Local results for flows whose speed or height satisfies a bound of the form $\frac{c}{t}$

Miles Simon

Universität Freiburg

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Abstract

In this paper we prove local results for solutions to the Ricci flow (heat flow) whose speed (height) is bounded by $\frac{c}{t}$ for some time interval $t \in (0, T)$. These results are contained in chapter 7 of [16]. In [17] further results from [16] may be found. In particular, there we construct short time solutions to Ricci flow for a class of compact Riemannian manifolds with isolated conelike singularities. The resulting solutions satisfy a bound of this form (the speed is bounded by $\frac{c}{t}$ for some time interval $t \in (0, T)$).

1 Local results for Ricci flow and reaction diffusion equations in general

In [4], R.Hamilton introduced the Ricci flow. A smooth family of metrics $(M, g(t))_{t \in [0,T)}$ is a solution to the Ricci flow with initial value g_0, or is a Ricci flow of g_0 if

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ricci}(g(t)) \ \forall \ t \in [0, T),$$
$$g(0) = g_0. \tag{1.1}$$

Ricci flow has been extensively studied, and has led to many results in topology and geometry: see for example [1],[3], [4] [5], [7], [8], [9], [12], [13] (see [2], [10] and [11] for good expositions of the works [12],[13]).

In Theorem 6.1 (and the proof thereof) of [17], we see that for certain compact Riemannian manifolds with cone-like singularities, it is possible to obtain a short time solution to the Ricci flow, and also to obtain control over the curvature, volume and diameter for some well defined time interval. This control on the curvature

(curvature behaves like $\frac{c}{t}$) is global, and suffices to prove the main theorem, Theorem 6.1. there. We are, however, also interested in the local behaviour of such solutions (see also [14] and [15] for further examples of solutions to Ricci flow whose speed is bounded by $\frac{c}{t}$).

The following is taken directly from Chapter 7 of [16]).

Problem 1.1. *For the class of solutions whose curvature is bounded by $\frac{c}{t}$ on some time interval $[0, T)$: if a region is diffeomorphic to a Euclidean ball and has bounded geometry at the curvature and C^0 level at time zero, does some smaller region (contained in the initial region) remain bounded geometrically at the curvature and C^0 level for a well controlled time interval?*

In the paper [12], G.Perelman proved a local result (see [12] Section 10) in the setting that one has a solution to Ricci flow which is compact, has bounded curvature at each time, and for which the absolute value of the Riemannian curvature on $B(g_0)(x_0, r_0)$ is bounded by $\frac{1}{r_0}$, and for which $B(g_0)(x_0, r_0)$ is close in some C^0 sense to the Euclidean ball of radius r_0 (see Theorem 10.3 of [12] for an exact statement of the theorem). There, the behaviour like $\frac{c}{t}$ is not assumed, but is in fact proved in a seperate theorem, Theorem 10.1 of [12]. In Theorem 10.1 he proves that the curvature at a point x_0 behaves like $\frac{c}{t}$ (for $t \in [0, \delta^2 r^2]$, $\delta = \delta(n) > 0$) under the even weaker assumptions that the scalar curvature on $B(g_0)(x_0, r_0)$ is bounded from below by $-r_0^2$, and $B(g_0)(x_0, r_0)$ is close in some C^0 sense to the Euclidean ball of radius r_0: see theorem 10.1 of [12] for an exact statement of the theorem.

Let us define more precisely what we mean by: *the geometry is bounded on the curvature and C^0 level.*

Definition 1.2. *The Euclidean n-ball of radius $r > 0$ $U = {}^I B_r(0)$ with a Riemannian metric g_0 is geometrically bounded by c if*

$$\sup_U {}^{g_0}|\mathrm{Riem}(g_0)| \leq \frac{c}{r} \tag{1.2}$$

$$cI \leq g_0 \leq \frac{1}{c}I \tag{1.3}$$

where I is the standard metric on ${}^I B_r(0)$.

Clearly the set is topologically the same as a ball (per definition). Hence a Euclidean ball with a metric g_0 is *bounded geometrically* if the given Riemannian metric g_0 can be compared to the standard metric I on the curvature and C^0 level.

In the following theorem we show that if U is a Euclidean ball of radius one, and (U, g_0) is bounded geometrically by some constant c, and $(\bar{U}, g(t))_{t \in [0,T)}$ is a smooth solution to the Ricci flow whose curvature is bounded above by $\frac{c}{t}$, then, for a well controlled time interval, a Euclidean ball of radius $r(c, n) > 0$ (contained in the initial ball with the same middle point) with the metric $g(t)$ is bounded geometrically by $4c$.

Theorem 1.3. *Let* $(U = {}^{I}B_1(0), g_0)$ *be geometrically bounded by* c_0. *Define*

$$^{I}\mathrm{dist}(x) = \mathrm{dist}(I)(\partial U, x),$$

where the distance is taken with respect to the Euclidean metric I. *Let* $({}^{I}B_{1+\delta}(0), g(t))_{t \in [0,T)}$ *($\delta > 0$) be any smooth solution to the Ricci flow satisfying* $g(0) = g_0$ *and*

$$t^{g(t)}|\mathrm{Riem}(x, t)| \qquad \le c_0, \tag{1.4}$$
$$\tag{1.5}$$

for all $x \in {}^{I}B_{1+\delta}(0)$ *and for all* $t \in [0, T)$. *Then there exists an* $N = N(c_0, n)$, *such that*

$$^{g(t)}|\mathrm{Riem}(x, t)|{}^{I}\mathrm{dist}^2(x) < N,$$

for all $x \in U$ *with* $^{I}\mathrm{dist}^2(x) \ge Nt$, *and* $t \le T$. *In particular,* $({}^{I}B_{\frac{1}{4N}}(0), g(t))$ *is bounded geometrically by* $e^1 c_0$ *for all* $t \le \min(\frac{1}{4N}, T)$.

Proof. For the proof, $|\mathrm{Riem}(h)|$ will always refer to $^h|\mathrm{Riem}(h)|$. Choose N big, and assume that the theorem does not hold. Then there must be a first time $t_0 \in [0, \min(T, \frac{1}{4N}))$ and point $x_0 \in U$ where the theorem does not hold.

$$^{I}\mathrm{dist}^2(x_0) \ge Nt_0,$$
$$x_0 \in U,$$
$$|\mathrm{Riem}(x_0, t_0)|{}^{I}\mathrm{dist}^2(x_0) = N,$$

and

$$|\mathrm{Riem}(x, t)|{}^{I}\mathrm{dist}^2(x) \le N \ \forall \ (x, t) \in U[0, t_0] \ \text{with} {}^{I}\mathrm{dist}^2(x) \ge Nt \tag{1.6}$$

Let us rescale: $\tilde{g}(x, s) := ag(x, \frac{s}{a})$, for $s \in [0, at_0]$, and $\tilde{I} = aI$. We define: $^{\tilde{I}}\mathrm{dist}^2(x) := \mathrm{dist}^2(\tilde{I})(\partial U, x)$ (that is, the distance from the boundary of U to the point x with respect to the metric $\tilde{I}$) which gives us $^{\tilde{I}}\mathrm{dist}^2(x) = a^{I}\mathrm{dist}^2(x)$. Set

$$a := \frac{N}{{}^{I}\mathrm{dist}^2(x_0)},$$

(a is then bigger than N) and for $t \in [0, t_0]$, let $\tilde{t} := at$. Then $^{\tilde{I}}\mathrm{dist}^2(x_0) = N$, and hence $N = |\mathrm{Riem}(x_0, t_0)|{}^{I}\mathrm{dist}^2(x_0) = |\widetilde{\mathrm{Riem}}(x_0, \tilde{t}_0)|{}^{\tilde{I}}\mathrm{dist}^2(x_0) = |\widetilde{\mathrm{Riem}}(x_0, \tilde{t}_0)|N$, which implies that $|\widetilde{\mathrm{Riem}}(x_0, \tilde{t}_0)| = 1$. Furthermore, $\tilde{t}_0 = \frac{t_0 N}{{}^{I}\mathrm{dist}^2(x_0)} \le 1$, since $^{I}\mathrm{dist}^2(x_0) \ge Nt_0$. We also have

$$|\widetilde{\mathrm{Riem}}(x, \tilde{t})|{}^{\tilde{I}}\mathrm{dist}^2(x) = |\mathrm{Riem}(x, t)|{}^{I}\mathrm{dist}^2(x) \le N \tag{1.7}$$

for all $(x, \tilde{t}) \in U[0, \tilde{t}_0]$ which satisfy $^{\tilde{I}}\mathrm{dist}^2(x) \ge N\tilde{t}$ in view of the facts

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(i) ${}^{\tilde{I}}\widetilde{\mathrm{dist}}^{2}(x) \geq N\tilde{t} \iff {}^{I}\mathrm{dist}^{2}(x) \geq Nt$ and

(ii) $\tilde{t} \leq \tilde{t}_0 \iff t \leq t_0$ and

(iii) equation (1.6) holds).

Now we consider two cases:

- case 1: ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) \geq 2N\tilde{t}_0 (\iff \tilde{t}_0 \leq \frac{1}{2})$

- case 2: ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) < 2N\tilde{t}_0 (\iff \tilde{t}_0 > \frac{1}{2})$

Assume case one holds. Then for y satisfying ${}^{\tilde{I}}\mathrm{dist}^{2}(y) \geq \frac{N}{2}$ we see that (use $\tilde{t}_0 \leq \frac{1}{2}$)

$$ {}^{\tilde{I}}\widetilde{\mathrm{dist}}^{2}(y) \geq N\tilde{t}_0 \geq N\tilde{t} $$

for all $\tilde{t} \leq \tilde{t}_0$, and hence

$$ |\widetilde{\mathrm{Riem}}(y,\tilde{t})|{}^{\tilde{I}}\widetilde{\mathrm{dist}}^{2}(y) \leq N, $$

in view of (1.7). Hence,

$$ |\widetilde{\mathrm{Riem}}(y,\tilde{t})| \leq 2 \; \forall \; \tilde{t} \leq \tilde{t}_0, \tag{1.8} $$

in view of the assumption on y (notice that ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) = N \geq \frac{N}{2}$ and so $y = x_0$ is valid in (1.8): that is, $|\widetilde{\mathrm{Riem}}(x_0,\tilde{t})| \leq 2$ for all $\tilde{t} \leq \tilde{t}_0$). That is,

$$ |\widetilde{\mathrm{Riem}}(y,\tilde{t})| \leq 2 \; \forall \; \tilde{t} \leq \tilde{t}_0, \quad \forall y \in \tilde{B}_{\frac{N}{4}}(x_0). $$

Using the fact that

$$ \frac{I}{c_0} \leq g_0 \leq c_0 I $$

we obtain that

$$ \frac{\tilde{I}}{c_0} \leq \tilde{g}_0 \leq c_0 \tilde{I} $$

and

$$ \sup_{y \in \tilde{B}_{\frac{N}{4}}(x_0)} |\mathrm{Riem}(\tilde{g}_0)|(y) \leq \frac{c_0}{N}, $$

since $\tilde{g}_0 = ag_0$, and $a = \frac{N}{{}^{I}\mathrm{dist}^{2}(x_0)} \geq N$ and $(U,g_0) = ({}^{I}B_1(0),g_0)$ is geometrically bounded by c_0. We also have $|\widetilde{\mathrm{Riem}}(x_0,\tilde{t}_0)| = 1$. This contradicts Lemma 1.4, if $N = N(c_0,n)$ is chosen large enough.

So assume case two holds. This is equivalent to $\frac{1}{2} < \tilde{t}_0$. Then, for $\tilde{t} \leq \frac{1}{2}$, we still have the estimate

$$ |\widetilde{\mathrm{Riem}}(y,\tilde{t})| \leq 2, \quad \forall y \; \text{ with } \; {}^{\tilde{I}}\mathrm{dist}^{2}(y) \geq \frac{N}{2} $$

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(this may be seen as follows:

$${}^{\tilde{I}}\mathrm{dist}^2(y) \geq \frac{N}{2} \geq N\tilde{t},$$

since $\tilde{t} \leq \frac{1}{2} < \tilde{t}_0$, and so, using (1.7), we obtain the estimate).

For $\tilde{t}_0 \geq \tilde{t} > \frac{1}{2}$, we have

$$|\tilde{\mathrm{Riem}}(y,\tilde{t})| \leq \frac{c_0}{\tilde{t}} \leq 2c_0,$$

in view of the assumption (1.4). W.l.o.g. $c_0 \geq 1$. Hence

$$|\tilde{\mathrm{Riem}}(y,\tilde{t})| \leq 2c_0, \forall t \in [0,\tilde{t}_0), \quad \forall\, y \in \tilde{B}_{\frac{N}{4}}(x_0).$$

Furthermore, $|\tilde{\mathrm{Riem}}(x_0,\tilde{t}_0)| = 1$. Once again, this leads to a contradiction for $N = N(c_0, n)$ chosen large enough, in view of Lemma 1.4. (as in case 1, we have

$$\frac{\tilde{I}}{c_0} \leq \tilde{g}_0 \leq c_0 \tilde{I},$$
$$\sup_{\tilde{B}_{\frac{N}{4}}(x_0)} |Riem(\tilde{g}_0)| \leq \frac{c_0}{N}, \tag{1.9}$$

and so we may apply the lemma). $\qquad\square$

The following lemma, which is used to help prove the locality theorem above, is a standard result from the theory of parabolic equations

Lemma 1.4. *Let* $(M, g(t))_{t \in [0,T)}$, *be a complete smooth solution to Ricci flow and* (U, x_0, I), $U \subset M$ *be isometric to to the (open) Euclidean ball* $B_N(0)$ *of radius* N, $(x_0 \sim 0)$ *and centre* 0. *Assume that:*

$$\frac{I}{c_0} \leq g_0 \leq c_0 I$$
$$\sup_U |\mathrm{Riem}(\cdot,t)| \leq c_1 \,\forall\, t \in [0,T),$$
$$\sup_U |\mathrm{Riem}|(\cdot,0) \leq \varepsilon.$$

Then for all $\sigma^2 \geq \varepsilon^2$ *there exists* $a = a(c_0, c_1, n)$ *and* $N_* = N_*(\sigma, c_0, c_1, n) > 0$ *such that if* $N \geq N_*$ *then*

$$|\mathrm{Riem}|^2(x_0,t) \leq \sigma^2 \exp^{at} \,\forall\, t \in [0,T) \tag{1.10}$$

In particular, if $\varepsilon \leq \exp^{-a} \delta$ *for some* $\delta > 0$ *then (choose* $\sigma = \exp^{-a} \delta$*: then* N_* *is a constant depending only on* c_0, c_1, n, δ *) there is a* $N_* = N_*(c_0, c_1, n, \delta) > 0$ *such that*

$$|\mathrm{Riem}(x_0,t_0)| \leq \delta^2, \,\forall\, t_0 \in [0,T) \cap [0,1)$$

if $N \geq N_*$.

Proof. Clearly, in view of the conditions in the assumption, and the equation of evolution for the metric, we have

$$\frac{1}{c_2(c_0, c_1, n)} I \le g(t) \le c_2(c_0, c_1, n) I \ \forall \ t \in [0, T)$$

Set

$$f(\cdot, t) := |\text{Riem}|^2(\cdot, t) - \sigma^2(1 + \rho^2) \exp^{at},$$

where a is to be chosen. and $\rho(x, t) := \text{dist}(g(t))(x, x_0)$. Then

$$
\begin{aligned}
\frac{\partial}{\partial t} f \le \ & {}^g\Delta f + 4|\text{Riem}|^3 - a\sigma^2(1 + \rho^2) \exp^{at} -2\sigma^2(\frac{\partial}{\partial t}\rho)\rho \exp^{at} + \sigma^2 \exp^{at} \, {}^g\Delta(\rho^2) \\
\le \ & {}^g\Delta f + 4c_1|\text{Riem}|^2 - a\sigma^2(1 + \rho^2) \exp^{at} \\
& + 4\sigma^2(n-1)c_1\rho^2 \exp^{at} + c_1\sigma^2 \exp^{at} c(c_1, n)\rho \\
\le \ & {}^g\Delta f + 4c_1|\text{Riem}|^2 - \frac{a}{2}\sigma^2(1 + \rho^2) \exp^{at}
\end{aligned}
$$

for all $x \in B_{\frac{N}{2c_2^2}}(0)$ for appropriately chosen $a = a(n, c_1)$, where here we have used the Hessian comparison principle in order to estimate ${}^g\Delta(\rho^2)$, and the fact that all distance minimising geodesics (in terms of $g(t)$) between 0 and points in ${}^I B_{\frac{N}{2c_2^2}}(0)$ must lie in ${}^I B_N(0)$ (this last fact may be seen as follows: the length of a ray from 0 to $p \in \partial B_{\frac{N}{2c_2^2}}(0)$ is trivially bounded from above by $c_2\frac{N}{2c_2^2} = \frac{N}{2c_2}$. For any curve starting from 0 which reaches the boundary of $B_N(0)$, the length is bounded from below by $\frac{1}{c_2}N$). Using the definition of f, we get

$$\frac{\partial}{\partial t} f \le {}^g\Delta f + 4c_1 f + 4c_1\sigma^2(1 + \rho^2) \exp^{at} -\frac{a}{2}\sigma^2(1 + \rho^2) \exp^{at} < 4c_1 f,$$

for appropriately chosen $a = a(n, c_1)$. Now for $\sigma^2 \ge \varepsilon^2$ we have $f(\cdot, 0) < 0$ on $B_{\frac{N}{2c_2^2}}(0)$. Now choose $N = N(\sigma, c_0, c_1, n)$ so large that $f(\cdot, t) < 0$ on $\partial B_{\frac{N}{2c_2^2}}(0)$ for all $t \in [0, T)$. The maximum principle then implies that f is less than zero for all $t \in [0, T)$, for all $x \in B_{\frac{N}{2c_2^2}}(0)$. Note that although ρ^2 is not smooth everywhere, using a trick of Calabi, we may still draw the same conclusion: see the proof of a Theorem 7.1 in [15] (essentially we define a new function $\tilde{\rho}(x, t) = \rho(x, q, t) + \rho(q, x_0, t)$ for some appropriately chosen q so that $\tilde{\rho}$ is smooth in a small neighbourhood of (p_0, t_0) where (p_0, t_0) is the first time and point where $f(p_0, t_0) = 0$. Thn we define:

$$\tilde{f}(\cdot, t) := |\text{Riem}|^2(\cdot, t) - \sigma^2(1 + \tilde{\rho}^2) \exp^{at},$$

and argue with $\tilde{f}$: due to the triangle inequality we have $\tilde{f} \le f$ and hence $\tilde{f} < 0$ for $t \le t_0$. Furthermore $\tilde{f}(x_0, t_0) = 0$, as q lies on a shortest geodesic between x_0 and p_0 at time t_0. Hence we may apply the maximum principle and still obtain a contradiction. Hence $f < 0$. $\qquad\square$

So we see that if a local region is relatively well controlled at time zero, and the curvature behaves globally like $\frac{c}{t}$ near time zero, then we can show that a (well defined) smaller region remains well behaved for a a well defined time interval.

The following remarks did not appear originally in [16].

Remark 1.5. *Notice that the condition $^I\mathrm{dist}^2(x) \geq Nt$ can be removed, since if $^I\mathrm{dist}^2(x) \leq Nt$ then $|\mathrm{Riem}(x,t)|^I\mathrm{dist}^2(x) \leq |\mathrm{Riem}(x,t)|Nt \leq c_0N$ if the conditions of the theorem are satisfied.*

Remark 1.6. $^I\mathrm{dist}^2(x) = (1 - |x|)^2$.

2 Local results for solutions f to the heat equation which satisfy a bound of the form $f \leq \frac{1}{t}$.

The following is taken directly from Chapter 7 of [16].

In fact, the important bound which leads to the locality result (Theorem 1.3), is the bound of the form $|\mathrm{Riem}| \leq \frac{c}{t}$. The argument was a scaling argument which used the parabolic maximum principle, and didn't really have anything to do with Ricci-flow. We illustrate this somewhat more precisely, by showing that a similar result holds for the heat flow.

Lemma 2.1. *Let f be a smooth solution to the heat flow on $B_2(0)[0,1]$ with*

$$ft|_{B_1(0)[0,1]} \leq 1 \quad \forall t \in [0,1], \tag{2.1}$$
$$f(x,t) \geq 0, \quad \forall x \in B_2(0), t \in [0,1] \tag{2.2}$$
$$\sup_{\bar{B}_1(0)} f(x,0)(1 - |x|^2)^2 \leq 1 \tag{2.3}$$

Then

$$\sup_{\bar{B}_1(0)} f(x,t)((1 - |x|^2)^2 - 50nt) \;\leq 50n$$

for all $t \in [0,1]$.

Proof. Set
$$l(x,t) := f(x,t)((1 - |x|^2)^2 - Mt) - M,$$
where $M = 50n$. Then $l \in C^\infty(\bar{B}_{\frac{3}{2}}(0)[0,1])$, and $l(x,0) \leq -50$, and $l(\cdot,t) \leq -50n < 0$ on $\partial(B_1(0))$ for all $t \in [0,1]$. Hence, if there is a time and point $(x,t) \in \bar{B}_1(0)[0,1]$ where $l(x,t) \geq 0$, then due to compactness, there must be a first time t_0 and point $x_0 \in B_1(0)$, where $l(x_0,t_0) = 0$ (possibly there is more than one point x_0, but there exists at least one point x_0). Assume that (x_0,t_0) is such a point, and (x_0,t_0) satisfies $(1 - |x_0|^2)^2 - Mt_0 \leq \frac{M}{2}t_0$. Then

$$l(x_0,t_0) \quad = f(x_0,t_0)((1 - |x_0|^2)^2 - Mt_0) - M$$

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$$\leq f(x_0, t_0)\frac{M}{2}t_0 - M$$
$$\leq -\frac{M}{2} < 0, \tag{2.4}$$

(in view of the conditions (2.1) and (2.2)) which contradicts the fact that $l(x_0, t_0) = 0$. So we may assume, without loss of generality, that

$$(1 - |x_0|^2)^2 - Mt_0 \geq \frac{M}{2}t_0 \tag{2.5}$$

Notice that this implies

$$\frac{2}{3}(1 - |x_0|^2)^2 \geq Mt_0$$

which further implies that

$$(1 - |x_0|^2)^2 - Mt_0 \geq \frac{1}{3}(1 - |x_0|)^2 \tag{2.6}$$

We calculate the evolution equation for l for such an (x_0, t_0).

$$
\begin{aligned}
\frac{\partial}{\partial t}l =\ & \Delta(l) - Mf - f\Delta((1 - |x|^2)^2) - 2\nabla_i f \nabla_i (1 - |x|^2)^2 \\
=\ & \Delta(l) - Mf - f(8|x|^2 - 4n(1 - |x|^2)) - 2\nabla_i f \nabla_i (1 - |x|^2)^2 \\
\leq\ & \Delta(l) - Mf + 4nf - 2\nabla_i f \nabla_i (1 - |x|^2)^2 \\
\leq\ & \Delta(l) - \frac{M}{2}f - 2\nabla_i f \nabla_i (1 - |x|^2)^2 \\
=\ & \Delta(l) - \frac{M}{2}f - \frac{2\nabla_i (f((1 - |x|^2)^2 - Mt))\nabla_i(1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)} \\
& + 2f\frac{\nabla_i(1 - |x|^2)^2\nabla_i(1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)} \\
=\ & \Delta(l) - \frac{M}{2}f - \frac{2\nabla_i (f((1 - |x|^2)^2 - Mt))\nabla_i(1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)} \\
& + 8f|x|^2\frac{(1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)} \tag{2.7}
\end{aligned}
$$

and hence, in view of the inequality (2.6),

$$
\begin{aligned}
\frac{\partial}{\partial t}l\ & \leq \Delta(l) - \frac{M}{2}f - \frac{2\nabla_i l \nabla_i (1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)} + 24f|x|^2 \\
& \leq \Delta(l) - \frac{2\nabla_i l \nabla_i (1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)} + 24f - \frac{M}{2}f \\
& < \Delta(l) - \frac{2\nabla_i l \nabla_i (1 - |x|^2)^2}{((1 - |x|^2)^2 - Mt)}. \tag{2.8}
\end{aligned}
$$

As $l(x_0, t_0)$ is a local maximum, we obtain a contradiction (note that $((1 - |x_0|^2)^2 - Mt_0) > 0$ due to the assumption (2.5)). $\quad\square$

In fact a scaled version of this lemma is true whenever we have at most polynomial growth of f in t, as the following lemma shows.

Lemma 2.2. *Let f be a smooth solution to the heat flow on $B_2(0)[0,1]$ with*

$$ft^p|_{B_1(0)[0,1]} \leq 1$$
$$f(x,t) \geq 0, \quad \forall x \in B_2(0), t \in [0,1] \tag{2.9}$$
$$\sup_{x \in \bar{B}_1(0)} (f(x,0)+1)^{\frac{1}{p}}(1-|x|^2)^2 \leq 2 \tag{2.10}$$

($p \geq 1$). Then

$$\sup_{x \in \bar{B}_1(0)} (f(x,t)+1)^{\frac{1}{p}}((1-|x|^2)^2 - M(n,p)t) \leq c(n,p) \quad \forall t \in [0,1].$$

Proof. If

$$ft^p \leq 1$$

then

$$(1+f)t^p \leq 2 \ \forall \ t \leq 1,$$

and hence

$$(1+f)^{\frac{1}{p}}t \leq 2.$$

Setting

$$\tilde{f} := (1+f)^{\frac{1}{p}},$$

we have

$$\tilde{f}t \leq 2,$$
$$\tilde{f}(x,0)(1-|x|^2)^2 \leq 2. \tag{2.11}$$

Using the fact that f solves the heat equation and $\tilde{f} \geq 1$, we calculate

$$\frac{\partial}{\partial t}\tilde{f} = \Delta(\tilde{f}) - \frac{1}{p}(\frac{1}{p}-1)\frac{|\nabla \tilde{f}|^2}{\tilde{f}}. \tag{2.12}$$

Define

$$\tilde{l} := \tilde{f}(x,t)((1-|x|^2)^2 - Mt) - M,$$

where $M = M(n,p)$ is a constant to be chosen. Let (x_0, t_0) be a first time and point where $\tilde{l} = 0$. Arguing as above, we obtain

$$\frac{\partial}{\partial t}\tilde{l} < \Delta(\tilde{l}) - 2(\nabla_i \tilde{l})V_i - \frac{M}{2}\tilde{f} - \frac{1}{p}(\frac{1}{p}-1)((1-|x|^2)^2 - Mt)\frac{|\nabla \tilde{f}|^2}{\tilde{f}} \tag{2.13}$$

where

$$V_i = \frac{\nabla_i(1-|x|^2)^2}{((1-|x|^2)^2 - Mt)}.$$

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Arguing as in the previous lemma, we see that we may assume that $((1 - |x|^2)^2 - Mt) \geq \frac{1}{3}(1 - |x|^2)^2$, and so V_i can be considered to be smooth in a small neighbourhood of (x_0, t_0). But the last term in the above inequality can also be estimated, as the following calculation shows.

$$\frac{c(p)}{\tilde{f}}((1 - |x|^2)^2 - Mt)\nabla_i \tilde{f} \nabla_i \tilde{f} = \quad \frac{c(p)}{\tilde{f}}\nabla_i(((1 - |x|^2)^2 - Mt)\tilde{f})\nabla_i \tilde{f}$$
$$-c(p)\nabla_i((1 - |x|^2)^2 - Mt)\nabla_i \tilde{f}. \quad (2.14)$$

Remembering that we may assume that $((1 - |x|^2)^2 - Mt) \geq \frac{1}{3}(1 - |x|^2)^2$, we get

$$-c(p)\nabla_i((1 - |x|^2)^2 - Mt)\nabla_i \tilde{f}$$
$$= \quad -\frac{c(p)}{((1 - |x|^2)^2 - Mt)}\nabla_i((1 - |x|^2)^2 - Mt)\nabla_i(\tilde{f}((1 - |x|^2)^2 - Mt))$$
$$+c(p)\frac{\tilde{f}}{((1 - |x|^2)^2 - Mt)}\nabla_i(1 - |x|^2)^2\nabla_i(1 - |x|^2)^2$$
$$\leq \quad -\frac{3c(p)}{((1 - |x|^2)^2}\nabla_i((1 - |x|^2)^2 - Mt)\nabla_i(\tilde{f}((1 - |x|^2)^2 - Mt))$$
$$+32c(p)\tilde{f} \quad (2.15)$$

Substituting this inequality into (2.14), we get

$$\frac{c(p)}{\tilde{f}}((1 - |x|^2)^2 - Mt)\nabla_i \tilde{f} \nabla_i \tilde{f} \leq W_i \nabla_i \tilde{l} + 12c(p)\tilde{f},$$

is a smooth vector field which is defined in a small neighbourhood (space and time) of the point (x_0, t_0). Substituting this inequality into equation (2.13) we get

$$\frac{\partial}{\partial t}(\tilde{l}) \quad < \quad \Delta(\tilde{l}) + \nabla_i(\tilde{l})Z_i. \quad (2.16)$$

at the point (x_0, t_0), where Z is a smooth vector field which is defined in a small neighbourhood (space and time) of the point (x_0, t_0). This gives us a contradiction, as (x_0, t_0) is a local maximum for $\tilde{l}$.

$\square$

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Mathematisches Institut, Eckerstr. 1, 79104 Freiburg im Br., Germany
e-mail: msimon at gmx.de