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Ricci flow of almost non-negatively curved three manifolds

By *Miles Simon* at Freiburg

Abstract. In this paper we study the evolution of almost non-negatively curved (possibly singular) three dimensional metric spaces by Ricci flow. The non-negatively curved metric spaces which we consider arise as limits of smooth Riemannian manifolds $(M_i, {}^i g)$, $i \in \mathbb{N}$, whose Ricci curvature is bigger than $-1/i$, and whose diameter is less than d_0 (independent of i) and whose volume is bigger than $v_0 > 0$ (independent of i). We show for such spaces, that a solution to Ricci flow exists for a short time $t \in (0, T)$, that the solution is smooth for $t > 0$, and has $\mathrm{Ricci}\big(g(t)\big) \geqq 0$ and $\mathrm{Riem}\big(g(t)\big) \leqq c/t$ for $t \in (0, T)$ (for some constant $c = c(v_0, d_0, n)$). This allows us to classify the topological type and the differential structure of the limit manifold (in view of the theorem of Hamilton [10] on closed three manifolds with non-negative Ricci curvature).

1. Introduction and statement of results

In the papers [9] and [10], R. Hamilton showed using the Ricci flow that

Theorem A ([10], Theorem 1.2). *If M^n, $n = 3(4)$ is a closed n-dimensional Riemannian manifold with non-negative Ricci curvature (non-negative curvature operator) then M^3 is diffeomorphic to a quotient of S^3, $S^2 \times \mathbb{R}$, or $\mathbb{R}^3$ by a group of fixed point free isometries acting properly discontinuously (M^4 is diffeomorphic to a quotient of one of the spaces S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $S^3 \times \mathbb{R}^1$, $S^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$ by a group of fixed point free isometries acting properly discontinuously) in the standard metric.*

It is interesting to note that in order to apply the theorem for $n = 3$ we only require information on the Ricci curvatures (not the sectional curvatures). The theorem implies that only certain three manifolds admit Riemannian metrics with non-negative Ricci curvature. This is not the case for negative Ricci curvature, as proved by Lohkamp in [16]: he proved that every closed manifold of dimension $n \geqq 3$ admits a Riemannian metric of negative Ricci curvature.

We say that a smooth family of metrics $\big(M, g(t)\big)_{t \in [0,T)}$ is a solution to the Ricci flow with initial value g_0, or is a Ricci flow of g_0 if

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ricci}\big(g(t)\big), \quad \forall t \in [0, T), \tag{1.1}$$
$$g(0) = g_0.$$

In three (and four) dimensions, there are similar results to Theorem A requiring less regularity of the starting metric (see Theorem B below).

Definition 1.1. Let M^n be closed, g a Lipschitz Riemannian metric on M. We say that $\mathrm{Ricci}(g) \geqq k$ ($\mathcal{R}(g) \geqq k$) if there exists smooth $({}^i g)_{i \in \mathbb{N}}$ with

(a) $|{}^i g - g|_{C^0(M)} \overset{i \to \infty}{\longrightarrow} 0$,

(b) $\sup_{i,j \in \mathbb{N}} {}^g|\Gamma({}^i g) - \Gamma({}^j g)|_{C^0(M)} < \infty$ and

(c) $\mathrm{Ricci}({}^i g) \geqq k - \frac{1}{i}$ $\quad (\mathcal{R}({}^i g) \geqq k - \frac{1}{i})$.

Here $\mathcal{R}$ refers to the curvature operator.

Theorem B ([21], Theorem 1.3). *Let $n = 3(4)$. The classification of Theorem* A *remains true if we allow Lipschitz metrics with non-negative Ricci curvature* (*non-negative curvature operator*) *in the sense of Definition* 1.1.

In this paper we will define a Ricci flow for a larger class of almost non-negatively Ricci curved (possibly singular) three dimensional metric spaces (M, d). The spaces we are interested in arise as Gromov-Hausdorff limits of sequences $(M_i, g_i) \in \mathcal{M}\left(n, d_0, v_0, -\frac{1}{i}\right)$, $i \in \mathbb{N}$ where

Definition 1.2. For $n \in \mathbb{N}$, $d_0 \in \mathbb{R}^+$, and $k \in \mathbb{R}$ let $\mathcal{M}(n, d_0, k)$ denote the space of smooth n-dimensional Riemannian manifolds of dimension n with diameter bounded above by d_0 and Ricci curvature not less than k. For $v_0 \in \mathbb{R}^+$, $\mathcal{M}(n, d_0, v_0, k)$ is the set of $(M, g) \in \mathcal{M}(n, d_0, k)$ which satisfy $\mathrm{vol}(M, g) \geqq v_0$.

It is well known that the space $\mathcal{M}(n, d_0, k)$ is precompact in the Gromov-Hausdorff space. That is, given a sequence of smooth n-dimensional Riemannian manifolds

$$(M_i^n, g_i)_{i \in \mathbb{N}} \in \mathcal{M}(n, d_0, k),$$

there exists a metric space (X, d_∞) and a subsequence of (M_i^n, g_i) (which we also call (M_i^n, g_i) for ease of reading) such that $\big(M_i^n, d(g_i)\big) \overset{i \to \infty}{\longrightarrow} (X, d_\infty)$, in the Gromov-Hausdorff sense, where here $d(g)$ denotes the distance function (metric) $d(g) : M \times M \to \mathbb{R}_0^+$ arising from the Riemannian metric g (see Appendix A). The Gromov-Hausdorff (space) distance between two metric spaces is defined in Appendix A. It is a very weak measure of how close two metric spaces are to being isometric to one another.

Definition 1.3. For $n \in \mathbb{N}$, $d_0 \in \mathbb{R}^+$, and $k \in \mathbb{R}$, $\bar{\mathcal{M}}(n, d_0, k)$ is the closure of $\mathcal{M}(n, d_0, k)$ with respect to the GH limit.

It is possible that the limit space (X, d_∞) does not enjoy the regularity properties of the spaces occurring in the converging sequence, as one sees in the following example.

Example 1.4. Let $(S^n, g_i)_{i \in \mathbb{N}}$ be a sequence of spheres with Riemannian metrics, where the metrics are chosen so that

- the sectional curvature is non-negative,

- the manifolds are becoming cone like in a fixed compact region (topologically a closed disc) as $i \to \infty$, and stay smooth away from this region (see the remark below),

- the diameter is bounded above by $0 < d_0 < \infty$ and the volume bounded below by $v_0 > 0$ where d_0, v_0 are constants independent of $i \in \mathbb{N}$.

Then $(S^n, d(g_i))$ converges in the Gromov-Hausdorff space to (S^n, d), where d is a (non-standard) metric on the sphere, and there exists a Riemannian metric g which is smooth away from the tip, induces d, but cannot be extended in a C^0 way to the tip. It is not possible to find a C^0 Riemannian metric g which induces d.

Remark 1.5. The induced Riemannian metric on the cone

$$C^n = \{(x, c^2|x|) \mid x \in \mathbb{R}^n\} \quad (c^2 > 0)$$

is C^∞ everywhere away from the tip $(\vec{0}, 0)$ of the cone, but cannot be extended continuously to this tip for $n \geqq 2$.

In [12], [27] and [28] the authors introduce other notions of "spaces with Ricci curvature bounded below". In those papers, the spaces that one considers are metric spaces (X, d) together with a measure m. One can measure the distance between two probability measures μ, ν using the L^2 Wasserstein-distance function d_W:

$$d_W(\mu, \nu) := \inf_q \left(\int_{M \times M} d^2(x, y)\, dq(x, y) \right)^{\frac{1}{2}},$$

where the infimum is taken over all *couplings* q of μ and ν. A *coupling* of μ and ν is a probability measure on $M \times M$ whose marginals (i.e. image measures under the projections) are the given measure μ and ν. Let $P_2(M)$ be the space of probability measures on M equipped with the distance d_W. The curvature bound from below is then defined using convexity properties of entropy functionals. For example, one definition in Sturm [27] is as follows: define the entropy

$$\mathrm{Ent}(\nu \mid m) := \int_M \frac{d\nu}{dm} \log\left(\frac{d\nu}{dm}\right) dm.$$

Then we say (X, d, m) has *Ricci curvature bounded from below by K in the weak sense* if for any pair $\nu_0, \nu_1 \in P_2(M)$ with non-infinite entropy, there exists a geodesic $\Gamma : [0, 1] \to P_2(M)$ connecting ν_0 and ν_1 such that

$$\mathrm{Ent}(\Gamma(t) \mid m) \leqq (1 - t)\,\mathrm{Ent}(\Gamma(0) \mid m) + t\,\mathrm{Ent}(\Gamma(1) \mid m) - \frac{K}{2} t(1 - t)\, d_W^2(\Gamma(0), \Gamma(1)),$$

for all $t \in [0,1]$ (see [27] for more details). A similar definition may be found in [17], Definition 0.7. Both of these definitions have the advantage of allowing very general spaces (not necessarily smooth Riemannian manifolds). A further advantage is that this condition is closed under Gromov-Hausdorff convergence (defined in Apendix A): if $(X_i, d_i, m_i) \to (X, d, m)$ as $i \to \infty$, and the (X_i, d_i, m_i) all have Ricci curvature bounded from below by K in the weak sense, then (X, d, m) has Ricci curvature bounded from below by K in the weak sense. This is not true in the smooth case, as the example above illustrates (the Ricci curvature is not defined on the tip of the cone in the above example). Furthermore, if (X, g) is a smooth Riemannian manifold, ${}^g d$ is the metric induced by g, and V_g is the volume form induced by g, then: $(X, {}^g d, V_g)$ has curvature bounded from below in the weak sense if and only if (X, g) has Ricci curvature bounded from below in the smooth sense.

In this paper we show that it is possible to evolve spaces $(X, d) \in \bar{\mathcal{M}}(n, d_0, v_0, k)$ by Ricci flow. In order to do this, we prove a number of estimates on the rate at which geometrical quantities change under the Ricci flow. Many of these estimates are obtained using the parabolic maximum principle in a smooth setting on a smooth manifold (for example, estimate (1.6) is obtained by examining the evolution equation of the Ricci curvature). For this reason, the setting of [17], [27] and [28] is not immediately appropriate for this paper. In particular, the underlying spaces in that setting are not necessarily manifolds (see [18] for results on Ricci flow in the setting of [17], [27] and [28]).

We prove Theorem 7.2.

Theorem 1.6. *Let $(M_i, {}^i g_0)$ be a sequence of closed three (or two) manifolds satisfying*

$$\operatorname{diam}(M_i, {}^i g_0) \leqq d_0,$$

$$\operatorname{Ricci}({}^i g_0) \quad (\sec({}^i g_0)) \geqq -\varepsilon(i)\, {}^i g_0,$$

$$\operatorname{vol}(M_i, {}^i g_0) \geqq v_0 > 0,$$

where $\varepsilon(i) \to 0$, as $i \to \infty$. Then there exists an $S = S(v_0, d_0) > 0$ and $K = K(v_0, d_0)$ such that the maximal solutions $\big(M_i, {}^i g(t)\big)_{t \in [0, T_i)}$ to Ricci-flow satisfy $T_i \geqq S$, and

$$\sup_{M_i} \big|\operatorname{Riem}\big({}^i g(t)\big)\big| \leqq \frac{K}{t},$$

for all $t \in (0, S)$. In particular the Hamilton limit solution

$$\big(M, g(t)\big)_{t \in (0,S)} = \lim_{i \to \infty} \big(M_i, {}^i g(t)\big)_{t \in (0,S)}$$

(see [13]) exists (after taking a subsequence). It satisfies the estimates

$$\sup_{M} \big|\operatorname{Riem}\big(g(t)\big)\big| \leqq \frac{K}{t}, \tag{1.2}$$

$$\operatorname{Ricci}\big(g(t)\big) \geqq 0 \quad \big(\sec\big(g(t)\big) \geqq 0\big), \tag{1.3}$$

for all $t \in (0, S)$ and $\big(M, g(t)\big)$ is closed. Hence, if $M = M^3$, then M^3 is diffeomorphic to a quotient of one of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by group of fixed point free isometries acting properly *discontinuously. Furthermore*

$$d_{GH}\big((M,d(g(t))),(X,d_\infty)\big)\to 0 \tag{1.4}$$

as $t\to 0$ *where* $(X,d_\infty)=\lim_{i\to\infty}(M_i,d({}^ig_0))$ *(the Gromov-Hausdorff limit).*

As a corollary we obtain the following classification theorem.

Corollary 1.7. *For all* $0<v_0<\infty$, $0<d_0<\infty$ *there exists an* $\varepsilon=\varepsilon(v_0,d_0)>0$ *such that if* (M^3,g) *is closed and* $(M,g)\in\mathcal{M}(3,d_0,v_0,-\varepsilon)$ *then* M *is diffeomorphic to a quotient of* S^3, $S^2\times\mathbb{R}$ *or* $\mathbb{R}^3$ *by a group of fixed-point free isometries acting properly discontinuously.*

Proof. Assume the corollary is not true. Then there exists a sequence $(M_i,{}^ig_0)\in\mathcal{M}(3,d_0,v_0,-\varepsilon(i))$, $i\in\mathbb{N}$, with $\varepsilon(i)\to 0$ as $i\to 0$ such that each of the M_i is not diffeomorphic to any of the manifolds listed in the theorem. But then we may apply Theorem 1.6 to obtain that a subsequence of $(M_i,{}^ig(t))_{t\in(0,S)}$ converges in the sense of Hamilton to a solution $(M,g(t))_{t\in(0,S)}$. This implies in particular that M_i is diffeomorphic to M for i big enough. This is a contradiction. □

A scale invariant form of this corollary is:

Corollary 1.8. *Let* d_0 *be given. There exist* $0<\varepsilon_2=\varepsilon_2(d_0)<\infty$ *such that if* (M^3,g) *satisfies*

$$\begin{aligned}\mathrm{Ricci}\cdot\mathrm{vol}^{\frac{2}{3}}&\geqq-\varepsilon_2,\\ \mathrm{diam}^3&\leqq d_0^3\cdot\mathrm{vol}\end{aligned}\tag{1.5}$$

then M *is diffeomorphic to a quotient of* S^3, $S^2\times\mathbb{R}$ *or* $\mathbb{R}^3$ *by a group of fixed-point free isometries acting properly discontinuously.*

In [29], [22], [23] and [8], Fukaya, Shioya and Yamaguchi obtained similar results (and more) for three manifolds with almost non-negative sectional curvature. For example, in [8] Fukaya and Yamaguchi proved:

Theorem C ([8], Corollary 0.13). *There exists an* $\varepsilon>0$ *such that if* (M^3,g) *is a Riemannian manifold whose diameter is not larger than* 1, *and has* $\sec\geqq-\varepsilon$, *then a finite covering of* M *is either*

- *homotopic to an* S^3 *or*
- *diffeomorphic to one of*
  - (a) T^3,
  - (b) $S^1\times S^2$,
  - (c) Nil.

Hence, using that the Poincaré Conjecture is correct (see Perelman's papers [19], [20]) (that is, a homotopy S^3 is homeomorphic to S^3), we have a good topological classification of 3-manifolds with $\sec \cdot \operatorname{diam}^2 \geqq -\varepsilon$ and ε small enough.

Notice that Theorem C does not require a bound from below on the volume.

Definition 1.9. If

$$\operatorname{vol}(M_i, g_i) \xrightarrow{i \to \infty} 0$$

for a sequence of smooth Riemannian manifolds (M_i, g_i) then we say that the sequence is a collapsing sequence, or that the sequence collapses. If there exists a $v_0 > 0$ such that

$$\operatorname{vol}(M_i, g_i) \geqq v_0, \quad \forall i \in \mathbb{N},$$

then we say that the sequence is a non-collapsing sequence, or that the sequence does not collapse.

The papers [29], [22], [23] and [8] use results and methods from the theory of convergence/collapse of Riemannian manifolds, and the theory of Alexandrov spaces (not Ricci flow).

In order to show that the Ricci-curvature of our solution is non-negative for all $t > 0$ (Equation (7.1)), we use the following lemma (Lemma 5.2 of this paper), which may be of independent interest.

Lemma 1.10. *Let g_0 be a smooth metric on a 3-dimensional manifold M^3 which satisfies*

$$\begin{gathered} \operatorname{Ricci}(g_0) \geqq -\frac{\varepsilon_0}{4} g_0 \\ (\sec(g_0) \geqq -\frac{\varepsilon_0}{4} g_0) \end{gathered} \tag{1.6}$$

for some $0 < \varepsilon_0 < 1/100$, and let $(M, g(\cdot, t))_{t \in [0,T)}$ be a solution to Ricci flow with $g(0) = g_0(\cdot)$. Then

$$\operatorname{Ricci}(g(t)) \geqq -\varepsilon_0(1 + kt)g(t) - \varepsilon_0(1 + kt)t\mathrm{R}(g(t))g(t), \qquad \forall t \in [0, T) \cap [0, T')$$

$$(\sec(g(t)) \geqq -\varepsilon_0\left(\frac{1}{2} + kt\right)g(t) - \varepsilon_0\left(\frac{1}{2} + kt\right)t\mathrm{R}(g(t))g(t), \quad \forall t \in [0, T) \cap [0, T'))$$

where $k = 100$ and $T' = T'(100) > 0$ is a universal constant.

2. Methods and structure of this paper

In this paper we will chiefly be concerned with metric spaces (X, d_∞) which arise as Gromov-Hausdorff limits of non-collapsing sequences of Riemannian manifolds $(M_i^3, g_i) \in \mathcal{M}_i(3, d_0, v_0 - \varepsilon(i))$ where $\varepsilon(i) \to 0$ as $i \to \infty$. In particular, we wish to flow such metric spaces (X, d_∞) by Ricci flow. As we saw in the previous section (see Example

1.4) such limits can be quite irregular (it is possible that the limit manifold is a non-C^0 Riemannian manifold). Nevertheless, they will be Alexandrov spaces and so do carry some structure (see Appendix A). In order to flow (X, d_∞) we will flow each of the (M_i^3, g_i) and then take a Hamilton limit of the solutions (see [13]). The two main obstacles to this procedure are:

- It is possible that the solutions $\big(M_i, g_i(t)\big)$ are defined only for $t \in [0, T_i)$ where $T_i \to 0$ as $i \to \infty$.

- In order to take this limit, we require that each of the solutions satisfy uniform bounds of the form

$$\sup_{M_i} \big|\mathrm{Riem}\big(g_i(t)\big)\big| \leqq c(t), \quad \forall t \in (0, T),$$

for some well defined common time interval $(0, T)$ ($c(t) \to \infty$ as $t \to 0$ would not be a problem here). Furthermore they should all satisfy a uniform lower bound on the injectivity radius of the form

$$\mathrm{inj}\big(M, g_i(t_0)\big) \geqq \sigma(t_0) > 0$$

for some $t_0 \in (0, T)$.

As a first step to solving these two problems, in Lemma 3.4 of Section 3 we see that a (three dimensional) smooth solution to the Ricci flow $\big(M, g(t)\big)_{t \in [0, T)}$ cannot become singular at time T as long as Ricci $\geqq -1$, the diameter remains bounded (by say d_0) and the volume stays bounded away from zero (say it is bigger than v_0). Furthermore, a bound of the form

$$\big|\mathrm{Riem}\big(g(t)\big)\big| \leqq \frac{c_0(d_0, v_0)}{t}, \quad \forall t \in [0, T) \cap [0, 1]$$

for such solutions is proved: that is, the curvature of such solutions is quickly smoothed out.

In Theorem 4.1 we present an application of the proof of 4.1. Notice that [19], Proposition 11.4, for the three dimensional case implies Lemma 4.1. Perelman's method of proof is somewhat different from that used in Lemma 4.1.

Section 5 is concerned with proving (for an arbitrary three dimensional solution to the Ricci flow) lower bounds for the Ricci curvature of the evolving metric, which depend on

- the bound from below for the Ricci curvature of the initial metric,

- the scalar curvature of the evolving metric.

One of the major applications is (see Lemma 1.10): if (M, g_0) satisfies $\mathrm{Ricci}(g_0) \geqq -\varepsilon_0$ (ε_0 small enough) and the solution satisfies $\mathrm{R}\big(g(t)\big) \leqq \frac{c_0}{t}$ for all $t \in (0, T)$ then

$$\mathrm{Ricci}\big(g(t)\big) \geqq -2c_0\varepsilon_0, \quad \forall t \in (0, T_*) \cap (0, T)$$

for some universal constant $T_* = T_* > 0$.

In Section 6, we consider smooth solutions to the Ricci flow which satisfy

$$\text{Ricci}\big(g(t)\big) \geqq -c_0, \tag{2.1}$$

$$\big|\text{Riem}\big(g(t)\big)\big|t \leqq c_0, \tag{2.2}$$

$$\text{diam}(M, g_0) \leqq d_0. \tag{2.3}$$

In Lemma 6.1, well known bounds on the evolving distance for a solution to the Ricci flow are proved for such solutions.

We combine this lemma with some results on Gromov-Hausdorff convergence and a theorem of Cheeger-Colding (from the paper [4]) to show (Corollary 6.2) that such solutions can only lose volume at a controlled rate.

In Section 7 we show (using the a priori estimates from the previous sections) that a solution to the Ricci flow of (X, d_∞) exists, where (X, d_∞) is the Gromov-Hausdorff limit as $i \to \infty$ of $\big(M_i, d(g_i)\big)$ where the (M_i, g_i) satisfy

$$\text{Ricci}(g_i) \geqq -\varepsilon(i),$$
$$\text{vol}(M_i, g_i) \geqq v_0,$$
$$\text{diam}(M_i, g_0) \leqq d_0.$$

More explicitly we prove Theorem 1.6.

The theorem which is essential in constructing such a solution is (Theorem 7.1 of this paper):

Theorem 2.1. *Let M be a closed three (or two) manifold satisfying*

$$\begin{aligned} \text{diam}(M, g_0) &\leqq d_0, \\ \text{Ricci}(g_0) \quad (\sec(g_0)) &\geqq -\varepsilon g_0, \\ \text{vol}(M, g_0) &\geqq v_0 > 0, \end{aligned} \tag{2.4}$$

where $\varepsilon \leqq \dfrac{1}{10c^2}$ and $c = c(v_0, d_0) \geqq 1$ is the constant from Lemma 3.4. Then there exists an $S = S(d_0, v_0) > 0$ and $K = K(d_0, v_0)$ such that the maximal solution $\big(M, g(t)\big)_{t \in [0,T)}$ to Ricci-flow satisfies $T \geqq S$, and

$$\sup_M \big|\text{Riem}\big(g(t)\big)\big| \leqq \frac{K}{t},$$

for all $t \in (0, S)$.

Appendix A contains definitions, results and facts about Gromov-Hausdorff space, which we require in this paper.

In Appendix B we define C-essential points, and δ-like necks, and consider discuss 0-like necks in the three dimensional case.

A proof of the (well known) Lemma 6.1 is contained in Appendix C.

Appendix D is a description of the notation used in this paper.

3. Bounding the blow up time from below using bounds on the geometry

An important property of the Ricci flow is that:

If certain geometrical quantities are controlled (bounded) on a half open finite time interval $[0, T)$, then the solution does not become singular as $t \nearrow T$ and may be extended to a solution defined on the time interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$. We are interested in the question:

Problem 3.1. *What elements of the geometry need to be controlled, in order to guarantee that a solution does not become singular*?

In [9], it was shown that for (M, g_0) a closed smooth Riemannian manifold, the Ricci flow equation

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\,\mathrm{Ricci}(g), \\ g(\cdot, 0) &= g_0, \end{aligned} \tag{3.1}$$

always has a solution $\big(M, g(t)\big)_{t \in [0, T)}$ for a short time. It was also shown that two such solutions defined on the same time interval must agree, if there initial values agree. Furthermore, for each smooth, closed (M, g_0) there exists a maximal time interval $[0, T_{\mathrm{Max}})$ ($T_{\mathrm{Max}} > 0$) for which, there exists a solution $\big(M, g(t)\big)_{t \in [0, T_{\mathrm{Max}})}$ to (3.1), and if $T_{\mathrm{Max}} < \infty$ then there is no solution $\big(M, g(t)\big)_{t \in [0, T_{\mathrm{Max}} + \varepsilon)}$ to (3.1) (for any $\varepsilon > 0$). Such a solution $\big(M, g(t)\big)_{t \in [0, T_{\mathrm{Max}})}$ is called a *maximal solution.*

Definition 3.2 (Maximal solutions). Let $\big(M, g(t)\big)_{t \in [0, T)}$ be a solution to Ricci flow. We say that the solution blows up at time T if

$$\sup_{M \times [0, T)} |\mathrm{Riem}| = \infty. \tag{3.2}$$

It was also shown in [9] that

Lemma 3.3. *Let $\big(M, g(t)\big)_{t \in [0, T)}$ be a closed, smooth solution to Ricci flow, with $g(0) = g_0$ and $T < \infty$, with*

$$\sup_{M \times [0, T)} |\mathrm{Riem}| < \infty. \tag{3.3}$$

Then, for some $\varepsilon > 0$, there exists a solution $\big(M, g(t)\big)_{t \in [0, T + \varepsilon)}$, with $g(0) = g_0$.

So we see that a bound on the supremum of the Riemannian curvature (that is, *control* of this geometrical quantity) on a finite time interval $[0,T)$ guarantees that this solution does not become singular as $t \nearrow T$. In the following lemma, we present other bounds on geometrical quantities which guarantee that a solution to the Ricci flow does not become singular as $t \nearrow T$.

Lemma 3.4. *Let* $\big(M^3, g(t)\big)_{t\in[0,T)}$, $T \leqq 1$ *be an arbitrary smooth solution to Ricci flow* (M^3 *closed*) *satisfying*

$$\begin{aligned} \operatorname{Ricci}(g) &\geqq -1, \\ \operatorname{vol}(M,g) &\geqq v_0 > 0, \\ \operatorname{diam}(g) &\leqq d_0 < \infty \end{aligned} \tag{3.4}$$

for all $t \in [0,T)$. *Then there exists a* $c = c(d_0, v_0)$, *such that*

$$\mathrm{R}\big(g(t)\big)t \leqq c$$

for all $t \in [0,T)$. *In particular,* $\big(M^3, g(t)\big)_{t\in[0,T)}$ *is not maximal.*

Corollary 3.5. *Let* $\big(M^3, g(t)\big)_{t\in[0,T)}$ *be an arbitrary smooth solution to Ricci flow satisfying*

$$\begin{aligned} \operatorname{Ricci}(g) &\geqq -1, \\ \operatorname{vol}(M,g) &\geqq v_0 > 0, \\ \operatorname{diam}(g) &\leqq d_0 < \infty \end{aligned} \tag{3.5}$$

for all $t \in [0,T)$. *Then there exists a* $c = c(d_0, v)$, *such that*

$$\mathrm{R}\big(g(t)\big) \leqq c\max\left(\frac{1}{t}, 1\right)$$

for all $t \in [0,T)$. *In particular,* $\big(M^3, g(t)\big)_{t\in[0,T)}$ *is not maximal.*

The proof of the corollary is a trivial iteration argument.

Proof. Fix $t_0 \in [0,T)$. We wish to show that

$$\mathrm{R}\big(g(t_0)\big) \leqq c\max\left(\frac{1}{t_0}, 1\right).$$

If $t_0 \leqq 1/2$ then we apply Lemma 3.4. If $(N+1)/2 > t_0 \geqq N/2$ ($N \in \mathbb{N}$) then we apply Lemma 3.4 to the solution $\left(M, g\left(\frac{(N-1)}{2} + t\right)\right)_{t\in[\frac{1}{2},1)}$ of Ricci flow (notice that $\frac{(N-1)}{2} + t = t_0$ implies that $1 > t \geqq 1/2$). $\square$

We now prove Lemma 3.4.

Proof. Assume to the contrary that there exist solutions $\big(M_i, {}^i g(t)\big)_{t \in [0, T_i)}$, $T_i \leqq 1$ to Ricci flow such that

$$(3.6) \qquad \sup_{(x,t) \in M_i \times (0, T_i)} {}^i\mathrm{R}(x,t)t \xrightarrow{i \to \infty} \infty,$$

or there exists some $j \in \mathbb{N}$ with

$$(3.7) \qquad \sup_{(x,t) \in M_j \times (0, T_j)} {}^j\mathrm{R}(x,t)t = \infty,$$

where ${}^i\mathrm{R} := \mathrm{R}({}^i g)$. It is then possible to choose points $(p_i, t_i) \in M_i \times [0, T_i)$ (or in $M_j \times [0, T_j)$: in this case we redefine $M_i = M_j$ and $T_i = T_j$ for all $i \in \mathbb{N}$ and hence we do not need to treat this case separately) such that

$$(3.8) \qquad \mathrm{R}(p_i, t_i)t_i = \sup_{(x,t) \in M_i \times (0, t_i]} {}^i\mathrm{R}(x,t)t \xrightarrow{i \to \infty} \infty.$$

Define

$$(3.9) \qquad {}^i\hat{g}(\cdot, \hat{t}) := c_i {}^i g\left(\cdot, t_i + \frac{\hat{t}}{c_i}\right),$$

where $c_i := {}^i\mathrm{R}(p_i, t_i)$. This solution to Ricci flow is defined for $0 \leqq t_i + \dfrac{\hat{t}}{c_i} < T_i$, that is, at least for $0 \geqq \hat{t} > -t_i c_i$. Let $A_i := t_i c_i$. Then the solution ${}^i\hat{g}(\hat{t})$ is defined at least for $\hat{t} \in (-A_i, 0)$. By the choice of (p_i, t_i) we see that the solution is defined for $\hat{t} > -A_i = -t_i c_i = -t_i {}^i\mathrm{R}(p_i, t_i) \xrightarrow{i \to \infty} -\infty$. Since $t_i \leqq T_i \leqq 1$, we also have

$$(3.10) \qquad c_i \xrightarrow{i \to \infty} \infty,$$

in view of the fact that

$$t_i c_i = t_i {}^i\mathrm{R}(p_i, t_i) \xrightarrow{i \to \infty} \infty.$$

Furthermore, letting $s(\hat{t}, i) := t_i + \dfrac{\hat{t}}{c_i}$, where $-A_i < \hat{t} \leqq 0$ we have

$$\begin{aligned}
(3.11) \qquad {}^i\hat{\mathrm{R}}(\cdot, \hat{t}) &= \frac{1}{c_i} {}^i\mathrm{R}\big(\cdot, s(\hat{t}, i)\big) \\
&= \frac{{}^i\mathrm{R}(\cdot, s)}{{}^i\mathrm{R}(p_i, t_i)} \\
&= \frac{{}^i\mathrm{R}(\cdot, s)s}{{}^i\mathrm{R}(p_i, t_i)t_i} \frac{t_i}{s} \\
&\leqq \frac{t_i}{s} \\
(3.12) \qquad &= \frac{t_i}{t_i + \dfrac{\hat{t}}{c_i}} \xrightarrow{i \to \infty} 1
\end{aligned}$$

in view of the definition of (p_i, t_i), and $0 \leqq s \leqq t_i$ (follows from the definition of s and the fact that $\hat{t} \leqq 0$), and (3.10). Due to the conditions (3.4) we see that there exist $l = l(v_0, d, n)$, and $\varepsilon = \varepsilon(v_0, d, n)$, such that

$$l \geqq \frac{\mathrm{vol}\big(B_r(p), {}^i g(t)\big)}{r^3} \geqq \varepsilon, \quad \forall r \leqq \mathrm{diam}\big(M_i, {}^i g(t)\big), \tag{3.13}$$

(in view of the Bishop-Gromov comparison principle) which implies the same result for any rescaling of the manifolds. Notice that the conditions (3.4) imply that

$$\mathrm{diam}\big(M, {}^i g(t)\big) \geqq d_1(n, v_0) > 0 \tag{3.14}$$

for some $\infty > d_1(n, v_0) > 0$. Otherwise, assume $\mathrm{diam}\big(M, {}^i g(t)\big) \leqq d_1$ for some small d_1, then $\mathrm{vol}\big(M, {}^i g(t)\big) \leqq c(n)\, d_1^3 \omega_n$ (Bishop-Gromov comparison principle), and hence $\mathrm{vol}\big(M, {}^i g(t)\big) < v_0$ if d_1 is too small, which would be a contradiction. Hence, $\mathrm{diam}\big(M, {}^i \hat{g}(0)\big) \xrightarrow{i \to \infty} \infty$, in view of the inequalities (3.14) and (3.10). Now using

$$l \geqq \frac{\mathrm{vol}\big(B_r(p), {}^i \hat{g}(t)\big)}{r^3} \geqq \varepsilon_0, \quad \forall r \leqq \mathrm{diam}\big(M_i, {}^i \hat{g}(\hat{t})\big), \tag{3.15}$$

we obtain a bound on the injectivity radius from below, in view of the theorem of Cheeger-Gromov-Taylor, [5] (the theorem of Cheeger-Gromov-Taylor says that for a complete Riemannian manifold (M, g) with $|\mathrm{Riem}| \leqq 1$, we have

$$\mathrm{inj}(x, g) \geqq r \frac{\mathrm{vol}\big(g, B_r(x)\big)}{\mathrm{vol}\big(g, B_r(x)\big) + \omega_n \exp^{n-1}},$$

for all $r \leqq \pi/4$). In particular, using that $\mathrm{diam}(M, g) \geqq d_1 > 0$ and $|\mathrm{Riem}| \leqq c$ (see [i] below) for the Riemannian manifolds in question, we obtain

$$\mathrm{inj}(x, g) \geqq \varepsilon \frac{s^{n+1}}{l s^n + \omega_n \exp^{n-1}} \geqq c^2(d_0, v_0, n) > 0 \tag{3.16}$$

for $s = \min\big((\omega_n \exp^{n-1})^{\frac{1}{n}}, \mathrm{diam}(M, g), \pi/4\big)$.

This allows us to take a pointed *Hamilton limit* (see [13]), which leads to a Ricci flow solution $\big(\Omega, o, g(t)_{t \in (-\infty, \omega)}\big)$, with $R \leqq R(o, 0) = 1$, and Ricci $\geqq 0$, $\omega > 0$ (at $t = 0$, as explained below, the full Riemannian curvature tensor of ${}^i \hat{g}(0)$ is bounded by $c(3)$ and so clearly each solution lives at least to a time $\omega > 0$ independent of i). More precisely:

- [i] The bound from below on the Ricci curvature, and the bound from above on the scalar curvature imply that the Ricci curvatures are bounded absolutely by the constant 5 for i big enough. In three dimensions, bounds from above and below on the Ricci curvatures imply bounds from above and below on the sectional curvatures and hence on the norm of the full Riemannian curvature tensor. This, together with the bound from below on the injectivity radius, allows us to a take a Hamilton limit of these Ricci flows.

- [ii] In fact the limit solution satisfies $\sec \geqq 0$, which can be seen as follows: Each rescaled solution ${}^i \hat{g}$ is defined on $M_i \times [-A_i, \omega]$ where $A_i \xrightarrow{i \to \infty} \infty$. They also each satisfy sec $\geqq -2$ and $|\mathrm{Riem}| \leqq c(n)$ for all $t \in (-S, 0)$ for any fixed S and all i big enough, in view of (3.12) and Ricci $\geqq -1$.

Let us translate in time by S, so that these solutions are defined on $M_i \times [-A_i + S, S]$ and satisfy sec $\geqq -2$ and $|\mathrm{Riem}| \leqq c(n)$ on $(0, S)$ (for i big enough). Without loss of generality, we assume that sec $\geqq -1$. We then use the improved pinching result of Hamilton [14] (see also [15]):

Theorem 3.6. *Let $g(t)$ be a solution to Ricci flow defined on $M \times [0, T)$, M closed. Assume at $t = 0$ that the eigenvalues $\alpha \geqq \beta \geqq \gamma$ of the curvature operator at each point are bounded below by $\gamma \geqq -1$. The scalar curvature is their sum $\mathrm{R} = \alpha + \beta + \gamma$, and $X := -\gamma$. Then at all points and all times we have the pinching estimate*

$$\mathrm{R} \geqq X[\log X + \log(1 + t) - 3],$$

whenever $X > 0$.

Notice that this estimate is also valid for the translated limit solution (defined on $[0, S)$) as it is valid for each i and the scalar curvature and X converge as $i \to \infty$ to the corresponding quantities of the translated (by S) limit solution.

Let $\delta > 0$ be any arbitrary small constant. Assume there exists $(x, t) \in \Omega \times \left[\frac{S}{2}, S\right)$ such that $X(x, t) \geqq \delta$. Then we get

$$\begin{aligned} \log(\delta) \leqq \log X(x, t) &\leqq \frac{\mathrm{R}(x, t)}{\delta} - \log(1 + t) + 3 \\ &\leqq \frac{c(n)}{\delta} - \log\left(1 + \frac{S}{2}\right) + 3 \end{aligned} \tag{3.17}$$

which is a contradiction for S big enough. Hence our initial limit solution (without any translations in time) has $X(x, 0) \leqq \delta$. As δ was arbitrary we get $X(\cdot, 0) \leqq 0$. A similar argument shows $X \leqq 0$ everywhere. That is, the limit space satisfies sec $\geqq 0$, $\forall t \in (-\infty, 0)$.

The volume ratio estimates

$$l \geqq \frac{\mathrm{vol}(B_r(p))}{r^3} \geqq \varepsilon_0, \quad \forall r > 0, \tag{3.18}$$

are also valid for (Ω, g), as these estimates are scale invariant, and $\mathrm{diam}(\Omega, g) = \infty$. At this point we could apply [19], Proposition 11.4, to obtain a contradiction. We prefer however to introduce an alternative method to Perelman in order to obtain a contradiction (this method may be of independent interest). We now consider the following two cases.

(Case 1) $\sup_{\Omega \times (-\infty, 0]} |t|\mathrm{R} = \infty$.

(Case 2) $\sup_{\Omega \times (-\infty, 0]} |t|\mathrm{R} < \infty$.

(Case 1) In the first case, in view of [7], Chapter 8, Section 6, we may assume w.l.o.g. that there exists a solution $\big(\Omega, o, g(t)_{t\in(-\infty,\infty)}\big)$, with

$$\sup_{\Omega\times(-\infty,\infty)} |\mathrm{R}(t)| \leqq 1 = |\mathrm{R}(0,o)|. \tag{3.19}$$

Note: we must slightly modify the argument there, by replacing Riem with R wherever it appears. We also use the fact (as mentioned above) that $|\mathrm{Riem}| \leqq c(3)\mathrm{R}$ in the case that $\mathrm{Ricci} \geqq 0$ (in dimension three) and that our scale invariant volume estimate (3.18) remains true for any rescalings of our solution: these two facts ensure that in the rescaling process of the argument in [3], Chapter 8, Section 6, an injectivity radius estimate is satisfied, and that the limit solution is well defined.

(Case 1.1) The sectional curvature is everywhere positive.

(Case 1.2) There exists $(p_0, t_0) \in \Omega \times (-\infty, \infty)$, and $v_{p_0}, w_{p_0} \in T_{p_0}\Omega$ with

$$\sec(p_0, t_0)(v_{p_0}, w_{p_0}) = 0.$$

First we consider (Case 1.1).

(Case 1.1) This means Ω is diffeomorphic to $\mathbb{R}^3$ in view of the soul theorem (see [6], Chapter 8) and in particular, Ω is simply connected. We may then apply the gradient soliton theorem of Hamilton [11] which implies, in view of (3.19), that $\big(\Omega, g(t)\big)_{t\in(-\infty,\infty)}$ is a gradient soliton. We may then, using the dimension reduction theorem of Hamilton, [12], Theorem 22.3, take a Hamilton limit of rescalings of this solution, to obtain a new solution, $\big(\mathbb{R}\times N, dx^2 \oplus \gamma(t)\big)_{t\in(-\infty,\infty)}$, or a quotient thereof by a group of fixed-point free isometries acting properly discontinuously, where dx^2 is the standard metric on $\mathbb{R}$, and $\big(N, \gamma(t)\big)_{t\in(-\infty,\infty)}$ is a solution to the Ricci flow, N is a surface, and $\mathrm{R}(\cdot, t) > 0$, on N. In the case that we have a quotient of $\big(\mathbb{R}\times N, dx^2 \oplus \gamma(t)\big)$ then we notice that $\big(\mathbb{R}\times N, dx^2 \oplus \gamma(t)\big)$ still satisfies (3.18) (the bound from below follows as the Riemannian covering map $f : \big(\mathbb{R}\times N, dx^2 \oplus \gamma(t)\big) \to \big(\Omega, g(t)\big)$ is a Riemannian submersion, and the bound from above follows in view of the Bishop-Gromov comparison principle) and so, without loss of generality, we may assume that we do not have a quotient. If N is compact, then $(\mathbb{R}\times N, dx^2 \oplus \gamma)$, does not satisfy the estimates (3.18), and so we obtain a contradiction. So w.l.o.g. we may assume that N is non-compact. Now we break this up into two cases:

(Case 1.1.1) $\sup_{N\times(-\infty,\infty)} |t|\,|\mathrm{R}(t)| = \infty$, and

(Case 1.1.2) $\sup_{N\times(-\infty,\infty)} |t|\,|\mathrm{R}(t)| < \infty$.

First we handle

(Case 1.1.1) Once again, w.l.o.g. ([7], Chap. 8, Sec. 6), we may assume

$$\sup_{N\times(-\infty,\infty)} \mathrm{R} \leqq 1 = \mathrm{R}(o, 0).$$

$R(t) > 0$, and N non-compact implies N is diffeomorphic to $\mathbb{R}^2$, which is simply connected. We may then use the gradient soliton theorem of Hamilton, [11], to obtain that (N, γ) is a gradient soliton, which implies ([12], Thm. 26.3), that (N, γ) is the cigar (Σ, cig). But $(\mathbb{R} \times \Sigma, dx^2 \oplus \mathrm{cig})$ do not satisfy the estimates (3.18), and so we obtain a contradiction.

(Case 1.1.2) $\sup_{N \times (-\infty, \infty)} |t|\,|R(t)| < \infty$. Hamilton, [12], Thm. 26.1, implies that $(N, \gamma) = (\mathbb{S}^2 \text{ or } \mathbb{R}^2, \gamma)/\Gamma$, where γ is the standard solution on S^2 or $\mathbb{R}^2$, and Γ is a finite group of isometries acting without fixed points on the standard S^2 or standard $\mathbb{R}^2$. $(\mathbb{R}^2, \gamma)$ cannot occur, since the surface should satisfy $R(t) > 0$ everywhere (the standard $(\mathbb{R}^2, \gamma)$ is flat). But then N is compact, and $(\mathbb{R} \times N, dx^2 \oplus \gamma)$, does not satisfy the estimates (3.18), and once again we obtain a contradiction.

(Case 1.2) There exists $(p_0, t_0) \in \Omega \times (-\infty, \infty)$, and $v_{p_0}, w_{p_0} \in T_{p_0}\Omega$ with

$$\sec(p_0, t_0)(v_{p_0}, w_{p_0}) = 0.$$

Then the maximum principle applied to the evolution equation of the curvature operator, implies that $\big(\Omega, o, g(t)\big)_{t \in (-\infty, \infty)} = \big(\mathbb{R} \times N, dx^2 \oplus \gamma(t)\big)_{t \in (-\infty, \infty)}$, or a quotient thereof by a group of isometries (see [10], Chapter 9) and $\sup_{N \times (-\infty, \infty)} R(t) \leqq 1 = R(o, 0)$. Without loss of generality, we may assume that we don't have a quotient, as explained in (Case 1.1). $R(t) > 0$, implies N is diffeomorphic to $\mathbb{S}^2/\Gamma$ or $\mathbb{R}^2$. In the case that N is diffeomorphic to $\mathbb{S}^2/\Gamma$, we obtain a contradiction, as then (Ω, g) does not satisfy (3.18). So w.l.o.g. N is diffeomorphic to $\mathbb{R}^2$, in particular N is simply connected. We may use the gradient soliton theorem of Hamilton [11], to get that (N, γ) is a soliton and it must be the cigar, in view of Theorem 26.3 of Hamilton [12]. This leads to a contradiction as then (Ω, g) does not satisfy (3.18) (similarly for the covering case).

(Case 2) $B := \sup_{\Omega \times (-\infty, 0]} |t|\,|\mathrm{Riem}(t)| < \infty.$

(Case 2.1) The asymptotic scalar curvature ratio $A = \limsup_{s \to \infty} R s^2 = \infty$. Remember that the asymptotic scalar curvature ratio is a constant in time for ancient solutions which have bounded curvature at each time and non-negative curvature operator. A is also independent of which origin we choose: see [12], Theorem 19.1. Then we use the dimension-reduction argument of Hamilton (see [12], Lemma 22.2 and the argument directly after the proof of Lemma 22.2) to obtain a new solution $(N \times \mathbb{R}, \gamma \oplus dx^2)$ or a quotient thereof by a group of isometries where (N, γ) is a solution to Ricci flow defined on $(-\infty, T]$ $(T > 0)$ (note, our injectivity radius estimate is still valid in view of the volume ratio estimate (3.18) which survives into every limit). If N is compact then we obtain a contradiction to (3.18). So we may assume that N is non-compact. We then consider the cases $\sup_{N \times (-\infty, \infty)} |t|\,|R(t)| = \infty$, and $\sup_{N \times (-\infty, \infty)} |t|\,|R(t)| < \infty$. Then, using the exact same arguments as in (Case 1.1.1) and (Case 1.1.2), we obtain a contradiction.

(Case 2.2) The asymptotic scalar curvature ratio $A = \limsup_{s \to \infty} R s^2 < \infty$. Remember that the asymptotic scalar curvature ratio is a constant in time for ancient solutions which have bounded curvature at each time and non-negative curvature operator. A is also independent of which origin we choose: see [12], Theorem 19.1.

Now we use another splitting argument of Hamilton (see [12], Theorem 24.7 for the compact version of this argument).

(Case 2.2.1) There exists a $C > 0$, s.t., for all $\tau \in (-\infty, 0)$, for all $\delta \in (0,1)$, there exists $(x,t) \in \Omega \times (-\infty, \tau)$ such that (x,t) is a C-essential δ-necklike point (see Appendix B). Let $\{\delta_i\}_{i \in \mathbb{N}}$ be a positive sequence, $\delta_i \xrightarrow{i \to \infty} 0$, and let (x_i, t_i) be chosen so that (x_i, t_i) is an C-essential δ_i-necklike point. Assume θ_i is a unit 2-form on $T_{x_i}\Omega$ with

$$|\mathrm{Riem}(x_i, t_i) - \mathrm{R}(x_i, t_i)(\theta_i \otimes \theta_i)| \leqq \delta_i |\mathrm{Riem}|(x_i, t_i).$$

Let ${}^i g(x,t) = \dfrac{1}{|t_i|} g(x, t_i + t|t_i|)$. Then

$$\begin{aligned}
(3.20) \qquad {}^{{}^i g}|{}^i\mathrm{Riem}(x,t)| &= |t_i|{}^{g}|\mathrm{Riem}(x, t_i + t|t_i|)| \\
&= |t_i|{}^{g}\big|\mathrm{Riem}\big(x, (t-1)|t_i|\big)\big| \\
&= \frac{|(t-1)|t_i||}{|1-t|}{}^{g}\big|\mathrm{Riem}\big(x, (t-1)|t_i|\big)\big| \\
&\leqq \frac{B}{|1-t|} \leqq 2B
\end{aligned}$$

for $t \leqq 1/2$. Notice that

$$(3.21) \qquad t_i + \frac{1}{2}|t_i| = t_i - \frac{1}{2}t_i = \frac{1}{2}t_i < 0$$

and so ${}^i g(t)$ is defined for (at least) $-\infty < t \leqq 1/2$. Furthermore,

$$(3.22) \qquad {}^{{}^i g}|{}^i\mathrm{Riem}(x_i, 0)| = |t_i|{}^{g}|\mathrm{Riem}(x_i, t_i)| \geqq C > 0,$$

since (x_i, t_i) is C-essential. Set

$$\psi_i := \frac{1}{|t_i|}\theta_i.$$

ψ_i is then a unit two form on $T_{x_i}\Omega$ with respect to $g^i(x,0)$. Then

$${}^{{}^i g}|{}^i\mathrm{Riem}(x_i, 0) - {}^i\mathrm{R}(x_i, 0)(\psi_i \otimes \psi_i)| \leqq \delta_i B.$$

Now taking a Hamilton pointed limit (our injectivity radius estimate is still valid) we obtain a solution $(\tilde{\Omega}, \tilde{g})$, defined for $t \leqq 1/2$ with

$${}^{\tilde{g}}|\widetilde{\mathrm{Riem}}(o, 0) - \tilde{\mathrm{R}}(o, 0)(\tilde{\psi} \otimes \tilde{\psi})| \leqq 0,$$

where $\tilde{\psi}$ is the unit two form (at time zero it has length one) defined on $T_o\tilde{\Omega}$, $\tilde{\psi} = \lim_{i \to \infty} (F_i)^* \psi_i$, for diffeomorphisms $F_i : B_i(o) \subset \tilde{\Omega} \to U_i \subset \Omega$. More precisely this ψ is obtained (in coordinates) as $\psi_{\alpha\beta}(o) := \lim_{i\to\infty} \frac{\partial(F_i)^r}{\partial x^\alpha}(o)\frac{\partial(F_i)^s}{\partial x^\beta}(o)(\psi_i)_{rs}(x_i)$, where $F_i : \big(B_i(o), \tilde{g}, o\big) \to (U_i, {}^i g, x_i) \subset M_i$, $F_i(o) = x_i$ are the diffeomorphisms occurring in the Hamilton limit process: $(F_i)^*({}^i g) \to \tilde{g}$ on $B_R(o)$ as $i \to \infty$ for all $R \geqq 0$ (notice then that for $t = 0$

$$
\begin{aligned}
(3.23) \qquad 1 &= ({}^i g)^{lm}({}^i g)^{rs}(\psi_i)_{rl}(\psi_i)_{sm}(x_i)\\
&= \big(F_i^*({}^i g)\big)^{lm}\big(F_i^*({}^i g)\big)^{rs}(F_i^*\psi_i)_{rl}(F_i^*\psi_i)_{sm}(o)\\
&\sim \tilde{g}^{lm}\tilde{g}^{rs}(F_i^*\psi_i)_{rl}(F_i^*\psi_i)_{sm}(o)
\end{aligned}
$$

for large i, and so $F_i^*\psi_i$ converges to a unit two form as $i \to \infty$, as stated). Furthermore $\mathrm{R}(o, 0) \geqq C > 0$ (in view of (3.22)) which implies (in view of the strong maximum principle applied to the evolution equation for R) that $\mathrm{R} > 0$. Hence, due to the maximum principle, $(\tilde{\Omega}, \tilde{g}) = (N \times \mathbb{R}, \gamma \oplus dx^2)$, or a quotient thereof by a group of isometries, where (N, γ) is a solution to the Ricci flow (see Appendix B for a more detailed explanation of this fact). If N is compact we obtain a contradiction to the volume ratio estimates. If N is non-compact, then we argue exactly as in (Case 1.1.1) and (Case 1.1.2) to obtain a contradiction.

(Case 2.2.2) For all $C > 0$, there exists $\tau \in (-\infty, 0)$, and $\delta \in (0, 1)$, such that for all $(x, t) \in \big(\Omega \times (-\infty, \tau)\big)$, (x, t) is not a C-essential δ-necklike point. Choose $C \leqq 1/16$, and let τ, δ be the τ, δ from the statement at the beginning of this case. Set

$$G := |t|^{\frac{\varepsilon}{2}} \frac{|\mathring{\mathrm{Riem}}|^2}{\mathrm{R}^{2-\varepsilon}},$$

with $\varepsilon \leqq \eta(\delta) := \frac{\delta}{100(3-\delta)}$ (notice that this function is well defined, as $\mathrm{R} > 0$ everywhere). Then, as Chow and Knopf show in [7] (see the proof of Theorem 9.19 there)

$$(3.24) \qquad \frac{\partial}{\partial t} G \leqq \Delta G + 2\frac{(1-\varepsilon)}{\mathrm{R}}\langle \nabla G, \nabla \mathrm{R}\rangle - \frac{\varepsilon}{2|t|} G,$$

for all $t \leqq \tau$. Let us examine G a little more carefully. For fixed $t < 0$ and a fixed x_0 we have the estimate

$$
\begin{aligned}
(3.25) \qquad \lim_{d(x,x_0,t)\to\infty} G(x,t) &= \lim_{d(x,x_0,t)\to\infty} |t|^{\frac{\varepsilon}{2}} \frac{|\mathring{\mathrm{Riem}}(x,t)|^2}{\mathrm{R}^2(x,t)} \mathrm{R}^{\varepsilon}(x,t)\\
&\leqq |t|^{\frac{\varepsilon}{2}} c(n) \lim_{d(x,x_0,t)\to\infty} \mathrm{R}^{\varepsilon}(x,t)\\
&= 0
\end{aligned}
$$

in view of the fact that the asymptotic scalar curvature ratio is less than infinity. Also, as Chow and Knopf point out, we have

$$(3.26) \qquad G = |t|^{\varepsilon}\mathrm{R}^{\varepsilon} \frac{|\mathring{\mathrm{Riem}}|^2}{\mathrm{R}^2} \frac{1}{|t|^{\frac{\varepsilon}{2}}} \leqq \frac{B^{\varepsilon} c(n)}{|t|^{\frac{\varepsilon}{2}}},$$

in view of the fact that $B := \sup_{\Omega\times(-\infty,\omega]} |t|\,|\mathrm{R}(t)| < \infty$, and hence

$$\lim_{t\to-\infty}\sup_{x\in M} G(x,t) = 0. \tag{3.27}$$

Let $\tau' < \tau - 2$ be a constant with $\sup_{\Omega} G(\cdot,t) < \varepsilon_0$ for all $t \leqq \tau'$. We know that

$$\sup_{M\times(-\infty,0]} |\mathrm{Riem}| \leqq c(n) \tag{3.28}$$

and without loss of generality

$$\sup_{M\times[\tau',\tau]} |\nabla\,\mathrm{Riem}|^2 + |\nabla^2\,\mathrm{Riem}|^2 \leqq c(n) \tag{3.29}$$

in view of the interior gradient estimates of Shi (see [12], Chapter 13). We also know that for given $\varepsilon_1 > 0$ and $s \in [\tau',\tau]$ there exists an $r(s,\varepsilon_1) > 0$ such that

$$\sup_{\{x\in M : d^2(x,x_0,s)\geqq r\}} |\mathrm{Riem}|(x,s) \leqq \varepsilon_1, \tag{3.30}$$

in view of the fact that the asymptotic scalar curvature ratio is finite. Hence, for all $\varepsilon_2 > 0$ there exists a $\delta > 0$, such that

$$\sup_{x\in M,\, t\in(s,s+\delta) : d^2(x,x_0,s)\geqq r} |\mathrm{Riem}|(x,t) \leqq \varepsilon_1 + \varepsilon_2,$$

in view of (3.28) and (3.29) and the evolution equation for $|\mathrm{Riem}|^2$. In particular if $\sup_M G(\cdot,s) < \varepsilon_0$, then $\sup_{M\times(s,s+\delta)} G(\cdot,t) < \varepsilon_0$, for small enough δ (outside of a fixed large compact set K, $G < \varepsilon_0$ for all $t \in (s,s+\delta)$ and inside K we use the fact that G is smooth). That is, the set

$$Z := \left\{ r : \sup_{\Omega} G(\cdot,t) < \varepsilon_0,\ \forall t \in [\tau', r) \right\}$$

is open. Hence either

$$\sup_{\Omega} G(\cdot,t) < \varepsilon_0$$

for all $t \in [\tau',\tau)$, or there is a first time $t_0 \in (\tau',\tau)$ such that $\sup_{\Omega} G(\cdot,t_0) = \varepsilon_0$. In the second case, we see (using equation (3.30) with $s = t_0$) that there must also be a point $x_0 \in M$ such that $G(x_0,t_0) = \varepsilon_0$. But this contradicts the maximum principle in view of (3.24).

This means that

$$\sup_{\Omega} G(\cdot,t) < \varepsilon_0,$$

for all $t \in (-\infty,\tau)$, and hence, since ε_0 was arbitrary,

$$G \equiv 0.$$

Hence $\Omega = S^3/\Gamma$, which is a contradiction to the fact that Ω is non-compact. $\square$

4. An application of the proof of Lemma 3.4

In certain cases, the proof of Lemma 3.4 is applicable even if M is non-compact. For example, the theorem below is proved similarly to Lemma 3.4. This theorem was initially proved (using other methods and for all dimensions) by Perelman [19], Proposition 11.4.

Theorem 4.1. *Let $(\Omega^3, g(t))_{t\in(-\infty,0]}$ be an ancient non-compact complete solution to Ricci flow, with (for some fixed origin $o \in M$)*

$$\sec \geqq 0,$$

$$\sup_{\Omega} |\mathrm{Riem}(g(t))| < \infty, \quad \forall t \in (-\infty, 0), \tag{4.1}$$

$$\mathscr{V}(\tau) := \lim_{r\to\infty} \frac{\mathrm{vol}(B_r(o,\tau))}{r^3} \geqq \mathscr{V}_0 > 0$$

for some time τ, $\tau \in (-\infty, 0)$. Then $(\Omega^3, g(t))$ is flat for all $t \in (-\infty, 0)$.

Remark 4.2. The limit in the statement of the theorem exists in view of the fact that $\frac{\mathrm{vol}(B_r(o,\tau))}{r^n}$ is non-increasing as r increases (in view of the Bishop-Gromov comparison principle).

Proof. Assume that the asymptotic scalar curvature ratio $A_\Omega = \limsup_{s\to\infty} \mathrm{R}s^2 = \infty$ (this is a constant independent of time). Translate in time so that $\tau = 0$.

Notice that for this solution, and any scaling of this solution which has bounded curvature by some fixed constant c in a ball of radius one around some origin o' at $t = 0$, we have a uniform bound on the injectivity radius from below at o', in view of (4.1) and [5]. We explain this here more precisely. We have the estimate

$$\frac{\mathrm{vol}(B_r(o',0))}{r^3} \geqq \mathscr{V}_0 > 0$$

for all $r > 0$ in view of (4.1) and the Bishop-Gromov volume comparison principle. Furthermore $\frac{\mathrm{vol}(B_r(o',0))}{r^3} \leqq \omega_3$ trivially using the Bishop-Gromov volume comparison principle. We may then apply the result of [5] to obtain our estimate for the bound on the injectivity radius, exactly as we did in the argument of Lemma 3.4. Also, the estimates

$$\omega_3 \geqq \frac{\mathrm{vol}(B_r(o,0))}{r^3} \geqq \mathscr{V}_0 > 0, \quad \forall r \geqq 0 \tag{4.2}$$

remain valid under scaling (as the inequality is scale invariant). Hence, we obtain a uniform bound from below on the injectivity radius estimate at o', for any scaling of this solution which has bounded curvature by some fixed constant c on a ball of radius one around o' at time zero.

We use the dimension-reduction argument of Hamilton (see [12], Lemma 22.2 and the argument directly after the proof of Lemma 22.2) to obtain a new solution (with non-negative sectional curvature and bounded curvature at each time) $(N \times \mathbb{R}, \gamma \times dx^2)(t)$, $t \in (-\infty, 0]$ or a quotient thereof by group of isometries. Also (4.2) remains true (at time zero) for the resulting solution, as we explained above. Without loss of generality, we may assume that we don't have a quotient of $(N \times \mathbb{R}, \gamma \times dx^2)(t)$: otherwise we lift the solution to the solution $(N \times \mathbb{R}, \gamma \times dx^2)(t)$ which still satisfies (4.2) at time zero, as explained in (Case 1.1) of the proof of Lemma 3.4.

Notice that the dimension-reduction argument of Hamilton is applicable here, in view of the bounds from below on the injectivity radius at the centres of the balls occurring in the argument (due to the argument at the beginning of this theorem). Without loss of generality the solution is defined on $(N \times \mathbb{R}, \gamma \times d\sigma^2)$ for $t \in (-\infty, \omega]$ for some $\omega > 0$, in view of the short time existence result of Shi, [21]. $\mathrm{R}(0, o) = 1 \neq 0$ due to the construction process in the dimension-reduction argument. $\mathrm{R}_N \geqq 0$ (for all times) since the sectional curvatures of $\big(\gamma(t) \times d\sigma^2, N^2 \times \mathbb{R}\big)$ are non-negative (for all times) and the curvature in the $\mathbb{R}$ direction is zero. Hence, due to the strong maximum principle again, $\mathrm{R}_N > 0$ for all $t \in (-\infty, \omega]$. Then, see [12], Lemma 26.2, we have

$$A_N = \limsup_{s \to \infty} \mathrm{R}_N s^2 < \infty$$

is a constant independent of $t \in (-\infty, \omega)$ on N.

This means that the asymptotic volume ratio $V_N(t)$ of $\big(N, \gamma(t)\big)$,

$$V_N(t) = \lim_{r \to \infty} \frac{\mathrm{vol}\big({}^{\gamma(t)}B_r(t)(\tilde{o}), \gamma(t)\big)}{r^2},$$

is independent of time (see [12], Theorem 18.3). Assume $o = (\tilde{o}, a) \in N \times \mathbb{R}$. This implies

$$\text{(4.3)} \qquad \begin{aligned} \frac{\mathrm{vol}\big({}^{\gamma(0)}B_r(\tilde{o}), \gamma(0)\big)}{r^2} &= \frac{\mathrm{vol}\big({}^{\gamma(0)}B_r(\tilde{o}) \times [a - r, a + r], \gamma(0) \oplus d\sigma^2\big)}{2r^3} \\ &\geqq \frac{\mathrm{vol}\big({}^{\gamma(0) \times d\sigma^2}B_r(o), \gamma(0) \times d\sigma^2\big)}{2r^3} \\ &\geqq \frac{\mathscr{V}_0}{2}, \end{aligned}$$

in view of (4.2) where here we have used that ${}^{\gamma(0) \times d\sigma^2}B_r(o) \subset {}^{\gamma(0)}B_r(\tilde{o}) \times [a - r, a + r]$. Hence $V_N(t) = V_N(0) \geqq \mathscr{V}_0/2$, which implies

$$\omega_2 \geqq \frac{\mathrm{vol}\big({}^{\gamma(t)}B_r(t)(\tilde{o}), \gamma(t)\big)}{r^2} \geqq V_N(t) \geqq \frac{\mathscr{V}_0}{2},$$

for all $r > 0$ and all $t \in (-\infty, \omega]$ in view of the monotonicity of the volume quotient (Bishop-Gromov volume comparison principle).

We then consider the following two cases:

(Case 1) $\sup_{N\times(-\infty,\omega]} |t| |R(\gamma(t))| = \infty$,

(Case 2) $\sup_{N\times(-\infty,\omega]} |t| |R(\gamma(t))| < \infty$,

exactly as in the proof of Lemma 3.4. Both cases lead to a contradiction.

In the case that $A_\Omega = \limsup_{s\to\infty} Rs^2 < \infty$ then we also know that

$$\mathscr{V}(t) =:= \lim_{r\to\infty} \frac{\mathrm{vol}(B_r(o,t))}{r^3}$$

is a constant on Ω independent of time, and in particular

$$\omega_3 \geqq \frac{\mathrm{vol}(B_r(o,t))}{r^3} \geqq \mathscr{V}_0 > 0, \quad \forall r \geqq 0 \quad \forall t \in (-\infty, 0).$$

Translate in time so that the solution is defined on $(-\infty, \omega)$, $\omega > 0$. We then consider the following two cases:

(Case 1) $\sup_{\Omega\times(-\infty,0]} |t|\,|R(t)| = \infty$,

(Case 2) $\sup_{\Omega\times(-\infty,0]} |t|\,|R(t)| < \infty$,

exactly as in the proof of Lemma 3.4. Both cases lead to a contradiction. □

5. Bounds on the Ricci curvature from below under Ricci flow

We prove quantitative estimates that tell us how quickly the Ricci curvature can decrease, if we assume that the Ricci curvature is not too negative at time zero. Both lemmas may be read independently of the rest of the results in this paper.

The first lemma is suited to the case that we have a sequence of solutions to Ricci flow $(M_i, {}^i g(t))_{t\in[0,T)}$ whose initial data satisfies

$$\mathrm{Ricci}({}^i g(0)) \geqq -\varepsilon_i \mathrm{R}({}^i g(0))\,{}^i g(0) - \varepsilon_i {}^i g(0), \tag{5.1}$$

where $\varepsilon_i \to 0$ as $i \to \infty$. One application of this lemma is: if a subsequence of subsets $(\Omega_i, {}^i g(t))$, $t \in [0, T)$ (Ω_i open) converges (in the sense of Hamilton, see [13]) to a smooth solution $(\Omega, g(t))$, $t \in (0, T)$, then the lemma tells us that the Ricci curvature of $(\Omega, g(t))$ is non-negative for all $t \in (0, T)$. This is very general, but does require that a limit solution exists.

The second lemma is suited to the case that we have a sequence of solutions to Ricci flow $(M_i, {}^ig(t))_{[0,T)}$ whose initial data satisfies

$$\text{Ricci}({}^ig(0)) \geqq -\varepsilon_i {}^ig(0), \tag{5.2}$$

where $\varepsilon_i \to 0$ as $i \to \infty$. Once again, one application of this lemma is: if a subsequence of subsets $(\Omega_i, {}^ig(t))$, $t \in (0,S)$ converges (in the sense of Hamilton, see [13]) to a smooth solution $(\Omega, g(t))$, $t \in (0,S)$, then the lemma tells us that the Ricci curvature of $(\Omega, g(t))$ is non-negative for all $t \in (0,S)$. Another useful application of the second lemma is: if a solution $(M, g(t))$, $t \in [0,T)$ satisfies

$$\begin{aligned} |\text{Riem}(g)| &\leqq \frac{c_0}{t}, \\ \text{Ricci}(g(0)) &\geqq -\varepsilon g(0) \end{aligned} \tag{5.3}$$

then for a well controlled time interval the solution satisfies

$$\text{Ricci}(g) \geqq -c_0 \varepsilon g.$$

As we saw in Lemma 3.4, such a bound is relevant to the question of existence of solutions to the Ricci flow. We apply this lemma in the main Theorem 7.1 and the Application 7.2.

Lemma 5.1. *Let g_0 be a smooth metric on a 3-dimensional manifold M^3 which satisfies*

$$\text{Ricci}(g_0) \geqq -\frac{\varepsilon_0}{4} g_0 - \frac{\varepsilon_0}{4} \text{R} g_0 \quad \left(\sec(g_0) \geqq -\frac{\varepsilon_0}{4} - \text{R}\frac{\varepsilon_0}{4}\right) \tag{5.4}$$

for some $0 < \varepsilon_0 < 1/100$, and let $(M, g(\cdot,t))_{t\in[0,T)}$ be a solution to Ricci flow with $g(0) = g_0(\cdot)$. Then

$$\text{Ricci}(g(t)) \geqq -\varepsilon_0(1+4t)g(t) - \varepsilon_0(1+4t)\text{R}(g(t))g(t), \quad \forall t \in [0,T) \cap \left[0, \frac{1}{8}\right)$$

$$\left(\sec(g(t)) \geqq -\varepsilon_0\left(\frac{1}{2}+t\right) - \varepsilon_0\left(\frac{1}{2}+t\right)\text{R}(g(t)), \quad \forall t \in [0,T) \cap \left[0, \frac{1}{8}\right)\right).$$

Proof. Define $\varepsilon = \varepsilon(t) = \varepsilon_0(1+4t)$, and the tensor $L(t)$ by

$$L_{ij} := \text{Ricci}_{ij} + \varepsilon \text{R} g_{ij} + \varepsilon g_{ij}.$$

We shall often write ε for $\varepsilon(t)$ (not to be confused with ε_0). Notice that $\varepsilon_0 < \varepsilon(t) \leqq 2\varepsilon_0$, for all $t \in [0, 1/8)$: we will use this freely. Then $L_i^j = (R_i^j + \varepsilon R \delta_i^j + \varepsilon \delta_i^j)$, and

$$\begin{aligned} \left(\frac{\partial}{\partial t} L\right)_{ij} &= \left(\frac{\partial}{\partial t} L_i^l\right) g_{jl} - 2L_i^l R_{jl} \\ &= g_{jl}\left(\frac{\partial}{\partial t}(R_{ik} g^{kl}) + \varepsilon \frac{\partial}{\partial t} R \delta_i^l + 4\varepsilon_0 R \delta_i^l + 4\varepsilon_0 \delta_i^l\right) - 2L_i^l R_{jl} \end{aligned}$$

$$= g_{jl}\frac{\partial}{\partial t}(R_{ik}g^{kl}) + \varepsilon g_{ij}\frac{\partial}{\partial t}R + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl}$$

$$= g_{jl}\big((\Delta\,\mathrm{Ricci})_i^l - Q_i^l + 2R_{ik}R_{sm}g^{km}g^{ls}\big)$$

$$+ \varepsilon g_{ij}(\Delta R + 2|\mathrm{Ricci}|^2) + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl}$$

$$= (\Delta L)_{ij} - Q_{ij} + 2R_{ik}R_{jm}g^{km} + 2\varepsilon|\mathrm{Ricci}|^2 g_{ij}$$

$$+ 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl},$$

where Q is the tensor

$$\begin{aligned} Q_{ij} &:= 6S_{ij} - 3RR_{ij} + (R^2 - 2S)g_{ij}, \\ S_{ij} &:= g^{kl}R_{ik}R_{jl} \end{aligned} \tag{5.5}$$

(see [9], Theorem 8.4). Clearly $L_{ij}(0) > 0$. Define N_{ij} by

$$N_{ij} := -Q_{ij} + 2R_{im}R_{sj}g^{ms} + 2\varepsilon|\mathrm{Ricci}|^2 g_{ij} + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl}.$$

We argue as in the proof of Hamilton's maximum principle, [9], Theorem 9.1.

We claim that $L_{ij}\big(g(t)\big) \geqq 0$. Assume there exist a first time and point (p_0, t_0) and a direction w_{p_0} for which $L(w,w)\big(g(t)\big)(p_0,t_0) = 0$. Choose coordinates about p_0 so that at (p_0,t_0) they are orthonormal, and so that Ricci is diagonal at (p_0,t_0). Clearly L is then also diagonal at (p_0,t_0). W.l.o.g.

$$\begin{aligned} R_{11} &= \lambda, \\ R_{22} &= \mu, \\ R_{33} &= v, \end{aligned} \tag{5.6}$$

and $\lambda \leqq \mu \leqq v$, and so

$$L_{11} = \lambda + \varepsilon(t_0)R + \varepsilon(t_0) \leqq L_{22} \leqq L_{33},$$

and so $L_{11} = 0$ (otherwise $L(p_0,t_0) > 0$: a contradiction). In particular,

$$N_{11}(p_0,t_0) = (\mu - v)^2 + \lambda(\mu + v) + 2\varepsilon\lambda^2 + 2\varepsilon\mu^2 + 2\varepsilon v^2 + 4\varepsilon_0 R + 4\varepsilon_0, \tag{5.7}$$

in view of the definition of Q (see [9], Corollary 8.2, Theorems 8.3, 8.4) and the fact that $L_{11} = 0$. Also, $L_{11} = 0 \Rightarrow \lambda = -\varepsilon R - \varepsilon$ at (p_0,t_0), and so, substituting this into (5.7), we get

$$\begin{aligned} N_{11}(p_0,t_0) &= (u - v)^2 + (-\varepsilon R - \varepsilon)(\mu + v) + 2\varepsilon(\lambda^2 + \mu^2 + v^2) + 4\varepsilon_0 R + 4\varepsilon_0 \\ &\geqq \varepsilon\big(-(\lambda + \mu + v)(\mu + v) + 2\lambda^2 + 2\mu^2 + 2v^2\big) + 4\varepsilon_0 R + 4\varepsilon_0 - \varepsilon(\mu + v) \\ &= \varepsilon\big(-(\lambda + \mu + v)(\mu + v) + 2\lambda^2 + 2\mu^2 + 2v^2\big) + 4\varepsilon_0 R + 4\varepsilon_0 - \varepsilon R + \varepsilon\lambda \\ &\geqq \varepsilon(\lambda - \lambda\mu - \lambda v + \mu^2 + v^2 + 2\lambda^2 - 2\mu v) + 4\varepsilon_0 R + 4\varepsilon_0 - \varepsilon R. \end{aligned}$$

To show $N_{11} > 0$, we consider a number of cases.

- *Case* 1. $\lambda \geqq 0$. This combined with $L_{11} = 0$ implies that $R < 0$. A contradiction to the fact that $\lambda \geqq 0$ and λ is the smallest eigenvalue of Ricci.

- *Case* 2. $\lambda \leqq 0$, $R \geqq 0$. This implies $\nu \geqq 0$ and hence

$$N_{11} \geqq \varepsilon(\lambda - \lambda\mu + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) + 4\varepsilon_0,$$

in view of the fact that $\varepsilon R \leqq 2\varepsilon_0 R$. In the case $\mu \geqq 0$ we obtain

$$N_{11} \geqq \varepsilon(\lambda + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) + 4\varepsilon_0 \geqq -\varepsilon + 4\varepsilon_0 > 0,$$

after an application of Young's inequality, and similarly in the case $\mu \leqq 0$ we get

$$N_{11} \geqq \varepsilon(\lambda - \lambda\mu + \mu^2 + \nu^2 + 2\lambda^2) + 4\varepsilon_0 > 0.$$

- *Case* 3. $\lambda \leqq 0$, $R \leqq 0$. We know that $R(g_0) \geqq -3\varepsilon_0$ will be preserved by Ricci flow, and hence $0 \geqq R\big(g(t)\big) \geqq -3\varepsilon_0$. We break Case 3 up into three Subcases 3.1, 3.2, 3.3.

  - *Case* 3.1. $\mu, \nu \leqq 0$. This with $R \geqq -3\varepsilon_0$ implies that $|\lambda|, |\mu|, |\nu| \leqq 3\varepsilon_0$ and hence

$$N_{11} \geqq -3\varepsilon\varepsilon_0 - 36\varepsilon\varepsilon_0^2 - 12\varepsilon_0^2 + 4\varepsilon_0 \geqq -100\varepsilon_0^2 + 4\varepsilon_0 > 0,$$

since $0 < \varepsilon_0 < 1/100$, $\varepsilon < 2\varepsilon_0 < 1$.

  - *Case* 3.2. $\mu \leqq 0$, $\nu \geqq 0$. Implies

$$N_{11} \geqq \varepsilon(\lambda - \lambda\mu + \mu^2 + \nu^2 + 2\lambda^2) - 12\varepsilon_0^2 + 4\varepsilon_0 > 0,$$

in view of Young's inequality, $\varepsilon_0 \leqq 1/100$, and $0 < \varepsilon < 2\varepsilon_0$.

  - *Case* 3.3. $\mu \geqq 0$ ($\Rightarrow \nu \geqq 0$). Then, similarly,

$$N_{11} \geqq \varepsilon(\lambda + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) - 12\varepsilon_0^2 + 4\varepsilon_0 > 0.$$

So in all cases $N_{11} > 0$. The rest of the proof is standard (see [9], Theorem 9.1): extend $w(p_0, t_0) = \frac{\partial}{\partial x^1}(p_0, t_0)$ in space to a vector field $w(\cdot)$ in a small neighbourhood of p_0 so that ${}^{g(t_0)}\nabla w(\cdot)(p_0, t_0) = 0$, and let $w(\cdot, t) = w(\cdot)$. Then

$$0 \geqq \left(\frac{\partial}{\partial t} L(w, w)\right)(p_0, t_0) \geqq \big(\Delta L(w, w)\big)(p_0, t_0) + N(w, w) > 0,$$

which is a contradiction.

The case for the sectional curvatures is similar: from [10], Sec. 5, we know that the reaction equations for the curvature operator are

$$\frac{\partial}{\partial t}\alpha = \alpha^2 + \beta\gamma,$$

$$\frac{\partial}{\partial t}\beta = \beta^2 + \alpha\gamma,$$

$$\frac{\partial}{\partial t}\gamma = \gamma^2 + \alpha\beta.$$

Note that

$$(5.8) \qquad R = \alpha + \beta + \gamma,$$

$$|\mathrm{Ricci}|^2 = \left(\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha+\gamma}{2}\right)^2 + \left(\frac{\beta+\gamma}{2}\right)^2\right)$$

$$= \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma).$$

Similar to the Ricci case, we examine the function $\alpha + \varepsilon(t)R + \varepsilon(t)$ where $\alpha \leqq \beta \leqq \gamma$ are eigenvalues of the curvature operator, and $\varepsilon(t) = \varepsilon_0\left(\frac{1}{2} + t\right)$. In order to make the following inequalities more readable, we write ε in place of $\varepsilon(t)$: that is, $\varepsilon = \varepsilon_0\left(\frac{1}{2} + t\right)$.

$$\frac{\partial}{\partial t}(\alpha + \varepsilon R + \varepsilon) = \varepsilon_0 + \varepsilon_0 R + \alpha^2 + \beta\gamma + 2\varepsilon|\mathrm{Ricci}|^2$$

$$= \varepsilon_0 + \varepsilon_0 R + \alpha^2 + \beta\gamma + \varepsilon(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma),$$

and so in the case that $\beta, \gamma \geqq 0$, or $\beta, \gamma \leqq 0$, $\frac{\partial}{\partial t}(\alpha + \varepsilon R + \varepsilon) \geqq \varepsilon_0(1 + R) > 0$. So assume that $\alpha \leqq \beta \leqq 0$, and $\gamma \geqq 0$. Combining these inequalities with $\varepsilon(t) \leqq \varepsilon_0$, we see that

$$\frac{\partial}{\partial t}(\alpha + \varepsilon R + \varepsilon) \geqq \varepsilon_0 + \varepsilon_0 R + \alpha^2 + \alpha\gamma + \varepsilon(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= \varepsilon_0 + \varepsilon_0 R + \alpha^2 + (\alpha + \varepsilon R + \varepsilon)\gamma$$

$$- \varepsilon R\gamma - \varepsilon\gamma + \varepsilon(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= \varepsilon_0 + \varepsilon_0 R + \alpha^2 + (\alpha + \varepsilon R + \varepsilon)\gamma - \varepsilon\gamma + \varepsilon(\alpha^2 + \beta^2 + \alpha\beta),$$

$$\geqq \alpha^2 + (\alpha + \varepsilon R + \varepsilon)\gamma + \varepsilon_0(1 + R - \gamma) + \varepsilon(\alpha^2 + \beta^2),$$

which, using $\varepsilon(t) \geqq \varepsilon_0/2$, is

$$\geqq \alpha^2 + (\alpha + \varepsilon R + \varepsilon)\gamma + \varepsilon_0\left(1 + \alpha + \beta + \frac{\alpha^2}{2} + \frac{\beta^2}{2}\right),$$

$$\geqq \alpha^2 + (\alpha + \varepsilon R + \varepsilon)\gamma,$$

in view of Young's inequality. At a point where $\alpha + \varepsilon R + \varepsilon = 0$, the last sum is strictly bigger than zero (if $\alpha = 0$, then, $R \geqq 0$, and hence $\alpha + \varepsilon R + \varepsilon \geqq \varepsilon > 0$: a contradiction). Then we argue as above. □

The above lemma shows us that if the Ricci curvature at time zero is bigger than $-\varepsilon$ (ε small) then the Ricci curvature divided by the scalar curvature is at most $-c\varepsilon$ at points where the scalar curvature is bigger than one (for a short well defined time interval). It can of course happen that the Ricci curvature becomes very large and negative in a short time, if the scalar curvature is very large and positive in a short time.

Now we prove an improved version of the above theorem, which allows for some scaling in time. In particular, for the class of solutions where $|\mathrm{Riem}|t \leqq c_0$ it tells us that: if the Ricci curvature at time zero is bigger than $-\varepsilon$ (ε small) then the Ricci curvature is at most $-c\varepsilon$ for some short well defined time interval.

Lemma 5.2. *Let g_0 be a smooth metric on a 3-dimensional manifold M^3 which satisfies*

$$\mathrm{Ricci}(g_0) \geqq -\frac{\varepsilon_0}{4} g_0 \qquad (5.9)$$
$$(\sec(g_0) \geqq -\frac{\varepsilon_0}{4})$$

for some $0 < \varepsilon_0 < 1/100$, and let $\big(M, g(\cdot,t)\big)_{t\in[0,T)}$ be a solution to Ricci flow with $g(0) = g_0(\cdot)$. Then

$$\mathrm{Ricci}\big(g(t)\big) \geqq -\varepsilon_0(1+kt)g(t) - \varepsilon_0(1+kt)t\mathrm{R}\big(g(t)\big)g(t), \quad \forall t \in [0,T) \cap [0,T')$$
$$(\sec\big(g(t)\big) \geqq -\varepsilon_0\left(\frac{1}{2}+kt\right) - \varepsilon_0\left(\frac{1}{2}+kt\right)t\mathrm{R}\big(g(t)\big), \quad \forall t \in [0,T) \cap [0,T'))$$

where $k = 100$ and $T' = T'(100) > 0$ is a universal constant.

Proof. The proof is similar to that above. Define $\varepsilon = \varepsilon(t) = \varepsilon_0(1+kt)$, and the tensor $L(t)$ by

$$L_{ij} := \mathrm{Ricci}_{ij} + \varepsilon t \mathrm{R} g_{ij} + \varepsilon g_{ij}.$$

We shall often write ε for $\varepsilon(t)$ (not to be confused with ε_0). Notice that $\varepsilon_0 < \varepsilon(t) \leqq 2\varepsilon_0$, for all $t \in [0, 1/k)$: we will use this freely. Then

$$L_i^j = (R_i^j + \varepsilon t R \delta_i^j + \varepsilon \delta_i^j),$$

and

$$\left(\frac{\partial}{\partial t}L\right)_{ij} = \left(\frac{\partial}{\partial t}L_i^l\right)g_{jl} - 2L_i^l R_{jl}$$
$$= g_{jl}\left(\frac{\partial}{\partial t}(R_{ik}g^{kl}) + \varepsilon \mathrm{R}\delta_i^l + \varepsilon t \frac{\partial}{\partial t}\mathrm{R}\delta_i^l + k\varepsilon_0 t \mathrm{R}\delta_i^l + k\varepsilon_0 \delta_i^l\right) - 2L_i^l R_{jl}$$

$$= g_{jl}\frac{\partial}{\partial t}(R_{ik}g^{kl}) + \varepsilon \mathrm{R} g_{ij} + \varepsilon t g_{ij}\frac{\partial}{\partial t}\mathrm{R} + k\varepsilon_0 t \mathrm{R} g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl}$$

$$= g_{jl}\big((\Delta\,\mathrm{Ricci})_i^l - Q_i^l + 2R_{ik}R_{sm}g^{km}g^{ls}\big) + \varepsilon \mathrm{R} g_{ij}$$

$$+ \varepsilon t g_{ij}(\Delta R + 2|\mathrm{Ricci}|^2) + k\varepsilon_0 t \mathrm{R} g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl}$$

$$= (\Delta L)_{ij} - Q_{ij} + 2R_{ik}R_{jm}g^{km} + \varepsilon \mathrm{R} g_{ij} + 2\varepsilon t|\mathrm{Ricci}|^2 g_{ij}$$

$$+ k\varepsilon_0 t R g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl},$$

where Q is the tensor defined in Equation (5.5). Clearly $L_{ij}(0) > 0$. Define N_{ij} by

$$N_{ij} := -Q_{ij} + 2R_{ik}R_{jm}g^{km} + \varepsilon \mathrm{R} g_{ij} + 2\varepsilon t|\mathrm{Ricci}|^2 g_{ij} + k\varepsilon_0 t R g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl}.$$

We argue as in the proof of Hamilton's maximum principle, [9], Theorem 9.1.

We claim that $L_{ij}\big(g(t)\big) > 0$ for all $t \in [0, T)$. Assume there exist a first time and point (p_0, t_0) and a direction w_{p_0} for which $L(w, w)\big(g(t)\big)(p_0, t_0) = 0$. Choose coordinates about p_0 so that at (p_0, t_0) they are orthonormal, and so that Ricci is diagonal at (p_0, t_0). Clearly L is then also diagonal at (p_0, t_0). W.l.o.g.

$$R_{11} = \lambda, \quad R_{22} = \mu, \quad R_{33} = \nu, \tag{5.10}$$

and

$$\lambda \leqq \mu \leqq \nu,$$

and so

$$L_{11} = \lambda + \varepsilon(t_0)t_0 R + \varepsilon(t_0) \leqq L_{22} \leqq L_{33},$$

and so $L_{11} = 0$ (otherwise $L(p_0, t_0) > 0$: a contradiction). In particular,

$$N_{11}(p_0, t_0) = (\mu - \nu)^2 + \lambda(\mu + \nu) + 2\varepsilon t\lambda^2 + 2\varepsilon t\mu^2 + 2\varepsilon t\nu^2 + \varepsilon \mathrm{R} g_{ij} + k\varepsilon_0 t R g_{ij} + k\varepsilon_0 g_{ij} \tag{5.11}$$

in view of the definition of Q (see [9], Corollary 8.2, Theorems 8.3, 8.4) and the fact that $L_{11} = 0$. We will show that $N_{11}(p_0, t_0) > 0$. $L_{11} = 0 \Rightarrow \lambda = -\varepsilon t_0 R - \varepsilon$ at (p_0, t_0), and so, substituting this into (5.7), we get

$$N_{11}(p_0, t_0) = (u - \nu)^2 + (-\varepsilon t_0 R - \varepsilon)(\mu + \nu) + 2\varepsilon t_0(\lambda^2 + \mu^2 + \nu^2)$$

$$+ \varepsilon \mathrm{R} + k\varepsilon_0 t \mathrm{R} g_{ij} + k\varepsilon_0$$

$$\geqq \varepsilon t_0\big(-(\lambda + \mu + \nu)(\mu + \nu) + 2\lambda^2 + 2\mu^2 + 2\nu^2\big) - \varepsilon(\mu + \nu)$$

$$+ \varepsilon \mathrm{R} + k\varepsilon_0 t_0 \mathrm{R} + k\varepsilon_0$$

$$
\begin{aligned}
&= \varepsilon t_0\big(-(\lambda+\mu+\nu)(\mu+\nu)+2\lambda^2+2\mu^2+2\nu^2\big)\\
&\quad+\big((-\varepsilon^2 t_0+k\varepsilon_0 t_0)\mathrm{R}-\varepsilon^2+k\varepsilon_0\big)\\
&\geqq \varepsilon t_0(-\lambda\mu-\lambda\nu+\mu^2+\nu^2+2\lambda^2-2\mu\nu)\\
&\quad+\big((-\varepsilon^2 t_0+k\varepsilon_0 t_0)\mathrm{R}-\varepsilon^2+k\varepsilon_0\big)
\end{aligned}
$$

where here we have used once again that

$$\lambda(x_0,t_0)=-\varepsilon(t_0)t_0R(x_0,t_0)-\varepsilon(t_0).$$

If $\mathrm{R}(x_0,t_0)\leqq 0$, then using the fact that $\mathrm{R}\geqq -3\varepsilon_0$ is preserved by the flow, we see that

$$\big(-\varepsilon^2(t_0)t_0+k\varepsilon_0(t_0)t_0\big)\mathrm{R}(x_0,t_0)-\varepsilon^2+k\varepsilon_0\geqq\frac{k}{2}\varepsilon_0.$$

Furthermore:

- [i] $\lambda=-\varepsilon R-\varepsilon\leqq\varepsilon$ (since $R\geqq-3\varepsilon_0$) and $\lambda=-\varepsilon R-\varepsilon\geqq-\varepsilon$, that is $|\lambda|\leqq\varepsilon$.
- [ii] Similarly $|\mu+\nu|=|R-\lambda|\leqq 4\varepsilon$.

Hence

$$\varepsilon t_0\big(-\lambda(\mu+\nu)+\mu^2+\nu^2+2\lambda^2-2\mu\nu\big)\geqq-50\varepsilon_0^2,$$

and so $N_{11}(p_0,t_0)>0$. Hence we must only consider the case $R(p_0,t_0)\geqq 0$.

- *Case* 1. $\lambda\geqq 0$. This combined with $L_{11}=0$ implies that $R(p_0,t_0)<0$. A contradiction.
- *Case* 2. $\lambda\leqq 0$, $\mu\geqq 0$, $\nu\geqq 0$. In this case we trivially obtain $N_{11}>0$.
- *Case* 3. $\lambda\leqq 0$, $\mu\leqq 0$, $\nu\geqq 0$. Implies

$$N_{11}>\varepsilon t_0(-\lambda\mu+\mu^2+\nu^2+2\lambda^2)\geqq 0,$$

in view of Young's inequality.

So in all cases $N_{11}>0$. The rest of the proof is standard (see [9], Theorem 9.1): extend $w(p_0,t_0)=\frac{\partial}{\partial x^1}(p_0,t_0)$ in space to a vector field $w(\cdot)$ in a small neighbourhood of p_0 so that ${}^{g(t_0)}\nabla w(\cdot)(p_0,t_0)=0$, and let $w(\cdot,t)=w(\cdot)$. Then

$$0\geqq\left(\frac{\partial}{\partial t}L(w,w)\right)(p_0,t_0)\geqq\big(\Delta L(w,w)\big)(p_0,t_0)+N(w,w)>0,$$

which is a contradiction.

The case for the sectional curvatures is similar: from [10], Sec. 5, we know that the reaction equations for the curvature operator are

$$\frac{\partial}{\partial t}\alpha = \alpha^2 + \beta\gamma,$$

$$\frac{\partial}{\partial t}\beta = \beta^2 + \alpha\gamma,$$

$$\frac{\partial}{\partial t}\gamma = \gamma^2 + \alpha\beta.$$

In what follows, we use the formulae (5.8) freely.

Similar to the Ricci case, we examine the function $\alpha + \varepsilon(t)tR + \varepsilon(t)$ where $\alpha \leqq \beta \leqq \gamma$ are eigenvalues of the curvature operator, and $\varepsilon(t) = \varepsilon_0\left(\frac{1}{2} + kt\right)$. In order to make the following inequalities more readable, we write ε in place of $\varepsilon(t)$: that is, $\varepsilon = \varepsilon_0\left(\frac{1}{2} + kt\right)$. We assume $t \leqq \frac{1}{2k}$ so that $\varepsilon_0 \frac{1}{2} \leqq \varepsilon(t) \leqq \varepsilon_0$.

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha + \varepsilon tR + \varepsilon) &= \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + \alpha^2 + \beta\gamma + 2\varepsilon t|\mathrm{Ricci}|^2\\ &= \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + \alpha^2 + \beta\gamma\\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma),\end{aligned}$$

and so in the case that $\beta, \gamma \geqq 0$, or $\beta, \gamma \leqq 0$,

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha + \varepsilon R + \varepsilon) &\geqq \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0\\ &\geqq -3\varepsilon_0^2 - 3\varepsilon_0^2 + k\varepsilon_0 > 0.\end{aligned} \tag{5.12}$$

So assume that $\alpha \leqq \beta \leqq 0$, and $\gamma \geqq 0$. Combining these inequalities with $\varepsilon(t) \leqq \varepsilon_0$, we see that

$$\begin{aligned}\frac{\partial}{\partial t}(\alpha + \varepsilon tR + \varepsilon) &\geqq \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + \alpha\gamma\\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + (\alpha + \varepsilon tR + \varepsilon)\gamma\\ &\quad - \varepsilon tR\gamma - \varepsilon\gamma + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \varepsilon R - \varepsilon\gamma + k\varepsilon_0 tR + k\varepsilon_0 + (\alpha + \varepsilon tR + \varepsilon)\gamma\\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta)\\ &= \varepsilon(\alpha + \beta) + k\varepsilon_0 tR + k\varepsilon_0 + (\alpha + \varepsilon tR + \varepsilon)\gamma\\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta)\\ &\geqq (2\varepsilon\alpha + k\varepsilon_0 tR + k\varepsilon_0) + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta)\end{aligned}$$

at a point where $\alpha + \varepsilon t R + \varepsilon = 0$. Using $\alpha + \varepsilon t R + \varepsilon = 0$ again, we get

$$\begin{aligned} 2\varepsilon\alpha + k\varepsilon_0 tR + k\varepsilon_0 &= 2\varepsilon(-\varepsilon t R - \varepsilon) + k\varepsilon_0 tR + k\varepsilon_0 \\ &= Rt(-2\varepsilon^2 + k\varepsilon_0) + k\varepsilon_0 - 2\varepsilon^2 \\ &> \frac{k}{2}\varepsilon_0, \end{aligned}$$

since $R \geqq -3\varepsilon_0$ is preserved by the flow, and $t \leqq 1/k$. Hence

$$\frac{\partial}{\partial t}(\alpha + \varepsilon t R + \varepsilon) \geqq \frac{k}{2}\varepsilon_0 + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta) > 0,$$

at a point where $\alpha + \varepsilon t R + \varepsilon = 0$. Then we argue as above. □

So although the Ricci curvature can become very large and negative under the Ricci flow, it can only do so at a controlled rate. In particular, as we mentioned before this lemma, if the curvature satisfies $|\mathrm{Riem}|t \leqq c_0$ for all $t \in [0, T)$ (in addition to the initial conditions) then $\mathrm{Ricci} \geqq -c_1(c_0)\varepsilon_0$, is true on some well defined time interval $[0, T')$ (in dimensions two and three).

6. Bounding the diameter and volume in terms of the curvature

The results of this section hold for all dimensions.

Lemma 6.1. *Let $(M^n, g(t))_{t\in[0,T)}$ be a solution to Ricci flow with*

$$\begin{aligned} &\mathrm{Ricci}(g(t)) \geqq -c_0, \\ (6.1) \qquad &|\mathrm{Riem}(g(t))|t \leqq c_0, \\ &\mathrm{diam}(M, g_0) \leqq d_0. \end{aligned}$$

Then

$$(6.2) \qquad d(p,q,0) - c_1(t, d_0, c_0, n) \geqq d(p,q,t) \geqq d(p,q,0) - c_2(n, c_0)\sqrt{t}$$

for all $t \in [0, T)$, where

$$c_1(t, d_0, c_0, n) \to 0$$

as $t \to 0$.

In particular if ${}^i g_0$ is a sequence of smooth metrics on manifolds M_i with

$$\begin{aligned} &\mathrm{diam}(M_i, {}^i g_0) \leqq d_0, \\ (6.3) \qquad &\mathrm{d_{GH}}\big((M_i, d({}^i g_0)), (X, d_X)\big) \xrightarrow{i\to\infty} 0 \end{aligned}$$

and $\left(M_i, {}^i g(t)\right)_{t \in [0, T_i)}$ *are solutions to Ricci flow with*

$$
\begin{aligned}
& {}^i g(0) = {}^i g_0, \\
(6.4) \qquad & \sec\left({}^i g(t)\right) \geqq -c_0 \quad \left(\operatorname{Ricci}\left({}^i g(t)\right) \geqq -c_0\right), \\
& \left|\operatorname{Riem}\left({}^i g(t)\right)\right| t \leqq c_0, \quad \forall t \in [0, T_i),
\end{aligned}
$$

then

$$\mathrm{d_{GH}}\left(\left(M_i, d\left({}^i g(t_i)\right)\right), (X, d_X)\right) \xrightarrow{i \to \infty} 0$$

for any sequence $t_i \in [0, T_i)$, $i \in \mathbb{N}$ *where* $t_i \xrightarrow{i \to \infty} 0$.

Proof. The first inequality

$$d(p, q, t) \geqq d(p, q, 0) - c_1(n, c_0)\sqrt{t}$$

is proved in [12], Theorem 17.2 (with a slight modification of the proof: see Appendix C). The second inequality follows easily from [12], Lemma 17.3: see Appendix C.

The second statement is a consequence of the first result, and the triangle inequality which is valid for the Gromov-Hausdorff distance:

$$
\begin{aligned}
(6.5) \qquad & \mathrm{d_{GH}}\left(\left(M_i, d\left({}^i g(t_i)\right)\right), (X, d_X)\right) \\
& \leqq \mathrm{d_{GH}}\left(\left(M_i, d\left({}^i g(t_i)\right)\right), \left(M_i, d\left({}^i g_0\right)\right)\right) + \mathrm{d_{GH}}\left(\left(M_i, d\left({}^i g_0\right)\right), (X, d_X)\right) \\
& \leqq c(t_i) + \mathrm{d_{GH}}\left(\left(M_i, d\left({}^i g_0\right)\right), (X, d_X)\right) \xrightarrow{i \to \infty} 0.
\end{aligned}
$$

Here we have used the characterisation of Gromov-Hausdorff distance given in A.9, and the fact that the identity map $I : \left(M_i, d\left({}^i g(t_i)\right)\right) \to \left(M_i, d\left({}^i g_0\right)\right)$, is an $c(t_i)$-Hausdorff approximation, where $c(t) \to 0$ as $t \to 0$: see Appendix A, Definition A.8 and Lemma A.9. □

Corollary 6.2. *Let* $\left(M^n, g(t)\right)_{t \in [0, T)}$ *be an arbitrary solution to Ricci flow* $(g(0) = g_0)$ *satisfying the conditions* (6.1) *and assume that there exists* $v_0 > 0$ *such that*

$$(6.6) \qquad \operatorname{vol}(M, g_0) \geqq v_0 > 0.$$

Then there exists an $S = S(d_0, c_0, v_0, n) > 0$ *such that*

$$\operatorname{vol}\left(M, g(t)\right) \geqq \frac{3 v_0}{4}, \quad \forall t \in [0, T) \cap [0, S).$$

Proof. If this were not the case, then there exist solutions $\left(M_i^n, {}^i g(t)\right)_{t \in [0, T_i)}$ satisfying the stated conditions and there exist $t_i \in [0, T_i)$, $t_i \xrightarrow{i \to \infty} 0$ such that

$$\operatorname{vol}\left(M_i, {}^i g(t_i)\right) = \frac{3 v_0}{4}.$$

But then

$$\mathrm{d}_{\mathrm{GH}}\big((M_i, d({}^i g(t_i))), (X, d_X)\big) \xrightarrow{i\to\infty} 0$$

from the lemma above. According to [2], Thm. 10.8 for the case that $\sec({}^i g(t)) \geqq -c_0$ (for the Ricci case we use [4], Theorem 5.4 of Cheeger-Colding) we also have

$$v_0 \leqq \mathrm{vol}(M_i, {}^i g_0) = \mathscr{H}^n\big(M_i, d({}^i g_0)\big) \xrightarrow{i\to\infty} \mathscr{H}^n(X, d_X)$$

which implies $\mathscr{H}^n(X, d_X) \geqq v_0$. Here $\mathscr{H}^n(X, d_X)$ is the n-dimensional Hausdorff mass of X with respect to the metric d_X. Similarly we have

$$\frac{3v_0}{4} = \mathscr{H}^n\big(M_i, d({}^i g(t_i))\big) \xrightarrow{i\to\infty} \mathscr{H}^n(X, d_X).$$

This implies $\mathscr{H}^n(X, d_X) = 3v_0/4$. A contradiction. □

7. Non-collapsed compact three manifolds of almost non-negative curvature

The results of this section are only valid for dimensions two and three.

Theorem 7.1. *Let M be a closed three (or two) manifold satisfying*

$$\mathrm{diam}(M, g_0) \leqq d_0,$$
$$\mathrm{Ricci}(g_0) \geqq -\varepsilon g_0,$$
$$\mathrm{vol}(M, g_0) \geqq v_0 > 0,$$

where $\varepsilon \leqq 1/10c^2$ and $c = c(v_0, d_0) \geqq 1$ is the constant from Lemma 3.4. *Then there exists an $S = S(d_0, v_0) > 0$ and $K = K(d_0, v_0)$ such that the maximal solution $\big(M, g(t)\big)_{t\in[0,T)}$ to Ricci-flow satisfies $T \geqq S$, and*

$$\sup_M \big|\mathrm{Riem}\big(g(t)\big)\big| \leqq \frac{K}{t},$$

for all $t \in (0, S)$.

Proof. Let $[0, T')$ be the maximal time interval for which

$$\mathrm{vol}\big(M, g(t)\big) > \frac{v_0}{2},$$
$$\mathrm{Ricci}\big(g(t)\big) > -1,$$
$$\mathrm{diam}\big(g(t)\big) < 5d_0.$$

If $T' \geqq 1$ then choose $S = 1/2$. The estimate for the curvature then follows from Lemma 3.4 and we are finished. So w.l.o.g. $T' \leqq 1$. Then the diameter condition will not be violated at time T' (as one easily sees by examining the evolution equation for distance under Ricci flow). So we assume w.l.o.g. $T' \leqq 1$ and the diameter condition is not violated. From Lemma 3.4, we know that there exists a $c = c(d_0, v_0)$ such that $R(t) \leqq \frac{c}{t}$, for all $t \in [0, T')$. Using Lemma 5.2 we see that there exists a $T'' = T''\big(c(d_0, v_0)\big) > 0$ such that Ricci $\geqq -1/2$ for all $t \in [0, T''] \cap [0, T')$. So the Ricci curvature condition is not violated on $[0, T''] \cap [0, T')$. Furthermore, in view of Corollary 6.2 there exists a $T''' = T'''\big(v_0, d_0, c(d_0, v_0)\big)$, such that $\mathrm{vol}\big(M, g(t)\big) > 3v_0/4$ for all $t \in [0, T'''] \cap [0, T''] \cap [0, T']$. Hence $T' \geqq \min\big(T''\big(c(d_0, v_0)\big), T'''(v_0, d_0)\big) > 0$, as required. The estimate for the curvature and the existence of S then follow from Lemma 3.4. □

Theorem 7.2. *Let $(M_i, {}^i g_0)$ be a sequence of closed three (or two) manifolds satisfying*

$$\mathrm{diam}(M_i, {}^i g_0) \leqq d_0,$$

$$\mathrm{Ricci}({}^i g_0) \quad (\sec({}^i g_0)) \geqq -\varepsilon(i)\, {}^i g_0,$$

$$\mathrm{vol}(M_i, {}^i g_0) \geqq v_0 > 0,$$

where $\varepsilon(i) \to 0$, as $i \to \infty$. Then there exists an $S = S(v_0, d_0) > 0$ and $K = K(v_0, d_0)$ such that the maximal solutions $\big(M_i, {}^i g(t)\big)_{t \in [0, T_i)}$ to Ricci-flow satisfy $T_i \geqq S$, and

$$\sup_{M_i} \big|\mathrm{Riem}\big({}^i g(t)\big)\big| \leqq \frac{K}{t},$$

for all $t \in (0, S)$. In particular the Hamilton limit solution

$$\big(M, g(t)\big)_{t \in (0, S)} = \lim_{i \to \infty} \big(M_i, {}^i g(t)\big)_{t \in (0, S)}$$

(see [13]) exists (after taking a subsequence). It satisfies the estimates

$$\sup_{M} \big|\mathrm{Riem}\big(g(t)\big)\big| \leqq \frac{K}{t}, \tag{7.1}$$

$$\mathrm{Ricci}\big(g(t)\big) \geqq 0 \quad \big(\sec\big(g(t)\big) \geqq 0\big), \tag{7.2}$$

for all $t \in (0, S)$ and $\big(M, g(t)\big)$ is closed. Hence, if $M = M^3$, then M^3 is diffeomorphic to a quotient of one of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by group of fixed point free isometries acting properly discontinuously. Furthermore

$$\mathrm{d_{GH}}\big(\big(M, d\big(g(t)\big)\big), (X, d_\infty)\big) \to 0 \tag{7.3}$$

as $t \to 0$ where $(X, d_\infty) = \lim_{i \to \infty} \big(M_i, d({}^i g_0)\big)$ (the Gromov-Hausdorff limit).

Proof. We apply the previous theorem. Then notice that Lemma 5.1 (or Lemma 5.2) implies that $\mathrm{Ricci}\big(g(t)\big) \geqq 0$ $\big(\sec\big(g(t)\big) \geqq 0\big)$ for this limit solution, for all $t \in (0, S)$. To prove that $\mathrm{d_{GH}}\big(\big(M, d\big(g(t)\big)\big), (X, d_\infty)\big) \to 0$ use the triangle inequality as in the proof of Lemma 6.1:

$$
\begin{aligned}
(7.4)\qquad & \mathrm{d_{GH}}\big((M, d(g(t))), (X, d_\infty)\big) \\
&\leqq \mathrm{d_{GH}}\big((M, d(g(t))), (M_i, d({}^i g(t)))\big) + \mathrm{d_{GH}}\big((M_i, d({}^i g(t))), (X, d_\infty)\big) \\
&\leqq \mathrm{d_{GH}}\big((M, d(g(t))), (M_i, d({}^i g(t)))\big) \\
&\quad + \mathrm{d_{GH}}\big((M_i, d({}^i g(t))), (M_i, d({}^i g_0))\big) + \mathrm{d_{GH}}\big((M_i, d({}^i g_0)), (X, d_\infty)\big) \\
&\leqq \mathrm{d_{GH}}\big((M, d(g(t))), (M_i, d({}^i g(t)))\big) + c(t) \\
&\quad + \mathrm{d_{GH}}\big((M_i, d({}^i g_0)), (X, d_\infty)\big) \xrightarrow{i \to \infty} c(t),
\end{aligned}
$$

for all $t > 0$, where $c(t) \to 0$ as $t \to 0$: here we have used (6.2), and the characterisation of Gromov-Hausdorff distance given in A.9 to obtain $c(t)$. □

A. Gromov-Hausdorff space and Alexandrov spaces

Definition A.1. Let (Z, d) be a metric space, $p \in Z$, $r > 0$.

$$B_r(p) := \{x \in Z : d(x, p) < r\}.$$

For two non-empty subsets $A, B \subset Z$

$$\mathrm{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$
$$B_r(A) := \{x \in Z : \mathrm{dist}(x, A) < r\}.$$

Definition A.2. For subsets $X, Y \subset (Z, d)$ we define the Hausdorff distance between X and Y by

$$\mathrm{d_H}(X, Y) := \inf\{\varepsilon > 0 : X \subset B_\varepsilon(Y) \text{ and } Y \subset B_\varepsilon(X)\}.$$

Then (see [1], Prop. 7.3.3):

Proposition A.3. • $\mathrm{d_H}$ *is a semi-metric on* 2^Z (*the set of all subsets of* Z).

• $\mathrm{d_H}(A, \bar{A}) = 0$ *for all* $A \subset Z$, *where* $\bar{A}$ *is the closure of* A (*in* (Z, d)).

• *If* A *and* B *are closed subsets of* (Z, d) *and* $\mathrm{d_H}(A, B) = 0$ *then* $A = B$.

Definition A.4. For a subset $X \subset Z$, (Z, d) a metric space, we define $d|_X$ to be the metric on X defined by

$$d|_X(a, b) = d(a, b).$$

We then define the Gromov-Hausdorff distance between two abstract metric spaces (X, d_X) and (Y, d_Y) as follows:

Definition A.5. $\mathrm{d_{GH}}\big((X, d_X), (Y, d_Y)\big)$ is the infimum over all $r > 0$ such that there exists a metric space (Z, d) and maps $f : X \to Z$, $X' := f(X)$, and $g : Y \to Z$,

$Y' := g(Y)$ such that $f : (X, d_X) \to (X', d|_{X'})$ and $g : (Y, d_Y) \to (Y', d_{Y'})$ are isometries and $\mathrm{d_H}(X', Y') < r$.

Fact A.6. $\mathrm{d_{GH}}$ *satisfies the triangle inequality, i.e.,*

$$\mathrm{d_{GH}}\big((X_1, d_1), (X_3, d_3)\big) \leqq \mathrm{d_{GH}}\big((X_1, d_1), (X_2, d_2)\big) + \mathrm{d_{GH}}\big((X_2, d_2), (X_3, d_3)\big)$$

for all metric spaces (X_1, d_1), (X_2, d_2), (X_3, d_3).

Proof. See [1], Prop. 7.3.16. □

Definition A.7. A v-Hausdorff approximation $f : X \to Y$ for metric spaces (X, d_X) and (Y, d_Y) is a map which satisfies

$$\begin{aligned} \big|d_Y\big(f(x), f(x')\big) - d_X(x, x')\big| &\leqq v, \\ B_v\big(f(X)\big) &= Y. \end{aligned} \tag{A.1}$$

Definition A.8. $\mathrm{Happrox}\big((X, d_X), (Y, d_Y)\big)$ is the infimum of v such that there exists a v-Hausdorff approximation $f : X \to Y$.

The proof of following well known lemma may also be found in [1].

Lemma A.9.

$$\mathrm{Happrox}\big((X, d_X), (Y, d_Y)\big) \leqq 2\mathrm{d_{GH}}\big((X, d_X), (Y, d_Y)\big) \leqq 4\,\mathrm{Happrox}\big((X, d_X), (Y, d_Y)\big).$$

Proof. See [1], Corollary 7.3.28. □

Now we state the compactness result of Gromov.

Proposition A.10. $\mathscr{M}(n, k, d_0)$ *is precompact in Gromov-Hausdorff space.*

Proof. See [1], Remark 10.7.5. □

Clearly $\mathscr{S}(n, k, d_0) \subset \mathscr{M}\big(n, (n-1)k, d_0\big)$ and so it is also precompact in Gromov-Hausdorff space.

In [2] (Theorem 10.8), the following fact about the convergence of Hausdorff measure was shown.

Theorem A.11. *Let* $(M_i, g_i) \in \mathscr{S}(n, k, d_0)$, $i \in \mathbb{N}$ *be a sequence of smooth Riemannian manifolds with* $\mathrm{vol}(M_i, g_i) \geqq v_0 > 0$, *for all* $i \in \mathbb{N}$ *and*

$$\big(M_i, d(g_i)\big) \xrightarrow{i \to \infty} (X, d_X)$$

in Gromov-Hausdorff space. Then

$$\mathrm{vol}(M_i, g_i) = \mathscr{H}_i(M_i) \xrightarrow{i \to \infty} \mathscr{H}(M),$$

where $\mathscr{H}_i : M_i \to \mathbb{R}_0^+$ *is n-dimensional Hausdorff measure with respect to* $d(g_i)$ *and* $\mathscr{H} : X \to \mathbb{R}_0^+$, *is n-dimensional Hausdorff measure with respect to* d_X.

Proof. See for example [1], Theorem 10.10.10. □

In [4] (Theorem 5.4) the same result was proved for $\mathscr{M}(n,k,d_0)$.

Theorem A.12. *Let* $(M_i, g_i) \in \mathscr{M}(n,k,d_0)$, $i \in \mathbb{N}$ *be a sequence of smooth Riemannian manifolds with* $\mathrm{vol}(M_i, g_i) \geqq v_0 > 0$ *for all* $i \in \mathbb{N}$, *and*

$$\big(M_i, d(g_i)\big) \xrightarrow{i\to\infty} (X, d_X)$$

in Gromov-Hausdorff space. Then

$$\mathrm{vol}(M_i, g_i) = \mathscr{H}_i(M_i) \xrightarrow{i\to\infty} \mathscr{H}(M),$$

where $\mathscr{H}_i : M_i \to \mathbb{R}_0^+$ *is n-dimensional Hausdorff measure with respect to* $d(g_i)$ *and* $\mathscr{H} : X \to \mathbb{R}_0^+$, *is n-dimensional Hausdorff measure with respect to* d_X.

Proof. See [4], Theorem 5.4. □

For further properties of Alexandrov spaces with curvature $\geqq k$ see [2] or the book [1]. For further properties of spaces with curvature bounded below see [4].

B. *C*-essential points and *δ*-like necks

Definition B.1. Let $\big(M, g(t)\big)_{t\in(-\infty,T)}$, $T \in \mathbb{R} \cup \{\infty\}$, be a solution to Ricci flow. We say that $(x,t) \in M \times (-\infty, T)$ is a C-essential point if

$$|\mathrm{Riem}(x,t)|\,|t| \geqq C.$$

Definition B.2. We say that $(x,t) \in M \times (-\infty,T)$ is a δ-necklike point if there exists a unit 2-form θ at (x,t) such that

$$|\mathrm{Riem} - R(\theta \otimes \theta)| \leqq \delta|\mathrm{Riem}|.$$

δ-necklike points often occur in the process of taking a limit around a sequence of times and points which are becoming singular. If $\delta = 0$, then the inequality reads

$$|\mathrm{Riem}(x,t) - R(x,t)(\theta \otimes \theta)| = 0.$$

In three dimensions this tells us that the manifold splits. This can be seen with the help of some algebraic lemmas.

Lemma B.3. *Let* $\omega \in \Omega^2(\mathbb{R}^3)$. *Then it is possible to write*

$$\omega = X \wedge V,$$

for two orthogonal vectors X *and* V.

Remark B.4. Here we identify one forms with vectors using

$$a\,dx^1 + b\,dx^2 + c\,dx^3 \equiv (a,b,c).$$

Proof. Assume

$$\omega = a\,dx^1 \wedge dx^2 + b\,dx^1 \wedge dx^3 + c\,dx^2 \wedge dx^3. \tag{B.1}$$

Without loss of generality $b \neq 0$. Then, we may write:

$$\omega = \left(dx^1 + \frac{c}{b}dx^2\right) \wedge (a\,dx^2 + b\,dx^3). \tag{B.2}$$

So $\omega = X \wedge Y$. Now let X, Z, W be an orthogonal basis all of length $|X|$. Then

$$Y = a_1 X + a_2 Z + a_3 W.$$

This implies

$$\begin{aligned} \omega &= X \wedge (a_1 X + a_2 Z + a_3 W) \\ &= X \wedge (a_2 Z + a_3 W) \end{aligned} \tag{B.3}$$

as required $(V = a_2 Z + a_3 W)$. □

Hence we may write the θ occurring above as

$$\theta = X \wedge V.$$

Hence

$$\mathrm{Riem}(x,t) = cX \wedge V \otimes X \wedge V,$$

with

$$\{X, V, Z\}$$

an orthonormal basis for $\mathbb{R}^3$.

The set $\{X \wedge V, X \wedge Z, V \wedge Z\}$ then forms an orthonormal basis and the curvature operator $\mathscr{R}$ can be written with respect to this basis as

$$\begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the manifold splits (if the solution is complete with bounded curvature and non-negative curvature operator) in view of the arguments in [10], Chapter 9.

C. Estimates on the distance function for Riemannian manifolds evolving by Ricci flow

For completeness, we prove some results which are implied or proved in [12] and stated in [3] as editor's note 24 from the same paper in that book. The lemma we wish to prove is

Lemma C.1. *Let* $\left(M^n, g(t)\right)_{t\in[0,T)}$ *be a solution to Ricci flow with*

$$\begin{aligned} &\mathrm{Ricci}\big(g(t)\big) \geqq -c_0, \\ &\big|\mathrm{Riem}\big(g(t)\big)\big| t \leqq c_0, \qquad \text{(C.1)} \\ &\mathrm{diam}(M, g_0) \leqq d_0. \end{aligned}$$

Then

$$d(p,q,0) - c_1(t, d_0, c_0, n) \geqq d(p,q,t) \geqq d(p,q,0) - c_1(n, c_0)\sqrt{t} \qquad \text{(C.2)}$$

for all $t \in [0, T)$, *where*

$$c_1(t, d_0, c_0, n) \to 0$$

as $t \to 0$.

Proof. The first inequality

$$d(p,q,t) \geqq d(p,q,0) - c_1(n, c_0)\sqrt{t}$$

is proved in [12], Theorem 17.2 after making a slight modification of the proof. If we examine the proof there (as pointed out in [3] as editor's note 24 of the same book), we see that in fact that what is proved is:

$$d(P,Q,t) \geqq d(P,Q,0) - C\int_0^t \sqrt{M(t)}$$

where $\sqrt{M(t)}$ is any integrable function which satisfies

$$\sup_M |\mathrm{Riem}(\cdot, t)| \leqq M(t).$$

In particular, in our case we may set

$$M(t) = \frac{c}{t}$$

which then implies the first inequality. The second inequality is also a simple consequence of results obtained in [12]. Lemma 17.3 tells us that

$$\frac{\partial}{\partial t} d(P,Q,t) \leqq -\inf_{\gamma\in\Gamma} \int_\gamma \mathrm{Ricci}(T,T)\, ds$$

where the inf is taken over the compact set Γ of all geodesics from P to Q realising the distance as a minimal length, T is the unit vector field tangent to γ. Then in our case Ricci $\geqq -c_0$ implies

$$\frac{\partial}{\partial t} d(P, Q, t) \leqq c_0 d(P, Q, t).$$

This implies that

$$d(P, Q, t) \leqq \exp^{c_0 t} d(P, Q, 0),$$

and as a consequence

$$\operatorname{diam}\bigl(M, g(t)\bigr) \leqq d_0 \exp^{ct}.$$

Hence

$$\begin{aligned}(\text{C.3}) \qquad d(P, Q, t) \leqq \exp^{c_0 t} d(P, Q, 0) &= d(P, Q, 0) + (\exp^{c_0 t} - 1) d(P, Q, 0) \\ &\leqq d(P, Q, 0) + (\exp^{c_0 t} - 1)\, d_0 \exp^{ct},\end{aligned}$$

which implies the result. □

D. Notation

$\mathbb{R}^+$ is the set of positive real numbers.

$\mathbb{R}_0^+$ is the set of non-negative real numbers.

For a Riemannian manifold (M, g), $\bigl(M, d(g)\bigr)$ is the metric space induced by g. For a tensor T on M, we write ${}^g|T|^2$ to represent the norm of T with respect to the metric g on M. For example if T is a $\binom{0}{2}$ tensor, then

$${}^g|T|^2 = g^{ij} g^{kl} T_{ik} T_{jl}.$$

${}^h\nabla T$ refers to the covariant derivative with respect to h of T.

^{h}Riem or Riem(h) refers to the Riemannian curvature tensor with respect to h on M.

^{h}Ricci or Ricci(h) or ${}^h R_{ij}$ refers to the Ricci curvature of h on M.

^{h}R or R(h) refers to the scalar curvature of h on M.

$\sec(p)(v, w)$ is the sectional curvature of the plane spanned by the linearly independent vectors v, w at p.

$\sec \geqq k$ means that the sectional curvature of every plane at every point is bounded from below by k.

$\mathcal{R}$ denotes the curvature operator.

$\mathscr{R} \geqq c$ means that the eigenvalues of the curvature operator are bigger than or equal to c at every point on the manifold.

$\Gamma(h)_{ij}^k$ or ${}^h\Gamma_{ij}^k$ refer to the Christoffel symbols of the metric h in the coordinates $\{x^k\}$,

$$ {}^h\Gamma_{ij}^k = \frac{1}{2} h^{kl} \left(\frac{\partial h_{il}}{\partial x^j} + \frac{\partial h_{jl}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^l} \right). $$

For a diffeomorphism $F : M \to N$ we will sometimes consider dF, a 1-form along F, defined by

$$ dF(x) := \frac{\partial F^\alpha}{\partial x^k} dx^k(x) \frac{\partial}{\partial y^\alpha}\bigg|_{(F(x))}. $$

For a general 1-form ω along F, $\omega = \omega_i^\alpha(x)\,dx^i(x) \otimes \frac{\partial}{\partial y^\alpha}\Big|_{(F(x))}$, we define the norm of ω with respect to l (a metric on M) and γ (a metric on N) by

$$ {}^{l,\gamma}|\omega|^2(x) = l^{ij}(x)\gamma_{\alpha\beta}\big(F(x)\big)\omega_i^\alpha(x)\omega_j^\beta(x). $$

For example,

$$ {}^{l,\gamma}|dF|^2(x) = l^{ij}(x)\gamma_{\alpha\beta}\big(F(x)\big)\frac{\partial F^\alpha}{\partial x^i}(x)\frac{\partial F^\beta}{\partial x^j}(x). $$

We define ${}^{g,h}\nabla\, dF$, a $\binom{0}{2}$ tensor along F, by

$$ ({}^{g,h}\nabla\, dF)_{ij}^\alpha := \left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k(g)\frac{\partial F^\alpha}{\partial x^k} + \Gamma_{\beta\sigma}^\alpha(h)\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\sigma}{\partial x^j} \right). $$

For a general $\binom{0}{2}$ tensor ψ along F, $\psi = \psi_{ij}^\alpha(x)\,dx^i(x) \otimes dx^j(x) \otimes \frac{\partial}{\partial y^\alpha}\Big|_{(F(x))}$, we define the norm of ψ with respect to l (a metric on M) and γ (a metric on N) by

$$ {}^{l,\gamma}|\psi|^2 = \gamma_{\alpha\beta}\big(F(x)\big) l^{ks}(x) l^{ij}(x)\eta_{ik}^\alpha(x)\eta_{js}^\beta(x). $$

For example

$$ {}^{l,\gamma}|{}^{g,h}\nabla\, dF|^2 = \gamma_{\alpha\beta}\big(F(x)\big) l^{ks}(x) l^{ij}(x) \left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^k} - \Gamma_{ik}^r(g)\frac{\partial F^\alpha}{\partial x^r} + \Gamma_{\eta\sigma}^\alpha(h)\frac{\partial F^\eta}{\partial x^i}\frac{\partial F^\sigma}{\partial x^k} \right) $$
$$ \times \left(\frac{\partial^2 F^\beta}{\partial x^j \partial x^s} - \Gamma_{js}^r(g)\frac{\partial F^\beta}{\partial x^r} + \Gamma_{\phi\rho}^\beta(h)\frac{\partial F^\phi}{\partial x^j}\frac{\partial F^\rho}{\partial x^s} \right). $$

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Mathematisches Institut, Eckerstr. 1, 79104 Freiburg im Br., Germany
e-mail: msimon@mathematik.uni-freiburg.de

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