy) \\
&= f(\frac{\partial}{\partial t} - \triangle)y + y(\frac{\partial}{\partial t} - \triangle)f - 2\nabla_i f \nabla_i y \\
&= -f(n-1)y^{-1} + y(-fk^2 - 2k^2(y_x)^2 f \\
&+(n-1)(y_x)^2 f^{-1} y^{-2}) - 2\nabla_i f \nabla_i y \\
&\ \text{(using } \nabla_1 f = -y_x k f, \\
&\ \nabla_1 y = y_x f^{-1} \text{)} \\
&= -f(n-1)y^{-1} - yfk^2 - 2yk^2(y_x)^2 f \\
&+(n-1)(y_x)^2 f^{-1} y^{-1} + 2(y_x)^2 k
\end{aligned}
$$

$$
\begin{aligned}
&= -f(n-1)y^{-1} - yfk^2 - 2yk^2(y_x)^2 f \\
&+(n-1)fy^{-1} - (n-1)f^{-1}y^{-1} + 2(y_x)^2 k \\
&= -yfk^2 - 2yk^2(y_x)^2 f - (n-1)f^{-1}y^{-1} + 2(y_x)^2 k
\end{aligned}
$$

Now note

$$
\begin{aligned}
&\nabla_1(fy) \\
&= f\nabla_1 y + y\nabla_1 f \\
&= y_x - yy_x kf
\end{aligned}
$$

Hence

$$a \cdot \nabla_1(fy) - 2(y_x)^2 f^{-1}y^{-1} + 2(y_x)^2 k = 0$$

where $a = 2y_x f^{-1}y^{-1}$. Hence

$$(\frac{\partial}{\partial t} - \triangle)(fy) = -yfk^2 - 2yk^2(y_x)^2 f - (n-1)f^{-1}y^{-1} + a \cdot \nabla_1(fy) - 2(y_x)^2 f^{-1}y^{-1} + 4(y_x)^2 k$$

Estimating the last term by Young's inequality:

$$4(y_x)^2 k \leq 2(y_x)^2 fk^2 y + 2(y_x)^2 f^{-1}y^{-1},$$

we finally obtain

$$(\frac{\partial}{\partial t} - \triangle)(fy) \leq -yfk^2 - (n-1)f^{-1}y^{-1} + a \cdot \nabla_1(fy) \leq a \cdot \nabla_1(fy).$$

Hence by the monotonicity formula (see Chapter 1) we have $fy \leq C_0$ as required.

Appendix A

Notes on rotationally symmetric surfaces

Let $y : \mathbf{R} \to \mathbf{R}$ generate the rotationally symmetric surface M. We wish to calculate the mean curvature of this rotationally symmetric surface M in terms of it's generator y, and the derivatives of y. To do this we choose a local co-ordinate map for M defined in terms of y, and calculate the metric g_{ij}, and the second fundamental form h_{ij} of the surface M with respect to this co-ordinate chart. Then we use the formula $H = g^{ij}h_{ij}$ to calculate the mean curvature of the surface M.

Let $\hat{f} : \mathbf{R} \times \mathbf{S}^{n-1} \to \mathbf{R}^{n+1}$ be the function defined by

$$\hat{f}(x, s_1, \ldots, s_n) = \begin{pmatrix} x_1 \\ y(x)s_1 \\ \vdots \\ y(x)s_n \end{pmatrix}$$

for some $y : \mathbf{R} \to \mathbf{R}$. Then $M = \hat{f}(\mathbf{R} \times \mathbf{S}^{n-1}) \subset \mathbf{R}^{n+1}$ is the n-dimensional rotationally symmetric surface sitting in $\mathbf{R}^{n+1}$, generated by the function y. A differentiable structure is given to M by composing $\hat{f}$ with charts from the differentiable structure for $\mathbf{S}^{n-1}$. e.g. for points on the top ridge of the surface (the top ridge is the set of points $\{ p \in M : p = (x_1, 0, \ldots, 0, y(x_1))\}$) we have

$$f(x, s_1, \ldots, s_n) = \begin{pmatrix} x \\ y(x)s_1 \\ \vdots \\ y(x)s_{n-1} \\ y(x)\sqrt{1 - s_1^2 - \cdots - s_{n-1}^2} \end{pmatrix}$$

is (the inverse of) a local co-ordinate map. Now we calculate:

$$D_1 f = \begin{pmatrix} 1 \\ s_1 y'(x) \\ \vdots \\ s_{n-1} y\prime(x) \\ y'(x)\sqrt{S} \end{pmatrix}$$

$$D_2 f = \begin{pmatrix} 0 \\ y(x) \\ 0 \\ \vdots \\ 0 \\ \frac{-y(x)s_1}{\sqrt{S}} \end{pmatrix}$$

...

$$D_n f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ y(x) \\ \frac{-y(x)s_{n-1}}{\sqrt{S}} \end{pmatrix}$$

where $S = 1 - s_1^2 - \cdots - s_{n-1}^2 = s_n^2$.

case 1 $j, i \neq 1$:

$$g_{ij} = D_i f \cdot D_j f = \frac{y^2(\delta_{ij} s_n^2 + s_{i-1} s_{j-1})}{s_n^2}$$

case 2 $i = 1, j \neq 1$ or $i \neq 1, j = 1$:

$$g_{j1} = g_{1j} = D_1 f \cdot D_j f = s_{j-1} y'(x) y(x) - y(x) s_{j-1} y'(x) = 0$$

case 3 $i = j = 1$

$$g_{11} = D_1 f \cdot D_1 f = 1 + s_1^2 y'(x)^2 + \cdots + s_{n-1}^2 y'(x)^2 + s_n^2 y'(x)^2 = 1 + y'(x)^2$$

$$\nu = \frac{D_1 f \times \cdots \partial_n f}{|D_1 f \times \cdots D_n f|} = \begin{pmatrix} \frac{-y'}{\sqrt{1+(y')^2}} \\ \frac{s_1}{\sqrt{1+(y')^2}} \\ \vdots \\ \frac{s_n}{\sqrt{1+(y')^2}} \end{pmatrix}$$

Now $h_{ij} = D_i\nu \cdot D_j f$. So calculate:

case 1 $i, j \neq 1$:

$$h_{ij} = \frac{y(s_n^2\delta_{ij} + s_{i-1}s_{j-1})}{\sqrt{1+(y')^2} \cdot s_n^2}$$

case 2 $i = 1, j \neq 1$ (or $i \neq 1, j = 1$)

$$h_{1j} = h_{j1} = \frac{-s_{j-1}y'y''y + ys_{j-1}y'y''}{(1+(y')^2)^{3/2}} = 0$$

case 3 $i = j = 1$

$$h_{11} = \frac{-y''}{\sqrt{1+(y')^2}}$$

Calculations give g^{-1} is defined by:

case1 $i = j = 1$:

$$g^{11} = \frac{1}{\sqrt{1+(y')^2}}$$

case 2 $i = 1, j \neq 1$ (or $j = 1, i \neq 1$)

$$g^{1j} = g^{j1} = 0$$

case 3 $i, j \neq 1$

$$g^{ij} = \frac{1}{y^2}(\delta_{ij} - s_i s_j)$$

Then $H = \sum$ eigen values of h_{ij} with respect to g_{ij}

$$= h_i^i \tag{A.1}$$

$$= g^{ij}h_{ij} \tag{A.2}$$

$$= \frac{(n-1)}{y\sqrt{1+(y')^2}} - \frac{y''}{(1+(y')^2)^{3/2}} \tag{A.3}$$

Note: Define

$$\tilde{y} = \lambda y(\frac{x}{\lambda}),$$

for some $\lambda > 0$. Then

$$\tilde{y}'(x) = (\lambda y(\frac{x}{\lambda}))' \tag{A.4}$$

$$= y'(\frac{x}{\lambda}) \tag{A.5}$$

$$
\begin{aligned}
\tilde{y}''(x) && \text{(A.6)}\\
&= (y'(\frac{x}{\lambda}))' & \text{(A.7)}\\
&= \frac{1}{\lambda}y''(\frac{x}{\lambda}) & \text{(A.8)}
\end{aligned}
$$

And so $\tilde{H}$ (the mean curvature of the rotationally symmetric surface generated by $\tilde{y}$), is

$$
\begin{aligned}
\tilde{H}(x) &= \frac{(n-1)}{\tilde{y}(x)\sqrt{1+(\tilde{y}'(x))^2}} - \frac{\tilde{y}''(x)}{(1+(\tilde{y}'(x))^2)^{3/2}} & \text{(A.9)}\\
&= \frac{(n-1)}{\lambda y(\frac{x}{\lambda})\sqrt{1+(y'(\frac{x}{\lambda}))^2}} - \frac{y''(\frac{x}{\lambda})}{\lambda(1+(y'(\frac{x}{\lambda}))^2)^{3/2}} & \text{(A.10)}\\
&= \frac{1}{\lambda}(\frac{(n-1)}{y(\frac{x}{\lambda})\sqrt{1+(y'(\frac{x}{\lambda}))^2}} - \frac{y''(\frac{x}{\lambda})}{(1+(y'(\frac{x}{\lambda}))^2)^{3/2}}) & \text{(A.11)}
\end{aligned}
$$

This is in fact true for hyper-surfaces in general, as is shown in Chapter 6, Lemma 6.2. i.e. if $\tilde{M} = cM$ for some $c > 0$, then $\tilde{H}(\tilde{p}) = H(p)$.

Appendix B

Notes on graphs

Let u be a $C^2(\Omega)$ function $u : \Omega \to \mathbf{R}$ for some hyperplane $\Omega \subset \mathbf{R}^{n+1}$. We wish to calculate the mean curvature of the surface $M = graph(u)$ in terms of u and it's derivatives. To do this we define a co-ordinate chart for M in terms of u and calculate the first and second fundamental form (g_{ij} and h_{ij}) of M in terms of this co-ordinate chart. We then use the formula $H = g^{ij}h_{ij}$ to calculate the mean curvature of the surface M.

$f : \Omega \to \mathbf{R}^{n+1}$, the graph of the function u, is given by

$$f(x) = \begin{pmatrix} x \\ u(x) \end{pmatrix}$$

f^{-1} is a co-ordinate chart for the surface $f(\Omega)$. We calculate:

$$D_k f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ D_k u \end{pmatrix}$$

where the 1 is in the k^{th} spot. Calculate

$$g_{ij} = D_i f \cdot D_j f = \delta_{ij} + D_i u \cdot D_j u$$

and

$$\nu = \frac{D_1 f \times \cdots \times D_n f}{|D_1 f \times \cdots \partial_n f|} = \begin{pmatrix} \frac{-D_1 u}{\sqrt{1+|Du|^2}} \\ \vdots \\ \frac{-D_1 u}{\sqrt{1+|Du|^2}} \\ \frac{1}{\sqrt{1+|Du|^2}} \end{pmatrix} \tag{B.1}$$

$$h_{ij} = D_i \nu \cdot D_j f = \frac{-D_{ij} u}{\sqrt{1+|Du|^2}}$$

$$g^{ij} = \begin{cases} 1 - \frac{(D_i u)^2}{1+|Du|^2} for i = j \\ -\frac{D_i u D_j u}{1+|Du|^2} for i \neq j \end{cases}$$

And so $H = \sum$ Eigenvalues of h_{ij} with respect to g_{ij}

$$\begin{aligned} &= h_i^i \\ &= g^{ij} h_{ij} \\ &= -D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right) \end{aligned} \tag{B.2}$$

How do graphs act under the flow? Well as mentioned in Chapter 1 the equation

$$\frac{\partial}{\partial t} \vec{F} = \vec{H} \tag{B.3}$$

is equivalent up to tangential diffeomorphisms to

$$(\frac{\partial}{\partial t} F)^{\perp} = \vec{H} \tag{B.4}$$

i.e. the surfaces we get under B.3 and B.4 are the same at each time. Now assume M_0 is an initial surface that can be written as the graph of a function $u_0 \in C^{2,0}(\Omega), u_0 : \Omega \to \mathbf{R}$, for some hyperplane Ω. If $\vec{F}$ is a solution to B.4 (with $M^n = \Omega$, $F_0(M = \Omega) = M_0$) which preserves the x-co-ordinate, then $F(x,t) = (x, u(x,t))$ for some function u defined on $\Omega \times [0,T)$, $u(\cdot,t) : \Omega \to \mathbf{R}$. So $\frac{\partial}{\partial t}\vec{F} = (0, u_t(x,t))$. Using the calculation of ν above (B.1) we see that

$$\langle \frac{\partial}{\partial t} \vec{F}, \nu \rangle = \frac{u_t}{\sqrt{1+|Du|^2}} \tag{B.5}$$

But from B.4 we have that

$$\langle \frac{\partial}{\partial t} \vec{F}, \nu \rangle = -H \tag{B.6}$$

So using the calculation of H above (B.2), and B.5 and B.6 we see that

$$u_t = \sqrt{1+|Du|^2} D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right)$$

This is the equation for the flow of a graph. Evolution of graphs has been extensively studied in [5] (where entire graphs are studied) and [4] (where interior estimates for surfaces are calculated using graphs to represent the surface locally).

Appendix C

Technical lemmas

As usual M^n is an n-dimensional surface sitting in $\mathbf{R}^{n+1}$, $\vec{x}$ is the position vector in $\mathbf{R}^{n+1}$.

Lemma C.1 *If $\vec{x}$ is the position vector in $\mathbf{R}^{n+1}$,* **then**

$$\nabla_{\vec{Y_p}}\vec{x} = \hat{\nabla}_{\vec{Y_p}}\vec{x} = \vec{Y_p}$$

for any vector $\vec{Y_p} \in T_pM$.

proof: Choose $\alpha : \mathbf{R} \to \mathbf{R}^{n+1}$ such that: $\alpha(t) \in M$, and $\alpha(0) = p$, and $\alpha'(0) = \vec{Y_p}$. Then

$$\begin{aligned} \hat{\nabla}_{Y_p}\vec{x} &= \frac{\partial}{\partial t}(\vec{x}(\alpha(t)))_{|t=0} \\ &= \frac{\partial}{\partial t}\alpha(t)_{|t=0} \\ &= \alpha'(0) \\ &= \vec{Y_p} \end{aligned}$$

$$\nabla_{Y_p}\vec{x} = \pi(\hat{\nabla}_{Y_p}\vec{x}),$$

where $\pi(Y_p)$ is projection of the vector Y_p onto the tangent space at p. So we are done. ∎

Lemma C.2 *If $\{\vec{a_i}\}$ is a local Riemannian orthonormal frame on M,*
(i.e. $g_{ij} = \langle\vec{a_i}, \vec{a_j}\rangle = \delta_{ij}$, and $\Gamma^l_{ij} = 0$), **then**

$$\hat{\nabla}_{\vec{a_i}}\vec{a_i} = \vec{H} = -H\nu \tag{C.1}$$

proof: Since the frame is Riemannian, we have $\Gamma^l_{ij} = 0$, and so

$$\langle\hat{\nabla}_{\vec{e_i}}\vec{a_i}, \vec{a_j}\rangle = \Gamma^j_{ii} = 0$$

Hence

$$\hat{\nabla}_{\vec{a}_i}\vec{a}_i = \langle \hat{\nabla}_{\vec{a}_i}\vec{a}_i, \vec{\nu}\rangle \vec{\nu}$$

Then note that

$$\begin{aligned} 0 &= \vec{a}_i \langle \vec{a}_i, \vec{\nu}\rangle \\ &= \langle \hat{\nabla}_{\vec{a}_i}\vec{a}_i, \vec{\nu}\rangle \\ &\quad + \langle \vec{a}_i, \hat{\nabla}_{\vec{a}_i}\vec{\nu}\rangle \\ \Rightarrow &\quad \langle \hat{\nabla}_{\vec{a}_i}\vec{a}_i, \vec{\nu}\rangle \\ &= -\langle \vec{a}_i, \hat{\nabla}_{\vec{a}_i}\vec{\nu}\rangle \end{aligned}$$

So

$$\begin{aligned} \hat{\nabla}_{\vec{a}_i}\vec{a}_i &= -\langle \vec{a}_i, \hat{\nabla}_{\vec{a}_i}\vec{\nu}\rangle \vec{\nu} \\ &= -h_{ii}\vec{\nu} \\ &= -g^{ij}h_{ij}\vec{\nu} \\ &= -H\vec{\nu} \\ &= \vec{H} \end{aligned}$$

Note: Mean curvature, H, as is defined generally in Differential Geometry is not given a sign. Here we give H the sign so that the above formula C.1 is true. So for a choice of unit normal $\vec{\nu}$, given our definition of $h_{ij} = \langle \vec{a}_i, \hat{\nabla}_{\vec{a}_j}\vec{\nu}\rangle$, we see that $H = g^{ij}h_{ij}$. ∎

Lemma C.3

$$\triangle \vec{x} = \vec{H} \tag{C.2}$$

proof: Choose a local orthonormal Riemannian frame $\{\vec{a}_i\}_{i=1}^n$ on M. Then using the previous two lemmas where appropriate

$$\begin{aligned} \triangle \vec{x} &= \hat{\nabla}_{\vec{a}_i}\hat{\nabla}_{\vec{a}_i}\vec{x} - \nabla_{\vec{a}_i}\vec{a}_i\vec{x} \\ &= \hat{\nabla}_{\vec{a}_i}\hat{\nabla}_{\vec{a}_i}\vec{x} \\ &\quad \text{(since the frame is Riemannian} \Rightarrow \\ &\quad \nabla_{\vec{a}_i}\vec{a}_i = \Gamma^j_{ii}\vec{a}_j = 0) \\ &= \hat{\nabla}_{\vec{a}_i}\vec{a}_i \\ &= \vec{H} \end{aligned}$$

(from the above lemma)). ∎

Lemma C.4

$$(\frac{\partial}{\partial t} - \triangle)|\vec{x}|^2 = -2n \tag{C.3}$$

proof: Choose a local orthonormal Riemannian frame $\{\vec{a_i}\}_{i=1}^n$ on M. Then

$$\begin{aligned}
\frac{\partial}{\partial t}|\vec{x}|^2 &= \frac{\partial}{\partial t}\langle\vec{x},\vec{x}\rangle \\
&= 2\langle\frac{\partial}{\partial t}\vec{x}, x\rangle \\
&= 2\langle\vec{H}, x\rangle
\end{aligned}$$

Using the previous lemmas where appropriate, we see:

$$\begin{aligned}
\triangle|\vec{x}|^2 &= \vec{a_i}(\vec{a_i}|\vec{x}|^2) \\
&= \vec{a_i}(\vec{a_i}\langle\vec{x},\vec{x}\rangle) \\
&= \vec{a_i}(2\langle\vec{a_i},\vec{x}\rangle) \\
&= 2\langle\hat{\nabla}_{\vec{a_i}}\vec{a_i},\vec{x}\rangle + 2\langle\vec{a_i},\hat{\nabla}_{\vec{a_i}}x\rangle \\
&= 2\langle\vec{H},\vec{x}\rangle + 2\langle\vec{a_i},\vec{a_i}\rangle \\
&= 2\langle\vec{H},\vec{x}\rangle + 2n
\end{aligned}$$

So $(\frac{\partial}{\partial t} - \triangle)|\vec{x}|^2 = -2n$.

Lemma C.5

$$|\nabla x^j|^2 \leq 1 \tag{C.4}$$

proof: Let $\{\ \vec{e_i}\ \}$ be a local orthonormal frame.

$$\begin{aligned}
|\nabla x^j|^2 &= \sum_i(\vec{e_i}(x^j))^2 \\
&= \sum_i(\vec{e_i}\langle\vec{x},\iota_j\rangle)^2 \\
&= \sum_i\langle\vec{e_i},\iota_j\rangle^2 \\
&= |\pi(\iota_j)|^2
\end{aligned}$$

where $\pi_p(\vec{Y})$ is projection of the vector $\vec{Y} \in \mathbf{R}^{n+1}$ onto the tangent space of M at p. So we are done. ■

Lemma C.6

$$(\frac{\partial}{\partial t} - \triangle)\vec{x} = 0 \tag{C.5}$$

proof:

$$\frac{\partial}{\partial t}\vec{x} = \vec{H},$$

and

$$\triangle\vec{x} = \vec{H}$$

from a previous lemma. ∎

Lemma C.7 *Let* $\{\ \vec{e_i}\ \}$ *be a local frame on* M*. Then we have*

$$\hat{\nabla}_{\vec{e_i}}\vec{\nu} = h_{ik}\vec{e_k} \tag{C.6}$$

proof:

$$\begin{aligned} 0 &= \vec{e_i}\langle\vec{\nu},\vec{\nu}\rangle \\ &= 2\langle\hat{\nabla}_{\vec{e_i}}\vec{\nu},\vec{\nu}\rangle \\ \Rightarrow &\quad \langle\hat{\nabla}_{\vec{e_i}},\vec{\nu}\rangle = 0 \end{aligned}$$

Hence $\hat{\nabla}_{\vec{e_i}}\vec{\nu}$ lies in the tangent space of the surface M. Hence

$$\begin{aligned} \hat{\nabla}_{\vec{e_i}}\vec{\nu} &= \langle\hat{\nabla}_{\vec{e_i}}\vec{\nu},\vec{e_k}\rangle\vec{e_k} \\ &= h_{ik}\vec{e_k} \end{aligned}$$

(by definition). ∎

Lemma C.8 *Let* $\{\vec{a_i}\}$ *be a local Riemannian orthonormal frame on* M*. Then we have*

$$\hat{\nabla}_{\vec{a_i}}\vec{a_k} = -h_{ik}\vec{\nu} \tag{C.7}$$

proof:

$$\hat{\nabla}_{\vec{a_i}}\vec{a_k} = \langle\hat{\nabla}_{\vec{a_i}}\vec{a_k},\vec{\nu}\rangle\vec{\nu} + \nabla_{\vec{a_i}}\vec{a_k}$$

In a Riemannian orthonormal frame the last term is 0 (since all the Christoffel symbols are 0). Also, see that

$$\begin{aligned} 0 &= \vec{a_i}\langle\vec{a_k},\vec{\nu}\rangle \\ &= \langle\hat{\nabla}_{\vec{a_i}}\vec{a_k},\vec{\nu}\rangle \\ &\quad +\langle\vec{a_k},\hat{\nabla}_{\vec{a_i}}\vec{\nu}\rangle \\ &= \langle\hat{\nabla}_{\vec{a_i}}\vec{a_k},\vec{\nu}\rangle + h_ik \\ \Rightarrow &\quad \langle\hat{\nabla}_{\vec{a_i}}\vec{a_k},\vec{\nu}\rangle = -h_{ik} \end{aligned}$$

Hence

$$\hat{\nabla}_{\vec{a_i}}\vec{a_k} = -h_{ik}\vec{\nu} \ \blacksquare$$

Lemma C.9

$$\langle \nabla H, \iota_1 \rangle = \triangle \langle \vec{\nu}, \iota_1 \rangle + |A|^2 \langle \vec{\nu}, \iota_1 \rangle \tag{C.8}$$

proof: As usual let { $\vec{a_i}$ } be a Riemannian orthonormal frame. Then

$$\begin{aligned}
\triangle\langle \vec{\nu}, \iota_1 \rangle &= \vec{a_i}(\vec{a_i}(\langle \vec{\nu}, \iota_1 \rangle)) \\
&= \vec{a_i}(\langle \hat{\nabla}_{\vec{a_i}}\vec{\nu}, \iota_1 \rangle) \\
&\quad \text{(then using Lemma C.6} \\
&= \vec{a_i}(\langle h_{ik}\vec{a_k}, \iota_1 \rangle) \\
&= \langle \hat{\nabla}_{\vec{a_i}}(h_{ik}\vec{a_k}), \iota_1 \rangle \\
&= \langle \vec{a_i}(h_{ik})\vec{a_k}, \iota_1 \rangle + \langle h_{ik}\hat{\nabla}_{\vec{a_i}}\vec{a_k}, \iota_1 \rangle \\
&= \langle \nabla_i(h_{ik})\vec{a_k}, \iota_1 \rangle + \langle h_{ik}(-h_{ik}\vec{\nu}), \iota_1 \rangle \\
&\quad \text{(since by Lemma C.7,} \\
&\quad \hat{\nabla}_{\vec{a_i}}\vec{a_k} = -h_{ik}\vec{\nu} \ , \\
&\quad \text{and also } \nabla_i(h_{jk}) = \vec{a_i}(h_{jk}) \\
&\quad \text{since the frame is Riemannian)} \\
&= \nabla_i(h_{ik})\langle \vec{a_k}, \iota_1 \rangle - (h_{ik})^2 \langle \vec{\nu}, \iota_1 \rangle \\
&= \nabla_i(h_{ik})\langle \vec{a_k}, \iota_1 \rangle - |A|^2 \langle \vec{\nu}, \iota_1 \rangle \\
&\quad \text{(using the Codazzi equations: } \nabla_i h_{jk} = \nabla_j h_{ik}\text{)} \\
&= \nabla_k(h_{ii})\langle \vec{a_k}, \iota_1 \rangle - |A|^2 \langle \vec{\nu}, \iota_1 \rangle
\end{aligned}$$

Now

$$\begin{aligned}
\nabla H &= \nabla_k(H)\vec{a_k} \\
&= \nabla_k(g^{ij}h_{ij})\vec{a_k} \\
&= g^{ij}\nabla_k(h_{ij})\vec{a_k} + h_{ij}\nabla_k(g^{ij})\vec{a_k} \\
&= \nabla_k(h_{ii})\vec{a_k}
\end{aligned}$$

So we are done. ■

Appendix D

Catenoid surfaces

In this appendix we present the well known family of catenoid minimal surfaces of dimension 2. Each catenoid surface in the family is rotationally symmetric, has mean curvature 0 and hence remains stationary under mean curvature flow. Catenoid surfaces behave exponentially as we go out towards ∞ on the x-axis. They can be explicitly described as follows.

Let $f_0 : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f_0 = r\cosh(\frac{x}{r})$$

for some $r > 0$. Let M_0 be the rotationally symmetric surface generated by rotating f around the x_1 - axis.
Now

$$f_{0_x} = \sinh(\frac{x}{r})$$

and

$$f_{0xx} = \frac{1}{r}\cosh(\frac{x}{r})$$

so that

$$
\begin{aligned}
-H \cdot \sqrt{1+(f_{0_x})^2} &= \frac{f_{0xx}}{1+(f_{0_x})^2} - \frac{1}{f} && \text{(D.1)}\\
&= \frac{\frac{1}{r}\cosh(\frac{x}{r})}{1+\sinh^2(\frac{x}{r})} - \frac{1}{r\cosh(\frac{x}{r})} && \text{(D.2)}\\
&= \frac{\cosh(\frac{x}{r})}{r\cosh^2(\frac{x}{r})} - \frac{1}{r\cosh(\frac{x}{r})} && \text{(D.3)}\\
&= 0 && \text{(D.4)}
\end{aligned}
$$

So $f = f_0$ is a solution to the evolution equation for a rotationally symmetric surface:

$$f_t = \frac{f_{xx}}{1 + (f_x)^2} - \frac{1}{f}$$

(calculated in Chapter 2 “Rotationally Symmetric Surfaces”). Numerically it has been shown that for dimension $n > 2$ there is also a family of rotationally symmetric catenoid like surfaces with mean curvature 0. It has been shown that for $n > 2$ each catenoid surface becomes vertical at finite distance from the origin.

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