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Ricci flow of $W^{2,2}$-metrics in four dimensions

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Abstract. In this paper we construct solutions to Ricci–DeTurck flow in four dimensions on closed manifolds which are instantaneously smooth but whose initial values g are (possibly) non-smooth Riemannian metrics whose components in smooth coordinates belong to $W^{2,2}$ and satisfy $\frac{1}{a}h \le g \le ah$ for some $1 < a < \infty$ and some smooth Riemannian metric h on M. A Ricci flow related solution is constructed whose initial value is isometric in a weak sense to the initial value of the Ricci–DeTurck solution. Results for a related non-compact setting are also presented. Various L^p-estimates for Ricci flow, which we require for some of the main results, are also derived. As an application we present a possible definition of scalar curvature $\ge k$ for $W^{2,2}$-metrics g on closed four manifolds which are bounded in the L^∞-sense by $\frac{1}{a}h \le g \le ah$ for some $1 < a < \infty$ and some smooth Riemannian metric h on M.

1. Introduction

In this paper we construct solutions to Ricci flow and Ricci–DeTurck flow, which are instantaneously smooth but whose initial values are (possibly) non-smooth Riemannian metrics whose components, in smooth coordinates, belong to certain Sobolev spaces.

For a given smooth Riemannian manifold (M,h), and an interval $I \subseteq \mathbb{R}$, a smooth family $g(t)_{t\in I}$ of Riemannian metrics on M is a solution to Ricci–DeTurck h-flow if

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= g^{ab}({}^h\nabla_a {}^h\nabla_b g_{ij}) - g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) \\ &\quad + \frac{1}{2} g^{ab} g^{pq} \big({}^h\nabla_i g_{pa} {}^h\nabla_j g_{qb} + 2 {}^h\nabla_a g_{jp} {}^h\nabla_q g_{ib} \\ &\qquad - 2 {}^h\nabla_a g_{jp} {}^h\nabla_b g_{iq} - 2 {}^h\nabla_j g_{pa} {}^h\nabla_b g_{iq} - 2 {}^h\nabla_i g_{pa} {}^h\nabla_b g_{jq}\big), \quad (1.1)\end{aligned}$$

in the smooth sense on $M \times I$, where here, and in the rest of the paper, ${}^h\nabla$ refers to the covariant derivative with respect to h. A smooth family $\ell(t)_{t\in I}$ of Riemannian metrics on M is a solution to Ricci flow if

$$\frac{\partial \ell}{\partial t} = -2\mathrm{Rc}(\ell) \quad (1.2)$$

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in the smooth sense on $M \times I$. The Ricci flow was first introduced and studied by R. Hamilton in [15]. Shortly after that, the Ricci–DeTurck flow was introduced and studied by D. DeTurck in [11]. Ricci–DeTurck flow and Ricci flow in the smooth setting are closely related: given a Ricci–DeTurck flow $g(t)_{t\in I}$ on a compact manifold and an $S \in I$ there is a smooth family of diffeomorphisms $\Phi(t)\colon M \to M$, $t \in I$ with $\Phi(S) = \mathrm{Id}$ such that $\ell(t) = (\Phi(t))^* g(t)$ is a smooth solution to Ricci flow. The diffeomorphisms $\Phi(t)$ solve the following ordinary differential equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Phi^\alpha(x,t) &= V^\alpha(\Phi(x,t),t) \quad \text{for all } (x,t) \in M^n \times I, \\ \Phi(x,S) &= x, \end{aligned} \tag{1.3}$$

where $V^\alpha(y,t) := -g^{\beta\gamma}({}^{\mathrm{g}}\Gamma^\alpha_{\beta\gamma} - {}^{\mathrm{h}}\Gamma^\alpha_{\beta\gamma})(y,t)$.

There are a number of papers on solutions to Ricci–DeTurck flow and Ricci flow starting from non-smooth Riemannian metric/distance spaces which immediately become smooth. Given a non-smooth starting space (M, g_0) or (M, d_0), it is possible in some settings, to find smooth solutions $g(t)_{t\in(0,T)}$ to (1.1), respectively $\ell(t)_{t\in(0,T)}$ to (1.2) defined for some $T > 0$, where the initial values are achieved in some weak sense. Here is a non-exhaustive list of papers, where examples of this type are constructed: [3, 8, 9, 16, 17, 20, 23, 27, 30–32, 35, 36]. The initial non-smooth data considered in these papers has certain structure, which when assumed in the smooth setting, leads to a priori estimates for solutions, which are then used to construct solutions in the class being considered. In some papers this initial structure comes from geometric conditions, in others from regularity conditions on the initial function space of the metric components in smooth coordinates. In the second instance, this is usually in the setting, that one has some C^0-control of the metric. That is, the metric is close in the L^∞-sense to the standard euclidean metric in smooth coordinates:

$$(1-\varepsilon)\delta \le g(0) \le (1+\varepsilon)\delta$$

for a sufficiently small ε. In the current paper, the structure of the initial metric $g(0)$ comes from the assumption, in the four-dimensional compact setting, that the components in coordinates are in $W^{2,2}$, and uniformly bounded from above and below:

$$\frac{1}{c}\delta \le g(0) \le c\delta$$

for some constant c. Closeness of the metric to δ is not assumed. With this initial structure, we show that a solution to Ricci–DeTurck flow exists. In the non-compact setting, we further require a uniform local *smallness* bound on the $W^{2,2}$-norm and a global uniform bound from above and below in the L^∞-sense, both with respect to a geometrically controlled background metric. We also investigate the question of how the initial values are achieved, in the metric and distance sense, as time goes back to zero. See Theorem 2.2 in the next section for details.

Using this solution to Ricci–DeTurck flow, we show without much trouble, that there is a Ricci flow related solution. The Ricci flow solution is related to the Ricci–DeTurck solution through a smooth family of isometries $(\Phi(t))_{t\in(0,T)}$ defined for a positive time interval, and having the property that $\Phi(S) = \mathrm{Id}$ for some $S > 0$. The convergence as time goes back to zero in the distance and metric sense is investigated for this Ricci Flow solution. We require some new estimates on convergence in the L^p-sense for solutions to Ricci flow, in order to show that there is indeed a limiting weak Riemannian metric, as time approaches to zero. We also show that the initial metric value of the Ricci flow that is achieved is isometric, in a weak sense, to the initial value $g(0)$ of the Ricci–DeTurck flow solution. See Theorem 2.3 in the next section for details.

Section 12 contains an application of the results obtained in the sections preceding it. We present a possible definition of 'the scalar curvature of g is bounded from below by $k \in \mathbb{R}$' for a metric

$$g \in W^{2,2} \cap L^\infty$$

with $\frac{1}{a}h \le g \le ah$ for some $1 < a < \infty$ and some smooth Riemannian metric h on a closed manifold M.

We conclude this introduction by noting that there are metrics

$$g_0 \in W^{2,n/2}(M) \cap L^\infty(M)$$

on compact n-dimensional manifolds, which satisfy $\frac{1}{a}h \le g_0 \le ah$ for some $0 < a < \infty$ and some smooth fixed Riemannian metric h on M, but are not continuous. In particular,

$$g_0 \in W^{2,2}(M) \cap L^\infty(M)$$

and $\frac{1}{a}h \le g_0 \le ah$, but g_0 is not continuous when $n = 4$. In the example we present below, there is a point $p \in M$, such that the values $\frac{1}{a}h$ and ah are achieved by g_0 infinitely often for *every* neighbourhood of p.

Let (M, h) be a smooth compact n-dimensional manifold, $U \subseteq M$ open, and $\varphi: U \to \varphi(U) = \mathbb{B}_1(0)$ be coordinates and $\tilde{h} := \varphi_* h$, the push forward of h with φ to $\mathbb{B}_1(0)$. For $\varepsilon, r > 0$, $c \in \mathbb{R}$, let

$$f = f_{\varepsilon,r,c}: \mathbb{B}_1(0) \to \mathbb{R}$$

be the $W^{2,n/2}(\mathbb{B}_1(0))$-function defined by

$$f(x) = r\Bigg(\frac{1}{\varepsilon}\Bigg(1 + \varepsilon + \sin\Bigg(c + \log\Bigg(\log\Bigg(\frac{2}{|x|}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)$$

for $x \neq 0$ and $f(0) = 0$. Then f is bounded from above and from below by

$$f(\cdot) \in \left[r, r\left(\frac{2+\varepsilon}{\varepsilon}\right)\right]$$

and the values r and $r((2+\varepsilon)/\varepsilon)$ are both achieved infinitely many times on any neighbourhood of $0 \in \mathbb{B}_1(0)$, and consequently we see that f is also not continuous. Now we set

$$\widetilde{g}(x) = (1-\eta(x))\widetilde{h}(x) + \eta(x)\widehat{g}(x),$$

where η is a smooth cut-off function $\eta \in [0,1]$ with support in $\mathbb{B}_{1/2}(0)$, where

$$\widehat{g}_{ij}(x) = f_{\varepsilon_i, r_i, c_i}(x)\delta_{ij},$$

where $\varepsilon_i, r_i, c_i \in \mathbb{R}$, $i, j \in \{1, \dots, n\}$, $\varepsilon_i, r_i > 0$. Then the metric g defined by $g = \varphi^*(\widetilde{g})$ on U, and $g = h$ on $M \setminus U$ is a metric on M with $g \in W^{2,n/2}(M) \cap L^\infty(M)$, $\frac{1}{a}h \leq g \leq ah$ for some $1 < a < \infty$, and g is not continuous.

2. Main results

The assumptions we make on the smooth background metric are as follows

$$\begin{aligned}
&(M,h) \text{ is a smooth, connected, complete manifold without boundary such that}\\
&v_i := \sup_M {}^h|{}^h\nabla^i \mathrm{Rm}(h)| < \infty \text{ for all } i \in \mathbb{N}_0, \text{ and}\\
&\mathrm{inj}(M,h) \geq i_0 > 0.
\end{aligned} \tag{2.1}$$

Such manifolds always satisfy a local uniform Sobolev inequality: there exist constants $0 < r_0(n,h)$, $C_S(n) < \infty$ such that

$$\left(\int_M f^{2n/(n-2)}\, dh\right)^{(n-2)/2} \leq C_S(n) \int_M |{}^h\nabla f|^2\, dh$$

and

$$\left(\int_M f^n\, dh\right)^{1/2} \leq C_S(n) \int_M |{}^h\nabla f|^{n/2}\, dh$$

for all smooth f whose support is contained in a ball of radius $r_0(n,h) > 0$. For the readers' convenience, we have included a proof in Section B; see Lemma B.1 and Remark B.2.

Ultimately we would like to construct solutions to (1.1) on four manifolds starting with initial data g_0 which are uniformly bounded from above and below by a multiple of h, g_0 is locally in $W^{2,2}$, and for which the homogeneous $W^{2,2}$-energy of g_0 is uniformly bounded,

$$E(g_0) := \int_M \left(|^h\nabla g_0|^2 + |^h\nabla^2 g_0|^2\right) dh < \infty.$$

That is, we assume that there exists $a > 0$ such that

$$\frac{1}{a}h \le g_0 \le ah, \quad E(g_0) := \int_M \left(|^h\nabla g_0|^2 + |^h\nabla^2 g_0|^2\right) dh < \infty. \tag{2.2}$$

In this setting we show the following theorem.

Theorem 2.1. *Let $1 < a < \infty$ and (M^4, h) be a four-dimensional smooth Riemannian manifold satisfying (2.1), and g_0 be a $W^{2,2} \cap L^\infty$ Riemannian metric, not necessarily smooth, which satisfies*

$$\frac{1}{a}h \le g_0 \le ah \tag{a}$$

and

$$\int_M \left(|^h\nabla g_0|^2 + |^h\nabla^2 g_0|^2\right) dh < \infty.$$

Then for any $0 < \varepsilon < 1$, there exist constants $0 < T = T(g_0, h, a, \varepsilon)$, $r = r(g_0, h, a, \varepsilon)$, $c_j = c_j(h, a, \varepsilon) < \infty$ for all $j \in \mathbb{N}_0$ and a smooth solution $(g(t))_{t\in(0,T]}$ to (1.1), where $g(t)$ satisfies

$$\frac{1}{400a}h \le g(t) \le 400ah, \tag{a_t}$$

$$\int_{B_r(x)} \left(|^h\nabla g(\cdot,t)|^2 + |^h\nabla^2 g(\cdot,t)|^2\right) dh \le \varepsilon, \tag{$\mathrm{b}_t(r)$}$$

$$|^h\nabla^j g(\cdot,t)|^2 \le \frac{c_j}{t^j} \tag{c_t}$$

for all $j \in \mathbb{N}_0$, $x \in M$, for all $t \in (0, T]$, where $c_j(h, a, \varepsilon) \to 0$ as $\varepsilon \searrow 0$ and

$$\int_{B_1(x_0)} \left(|g_0 - g(t)|^2 + |^h\nabla(g_0 - g(t))|^2 + |^h\nabla^2(g_0 - g(t))|^2\right) dh \to 0 \quad \text{as } t \searrow 0 \tag{d_t}$$

$$\sup_{x\in B_1(x_0)} |^h\nabla^j g(\cdot,t)|^2 t^j \to 0 \quad \text{for } t \searrow 0, \tag{e_t}$$

and for all $2 \ge R > 1$, there exists a $V(a, R) > 0$ such that

$$\int_{B_1(x_0)} \left(|^h\nabla g(\cdot,t)|^2 + |^h\nabla^2 g(\cdot,t)|^2\right) dh \le \int_{B_R(x_0)} \left(|^h\nabla g_0(\cdot)|^2 + |^h\nabla^2 g_0(\cdot)|^2\right) dh + V(a, R)t \tag{f_t}$$

for all $x_0 \in M$ and for all $t \leq T$. Furthermore, there exists $\varepsilon_0 = \varepsilon_0(g_0, h, a)$ such that if $\varepsilon \leq \varepsilon_0$, then the solution is unique in the class of solutions which satisfy (a_t), ($\mathrm{b}_t(r)$), (c_t), (d_t) *for the $r = r(g_0, h, a, \varepsilon) > 0$ defined above.*

Proof. See Theorem 6.5, the proof is given there. ∎

Assume (2.1) and (2.2) and that M is four-dimensional. Then for any $1 > \varepsilon > 0$, we can find an $r > 0$ such that

$$\frac{1}{a}h \leq g_0 \leq ah, \quad \sup_{x \in M} \int_{B_r(x)} \left(|{}^h\nabla g_0|^4 + |{}^h\nabla^2 g_0|^2 \right) dh < \varepsilon, \tag{2.3}$$

see Theorem B.3 in Section B for a proof. After scaling h and g_0 once, and still calling the resulting metrics g_0 and h, we may assume

$$\begin{aligned}
&(M, h) \text{ is a smooth, connected, complete manifold without boundary}\\
&\text{such that}\\
&\sup_M {}^h|{}^h\nabla^i \mathrm{Rm}(h)| < \infty \text{ for all } i \in \mathbb{N}_0,\\
&\textstyle\sum_{i=0}^4 \sup_M {}^h|{}^h\nabla^i \mathrm{Rm}(h)| \leq \delta_0(a),\\
&\mathrm{inj}(M, h) \geq 100,
\end{aligned} \tag{2.4}$$

for a small positive constant $\delta_0(a)$ of our choice, in place of the assumptions (2.1), and the scale invariant condition

$$\frac{1}{a}h \leq g_0 \leq ah$$

and

$$\sup_{x \in M} \int_{B_1(x)} \left(|{}^h\nabla g_0|^4 + |{}^h\nabla^2 g_0|^2 \right) dh < \frac{\varepsilon}{2},$$

is still correct, and hence, using Hölder's inequality, we have

$$\frac{1}{a}h \leq g_0 \leq ah, \quad \sup_{x \in M} \int_{B_1(x)} \left(|{}^h\nabla g_0|^2 + |{}^h\nabla^2 g_0|^2 \right) dh < c(n)\sqrt{\varepsilon}. \tag{2.5}$$

Note further, if we assume (2.1), then (2.3) is a stronger assumption than (2.2): (2.1) and (2.2) imply, for any $\varepsilon > 0$, there exists an $r > 0$ such that (2.3) holds, but for any given $\varepsilon > 0$, there are g_0 and h and $r > 0$ for which (2.1) and (2.3) hold, but $E(g_0) := \infty$.

The main estimates required for the construction of solutions to (1.1) in the $W^{2,2}$ setting in this paper are proved in this setting, that is under the assumptions (2.5) (with $c(n)\sqrt{\varepsilon}$ replaced by ε) and (2.4), and we also prove an existence result in this setting.

Theorem 2.2. *For any $1 < a < \infty$, there exists a constant $\varepsilon_1 = \varepsilon_1(a) > 0$ with the following properties. Let (M^4, h) be a smooth four-dimensional Riemannian manifold which satisfies (2.4). Let g_0 be a $W^{2,2}_{\text{loc}} \cap L^\infty$-Riemannian metric, not necessarily smooth, which satisfies*

$$\frac{1}{a} h \leq g_0 \leq a h, \tag{a}$$

$$\int_{B_2(x)} \left(|{}^h\nabla g_0|^2 + |{}^h\nabla^2 g_0|^2 \right) dh \leq \varepsilon \quad \text{for all } x \in M, \tag{b}$$

where $\varepsilon \leq \varepsilon_1$. Then there exist a constant $T = T(a, \varepsilon) > 0$ and a smooth solution $(g(t))_{t \in (0,T]}$ to (1.1) such that

$$\frac{1}{400a} h \leq g(t) \leq 400 a h, \tag{a_t}$$

$$\int_{B_1(x)} (|{}^h\nabla g(\cdot, t)|^2 + |{}^h\nabla^2 g(\cdot, t)|^2)\, dh \leq 2\varepsilon, \tag{b_t}$$

$$|{}^h\nabla^j g(\cdot, t)|^2 \leq \frac{c_j(h, a, \varepsilon)}{t^j} \tag{c_t}$$

for all $x \in M$, $t \in [0, T]$, where $c_j(h, \varepsilon, a) \to 0$ as $\varepsilon \to 0$, and

$$\int_{B_1(x)} \left(|g_0 - g(t)|^2 + |{}^h\nabla(g_0 - g(t))|^2 + |{}^h\nabla^2(g_0 - g(t))|^2 \right) dh \to 0$$
$$\text{as } t \searrow 0 \text{ for all } x \in M. \tag{d_t}$$

The solution is unique in the class of solutions which satisfy (a_t), (b_t), (c_t), and (d_t). The solution also satisfies the local estimates

$$\sup_{x \in B_1(x_0)} |{}^h\nabla^j g(\cdot, t)|^2 t^j \to 0 \quad \text{for } t \to 0, \tag{e_t}$$

and for all $1 < R \leq 2$, there exists a $V(a, R) > 0$, such that

$$\int_{B_1(x_0)} \left(|{}^h\nabla g(\cdot, t)|^2 + |{}^h\nabla^2 g(\cdot, t)|^2 \right) dh$$
$$\leq \int_{B_R(x_0)} \left(|{}^h\nabla g_0(\cdot)|^2 + |{}^h\nabla^2 g_0(\cdot)|^2 \right) dh + V(a, R) t \tag{f_t}$$

for all $x_0 \in M$, $2 \geq R > 1$, and for all $t \leq T$.

Proof. The theorem follows from Theorem 6.3 and Remark 6.4. ∎

With a solution of this type at hand, we can without much trouble now construct a solution to the Ricci flow

$$(\ell(t))_{t \in (0,T)} = ((\Phi(t))^* g(t))_{t \in (0,T)}$$

with $\ell(S) = g(S)$ and $\Phi(S) = \mathrm{Id}$ for any given fixed $S > 0$. After some work it becomes clear, that the Ricci Flow solution has initial starting data corresponding in some weak isometric sense to the starting data g_0 of the Ricci–DeTurck flow solution. More specifically, we show for all $p \in [1, \infty)$, that there is a weak limit $\ell_0 := \lim_{t\searrow 0} \ell(t)$ in the L^p_{loc}-sense and that ℓ_0 is isometric to g_0 with the help of a $W^{1,p}$ isometry, and that there is a uniform limit $d_0 := \lim_{t\searrow 0} d_t$ for $d_t := d(g(t))$, where d_0 can be explicitly calculated from the starting data g_0. These facts, and more details, are contained in the following theorem.

Theorem 2.3. *Let $1 < a < \infty$, $M = M^4$ be a four-dimensional manifold, and g_0 and h satisfy the assumptions (2.4), (a) and (b) with $\varepsilon \le \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(a) > 0$ is the constant coming from Theorem 2.2, and let $(M, g(t))_{t\in(0,T]}$ be the smooth solution to (1.1) constructed in Theorem 2.2. Then*

(i) *there exists a constant $c(a)$ and a smooth solution $\Phi\colon M \times (0, T] \to M$ to (1.3) with $\Phi(T/2) = \mathrm{Id}$ such that*

$$\Phi(t) := \Phi(\cdot, t)\colon M \to M$$

is a diffeomorphism, and

$$d_h(\Phi(t)(x), \Phi(s)(x)) \le c(a)\sqrt{|t - s|}$$

for all $x \in M$. The metrics $\ell(t) := (\Phi(t))^* g(t)$, $t \in (0, T]$ solve the Ricci flow equation. Furthermore, there are well-defined limit maps

$$\Phi(0)\colon M \to M, \quad \Phi(0) := \lim_{t\searrow 0} \Phi(t), \quad W(0)\colon M \to M, \quad W(0) := \lim_{t\searrow 0} W(t),$$

where $W(t)$ is the inverse of $\Phi(t)$ and these limits are obtained uniformly on compact subsets, and $\Phi(0)$, $W(0)$ are homeomorphisms inverse to one another.

(ii) *For the Ricci flow solution $\ell(t)$ from* (i), *there is a value*

$$\ell_0(\cdot) = \lim_{t\searrow 0} \ell(\cdot, t)$$

well defined up to a set of measure zero, where the limit exists in the L^p_{loc}-sense, for any $p \in [1, \infty)$, such that ℓ_0 is positive definite and in $W^{1,2}_{\mathrm{loc}}$, and for any $x_0 \in M$ and $0 < s < t \le T$, we have

$$\int_{B_1(x_0)} |\ell(s) - \ell_0|^p_{\ell(t)}\, d\ell(t) \le c(g_0, h, p, x_0)s,$$

$$\int_{B_1(x_0)} |(\ell(0))^{-1} - (\ell(s))^{-1}|^p_{\ell(t)}\, d\ell(t) \le c(g_0, h, p, x_0)|s|^{1/4},$$

$$\int_{B_1(x_0)} |\nabla \ell_0|^2_{\ell(t)}\, d\ell(t) \le c(g_0, h, p, x_0)t^{\sigma},$$

$$\int_{B_1(x_0)} |\mathrm{Rm}(\ell)|^2(x,t)\, d\ell(x,t) + \int_0^t \int_{B_{\ell(s)}(x_0,1)} |\nabla \mathrm{Rm}(\ell)|^2(x,s)\, d\ell(x,s)\, ds \le c(g_0, h, p, x_0),$$

$$\sup_{B_1(x_0)} |\nabla^j \mathrm{Rc}(\ell(t))|^2 t^{j+2} \to 0 \quad \text{as } t \searrow 0 \text{ for all } j \in \mathbb{N}_0$$

for a universal constant $\sigma > 0$, where ∇ refers to the gradient with respect to $\ell(t)$, $c(g_0, h, p, x_0)$ is a constant depending on g_0, h, p, x_0, but not on t or s.

(iii) *The limit maps*

$$\Phi(0)\colon M \to M, \quad \Phi(0) := \lim_{t\searrow 0} \Phi(t), \quad W(0)\colon M \to M, \quad W(0) := \lim_{t\searrow 0} W(t)$$

from (i) *are also obtained in the $W^{1,p}_{\mathrm{loc}}$-sense for $p \in [1,\infty)$. Furthermore, for any smooth coordinates $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$ with $W(0)(V) \subset\subset U$, the functions*

$$(\ell_0)_{ij} \circ W(0)\colon V \to \mathbb{R}$$

are in L^p_{loc} for all $p \in [1,\infty)$ and $(g_0)_{\alpha\beta}\colon V \to \mathbb{R}$ and $(\ell_0)_{ij}\colon U \to \mathbb{R}$ are related by the identity

$$(g_0)_{\alpha\beta} = D_\alpha (W(0))^i D_\beta (W(0))^j ((\ell_0)_{ij} \circ W(0)),$$

which holds almost everywhere. In particular, ℓ_0 is isometric to g_0 almost everywhere through the map $W(0)$, which is in $W^{1,p}_{\mathrm{loc}}$ for all $p \in [1,\infty)$.

(iv) *We define*

$$d_t(x,y) = d(g(t))(x,y), \quad \tilde{d}_t(p,q) = d(\ell(t))(p,q)$$

for all $x, y, p, q \in M$ and $t \in (0,T)$. There are well-defined limit metrics

$$d_0, \tilde{d}_0\colon M \times M \to \mathbb{R}_0^+, \quad d_0(x,y) = \lim_{t\searrow 0} d_t(x,y),$$

$$\tilde{d}_0 := M \times M \to \mathbb{R}_0^+, \quad \tilde{d}_0(p,q) = \lim_{t\searrow 0} \tilde{d}_t(p,q),$$

and they satisfy

$$\tilde{d}_0(x,y) = d_0(\Phi(0)(x), \Phi(0)(y)).$$

That is, $(M, \tilde{d}_0)$ and (M, d_0) are isometric to one another through the map $\Phi(0)$.

The metric d_0 satisfies

$$d_0(x,y) := \liminf_{\varepsilon\searrow 0} \inf_{\gamma \in C_{\varepsilon,x,y}} L_{g_0}(\gamma),$$

where $C_{\varepsilon,x,y}$ is the space of ε-approximative Lebesgue curves *between x and y with respect to g_0: This space is defined/examined in Definition* 8.2.

Proof. See Theorem 8.3, the proof is given there. ∎

Remark 2.4. An attempt to construct a Ricci flow solution $\ell(t)$ with

$$\Phi(0) = \mathrm{Id} \quad \text{and} \quad \ell(0) = \Phi(0)^* g(0) = g(0),$$

using similar methods to those we used to construct the Ricci flow solution in Theorem 2.3, could lead to a non-smooth Ricci flow solution, which does not immediately become smooth (we say the solution is in *a non-smooth gauge*), as we now explain. The solutions $g(t)$ constructed in Theorem 2.1 are limits of solutions $g_i(t)$ with initial data $g_i(0)$, where $g_i(0) \to g(0)$ in $W^{2,2}_{\mathrm{loc}}$. For $M = \mathbb{T}^4$, the four-dimensional torus, whose circles have radius 10, with h the standard flat metric on $\mathbb{T}^4$. Let $g_i(0) = \varphi(i)^* h$, where $\varphi(i)\colon \mathbb{T}^4 \to \mathbb{T}^4$ are diffeomorphisms, equal to the identity outside a ball $\mathbb{B}_1(0)$ of radius one (which we identify with the standard euclidean ball of radius one), and

$$\varphi(i)|_{\mathbb{B}_1(0)}\colon \mathbb{B}_1(0) \to \varphi(i)(\mathbb{B}_1(0)) = \mathbb{B}_1(0)$$

are uniformly bi-Lipschitz diffeomorphisms,

$$\frac{1}{B}|x-y| \le |\varphi(i)(x) - \varphi(i)(y)| \le B|x-y|$$

for all $x, y \in \mathbb{B}_1(0)$, with $\varphi_i(0) \to \psi$ as $i \to \infty$ in the $W^{3,2}$-sense. Assume that ψ is not smooth. For example, we can take

$$\varphi(i)(x) = x(1 + \eta\sigma f_i(x))$$

with

$$f_i(x) := \left(2 + \sin\left(\log\left(\log\left(\frac{2}{\sqrt{|x|^2 + 1/i}}\right)\right)\right)\right),$$

σ a small positive constant, and η a smooth cut-off function with $\eta = 1$ on $\mathbb{B}_{1/2}(0)$ and $\eta = 0$ on $(\mathbb{B}_{3/4}(0))^c$. Notice that the $\varphi(i)$ are uniformly Bi-Lipschitz, as we now explain. Assume that $|x| \le |y|$. Then

$$\begin{aligned}
|\varphi(i)(x) - \varphi(i)(y)| &= |(x-y) + x\eta\sigma f_i(x) - y\eta\sigma f_i(y)| \\
&= |(x-y) + x\sigma(\eta(x) f_i(x) - \eta(y) f_i(y)) + (x-y)\eta\sigma f_i(y)| \\
&\ge \frac{9}{10}|x-y| - 2\sigma|x||\eta(x) - \eta(y)| - 2\sigma|x-y| - \sigma|x||f_i(x) - f_i(y)| \\
&\ge \frac{9}{10}|x-y| - 2\sigma|x||D_v\eta(c)||x-y| - 2\sigma|x-y| - \sigma|x||D_v f_i(b)||x-y| \\
&\ge \frac{1}{2}|x-y| - \sigma|x||D_v f_i(b)||x-y|,
\end{aligned}$$

where b and c are points in the line between x and y and v is a length one vector pointing in the direction of the line between x and y. A calculation shows us that

$$\begin{aligned}|D_v f_i(b)| &= |\cos(\dots)|\Big|\frac{1}{|\log(1/\sqrt{|b|^2+1/i})|}2\frac{\langle b,v\rangle|}{(|b|^2+1/i)}\\ &\le |\cos(\dots)|\frac{1}{|b|},\end{aligned}$$

which, combined with the fact that $|x| \le |b|$, gives us

$$\sigma|x||D_v f_i(b)||x-y| \le \sigma|x-y|,$$

and hence

$$|\varphi(i)(x) - \varphi(i)(y)| \ge \frac{1}{4}|x-y|.$$

A similar calculation shows us that

$$|\varphi(i)(x) - \varphi(i)(y)| \le 4|x-y|.$$

The definition of the $\varphi(i)$'s guarantees that $\varphi(i)\colon \mathbb{B}_1(0) \to \mathbb{R}^n$ are smooth bi-Lipschitz diffeomorphisms whose image lies in $B_1(0)$. Furthermore, $\varphi(i)(tx)$ is a continuous line for t between 0 and 1 lying on the standard line tx between 0 and x. Hence,

$$\{\varphi(i)(tx) \mid t \in [0,1]\} = \{tx \mid t \in [0,1]\}.$$

This shows that $\varphi(i)\colon \mathbb{B}_1(0) \to \mathbb{B}_1(0)$ is also onto.

Then $\varphi(i) \to \psi$ in the $W^{3,2}$-sense, with

$$\psi(x) = x(1+\eta\sigma f(x)), \quad f(x) := \Big(2+\sin\Big(\log\Big(\log\Big(\frac{2}{|x|}\Big)\Big)\Big)\Big)$$

for $x \ne 0$, $f(0) := 0$, $g_i(0) \to g(0)$ in the $W^{2,2}$-sense, but $g(0)$ is not smooth, and there exists an $1 < a = a(B,K) < \infty$ such that $\frac{1}{a}h \le g_i(0) \le ah$. Hence, Theorem 2.1 is applicable and a limit solution

$$g(t)_{t\in(0,T)} = \lim_{i\to\infty} g_i(t)_{t\in(0,T)}$$

exists with $g(t) \to g(0)$ in the $W^{2,2}$-sense as $t \searrow 0$. However, the Ricci flow of $\ell_i(0) = g_i(0)$ is $\ell_i(t) = \ell_i(0)$, as the metric $g_i(0)$ is flat. Hence, $\ell_i \to \ell$ in the $W^{2,2}$-sense, where $\ell(t) = \ell(0) = g(0)$ for all $t \in (0,T)$. By construction $g(0)$ is non-smooth. We avoid these non-smooth gauges by choosing $\Phi(S) = \mathrm{Id}$ for some $S > 0$ in Theorem 2.3.

In order to prove the relationships of Theorem 2.3, in particular the existence of the limit ℓ_0, we require some new estimates which hold for solutions to Ricci flow of the type constructed here, and for a more general class. The theorems and lemmata that we use to prove these estimates are contained in Section 7.

The existence of the weak metric ℓ_0 is achieved with the following theorem.

Theorem 2.5. *For all $p \in [2,\infty)$ and $n \in \mathbb{N}$, there exists an $\alpha_0(n,p) > 0$ such that the following holds. Let Ω be a smooth n-dimensional manifold and $(\Omega^n, \ell(t))_{t\in(0,T]}$ be a smooth solution to Ricci flow satisfying*

$$\int_\Omega |\mathrm{Rc}(\ell(t))| d\ell(t) \leq \varepsilon, \quad |\mathrm{Rc}(\ell(t))| \leq \frac{\varepsilon}{t} \text{ on } \Omega$$

for all $t \in (0,T]$, where $\varepsilon \leq \alpha_0$. Then there exists a unique, positive definite, symmetric two-tensor $\ell_0 \in L^p$ such that $\ell(s) \to \ell_0$ in $L^p(\Omega)$ as $s \searrow 0$ where ℓ_0, and $\ell^{-1}(s) \to (\ell_0)^{-1}$ in $L^p(\Omega)$ as $s \searrow 0$.

Proof. See Theorem 7.1, the proof is given there. ∎

The proof of the existence of a homeomorphism $\Phi(0)$ at time zero in Theorem 2.3 can also be applied with no change, to the setting of a Ricci–DeTurck flow coming out of a C^0-metric on an n-dimensional Riemannian manifold, respectively for the Ricci flow related solution. This fact is stated in the following theorem.

Theorem 2.6. *For any $n \in \mathbb{N}$, there exists an $\delta_0(n) > 0$ such that the following holds. Let (M^n, h) be a smooth n-dimensional manifold satisfying the assumptions (2.4), where now $\delta_0 = \delta_0(n)$ is a small constant of our choice, and assume g_0 is a C^0-metric satisfying*

$$(1-\delta_0(n))h \leq g_0 \leq (1+\delta_0(n))h.$$

Let $(M, g(t))_{t\in(0,T)}$ be the solution to (1.1), where $g(t) \to g_0$ as $t \searrow 0$ in the C^0_{loc}-sense constructed in [30] or [17], and let $\Phi\colon M \times (0,T) \to M$ be the solution to (1.3), with $\Phi(\cdot, T/2) = \mathrm{Id}(\cdot)$. Then there exists a homeomorphism $\Phi(0)\colon M \to M$ such that $\Phi(t) \to \Phi(0)$ locally uniformly, and $d(g(t)) \to d(g(0)) =: d_0$ locally uniformly and $d(\ell(t)) \to \tilde{d}_0$ locally uniformly as $t \searrow 0$, where

$$\tilde{d}_0(\tilde{x}, \tilde{y}) = d_0(\Phi(0)(\tilde{x}), \Phi(0)(\tilde{y}))$$

for all $\tilde{x}, \tilde{y} \in M$.

Proof. The solutions constructed in [30], respectively [17], satisfy

$$|{}^h\nabla g|_h^2(t) + |{}^h\nabla^2 g|_h(t) \leq \frac{c(\delta_0, n)}{t},$$

where $c(\delta_0, n) \to 0$ as $\delta_0 \to 0$. These facts are required in the proof of Theorem 8.3 (i). We may now copy and paste the proof of Theorem 8.3 (i) to here, and in doing so we obtain the existence of a homeomorphism $\Phi(0)$ which is obtained locally uniformly as the limit, in the C^0-norm, of $\Phi(t)$ with $t \to 0$.

Also, the solutions constructed in [30], respectively [17], satisfy $g(t) \to g(0)$ locally uniformly in the C^0-norm as $t \to 0$, and hence $d(g(t)) \to d(g(0))$ locally uniformly, and consequently,

$$d(\ell(t)) = (\Phi(t))^*(d(g(t))) \to \tilde{d}_0 = (\Phi(0))^*(d_0)$$

locally uniformly. ∎

In Section 12 we prove the following theorem (Theorem 2.8), which is an application of the above results. Compare the paper [4], where sequences of smooth Riemannian metrics with scalar curvature bounded from below which approach a C^0-metric with respect to the C^0-norm are considered. We consider $W^{2,2}$-metrics which have scalar curvature bounded from below in the following weak sense.

Definition 2.7. Let M be a four-dimensional smooth closed manifold and g be a $W^{2,2}$-Riemannian metric (positive definite everywhere) and let $k \in \mathbb{R}$. Locally the scalar curvature may be written

$$\mathrm{R}(g) = g^{jk}\big(\partial_i \Gamma(g)^i_{jk} - \partial_j \Gamma(g)^i_{ik} + \Gamma(g)^i_{ip}\Gamma(g)^p_{jk} - \Gamma(g)^i_{jp}\Gamma(g)^p_{ik}\big),$$

where

$$\Gamma(g)^m_{ij} = \frac{1}{2} g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}),$$

and hence $\mathrm{R}(g)$ is well defined in the L^2-sense for a $W^{2,2}$ Riemannian metric. Let $k \in \mathbb{R}$. We say the scalar curvature $\mathrm{R}(g)$ is weakly bounded from below by k, such that $\mathrm{R}(g) \geq k$, if this is true almost everywhere, for all local smooth coordinates.

Theorem 2.8. *Let (M, h) be four-dimensional closed and satisfy (2.4). Assume that (M, g_0) is a $W^{2,2}$-metric such that $\frac{1}{a}h \leq g_0 \leq ah$ for some $\infty > a > 1$ and $\mathrm{R}(g_0) \geq k$ in the weak sense of Definition 2.7. Then the solution $g(t)_{t\in(0,T)}$ to the Ricci–DeTurck flow, respectively $\ell(t)_{t\in(0,T)}$ to the Ricci flow constructed in Theorem 8.3, with initial value $g(0) = g_0$, has $\mathrm{R}(g(t)) \geq k$ and $\mathrm{R}(\ell(t)) \geq k$ for all $t \in (0, T)$.*

Proof. See Theorem 12.2, the proof is given there. ∎

Remark 2.9. From this theorem we see that for a metric $g_0 \in L^\infty \cap W^{2,2}(M^4)$ with $\frac{1}{a}h \leq g_0 \leq ah$ for some positive constant $a > 0$, g_0 has scalar curvature $\geq k$ in the weak sense of Definition 2.7 if and only if there exists a sequence of smooth Riemannian metrics $g_{i,0}$ with $\frac{1}{b}h \leq g_{i,0} \leq bh$ for some $1 < b < \infty$ and $\mathrm{R}(g_{i,0}) \geq k$, and $g_{i,0} \to g_0 \in W^{2,2}(M^4)$ as $i \to \infty$ if and only if the Ricci–DeTurck flow of g_0 constructed in Theorem 2.1 has $\mathrm{R}(g(t)) \geq k$ for all $t \in (0, T)$.

3. Outline of the paper

The paper contains twelve sections and four appendices. Section 1 is an introduction and Section 2 contains statements of the main results, and this section gives an outline of the paper.

In Section 4 we prove a priori C^1 and L^∞-estimates for *smooth* solutions to the Ricci–DeTurck flow. The L^∞-estimates we are concerned with in this paper take the form $\frac{1}{b}h \leq g \leq bh$ for some constant $1 < b < \infty$, for the fixed background metric h, which is used to define the Ricci–DeTurck flow in (1.1). In particular, we show in Theorem 4.2, that smooth compact solutions which a priori have small local $W^{2,2}$-energy along the flow and satisfy an initial L^∞-estimate must also satisfy L^∞ and C^1-estimates along the flow. In the non-compact setting, we require further that the smooth solution satisfies a regularity condition in order to obtain the same result; see Theorem 4.2.

In Section 5 we prove various local estimates for integral quantities, assuming our solution satisfies an L^∞ bound and has small local $W^{2,2}$-energy; see Theorem 5.1. This also leads to estimates on the convergence as time goes back to zero of the solution, as explained in, for example, Theorem 5.8 and Corollary 5.7.

Section 6 uses the a priori estimates of the previous sections with well-known existence theory for parabolic equations to show that solutions in the classes considered in those sections exist, even when the initial data is non-smooth. That is, solutions to Ricci–DeTurck flow exist, if the initial metric is locally in $W^{2,2}$ and has small local initial energy and satisfies an L^∞ bound with respect to h. The solutions obtained continue to have small local energy and satisfy an L^∞ bound.

In Section 7 estimates are proved for solutions to Ricci flow in a setting which includes the class of Ricci flows we construct using the Ricci–DeTurck flow of Section 6. In particular, it is shown in the setting of Section 7, that a weak initial value of the Ricci flow exists.

In Section 8 a Ricci flow is constructed from the Ricci–DeTurck flow of Section 6 and in Theorem 8.3, the relationship between the two solutions is investigated. In particular, relations between the distance and the weak Riemannian metrics at time zero, as well as the convergence properties as time goes to zero are stated. Further necessary lemmata, theorems, etc., which we require to prove this theorem are contained in Sections 7, 9, and 10.

Section 9 is concerned with convergence properties of Riemannian metrics in certain Sobolev spaces, and Section 10 is concerned with a definition of distance, respectively convergence properties of distances, for Riemannian metrics defined in certain Sobolev spaces. Theorem 11.2 proves uniqueness of the Ricci–DeTurck solutions in a class which includes the class of solutions that are constructed in this paper.

In Section 12 we present an application for $W^{2,2} \cap L^\infty$-metrics with scalar curvature bounded from below in the weak sense.

Sections A–D are technical appendices containing certain estimates, statements, the calculation/verification of which, are not included in the other sections of the paper, in order to facilitate reading.

Section A contains a short time existence result for Ricci–DeTurck flow, using the method of W.-X. Shi.

In Section B we state and prove some facts about Sobolev inequalities and norms thereof adapted to the setting of the paper.

Section C contains estimates for ordinary differential equations which are required at many points in the paper.

Section D contains statements which compare pointwise norms and L^p-norms of different Riemannian metrics. The estimates contained in the statements are also used at many points in the paper.

4. L^∞- and C^1-estimates of the Ricci–DeTurck flow

In this section we derive an a priori L^∞ time independent bound on the evolving metric g, and show that the gradient thereof is bounded by $1/\sqrt{t}$ under the a priori assumptions that: we have an L^∞ bound $\frac{1}{a}h \leq g_0 \leq ah$ at time zero, the $W^{2,2}$-norm of the solution restricted to balls of radius one are *small*, and the time interval being considered has *small* length, where here the notion of *small* depends on n and a.

As a first lemma, we show that if we already have an L^∞ and a time dependent gradient bound, then all other derivatives may be estimated.

Lemma 4.1. *Let (M, h) be n-dimensional and satisfy (2.4) and $g(\cdot, t)_{t\in[0,T)}$, $T \leq 1$ be a smooth family of metrics which solves (1.1) and satisfies the a priori bounds*

$$\frac{1}{a}h \leq g(t) \leq ah, \quad \sup_{x\in M} |{}^h\nabla g|^2(x,t) \leq \frac{b}{t}, \tag{4.1}$$

for all $t \in [0, T)$ and some $1 < a, b < \infty$. Then for all $i \in \mathbb{N}$, there exist constants $N_i = N_i(a, b, n, h)$ such that

$$\sup_{x\in M} |{}^h\nabla^i g|^2(x,t) \leq \frac{N_i}{t^i} \tag{4.2}$$

for all $t \in [0, T)$.

Proof. We start with the case $i = 2$. Let $g(\cdot, t)_{t\in[0,T)}$, $T \leq 1$ be a solution to (1.1) which satisfies (4.1). Let $N_2 \in \mathbb{N}$ be large (to be determined in the proof) and assume that (4.2) does not hold. That is, for $N := N_2$, there exists a $0 < t_0 < T$ and an $x_0 \in M$ such that

$$|^h\nabla^2 g|^2(x_0, t_0) > \frac{N}{t_0^2}.$$

Define $\widetilde{g}(x,t) := cg(x,t/c)$ and $\widetilde{h}(y) = ch(y)$ for a $c > 0$ to be chosen. Then $\widetilde{g}(t)$ solves the $\widetilde{h}$ -flow for $t \in [0, Tc)$ and we have the scaling relations

$$|^{\widetilde{h}}\nabla^i \widetilde{g}|^2(x,t) = c^{-i}|^h\nabla^i g|^2\Big(x, \frac{t}{c}\Big).$$

By choosing $c = \sqrt{N}/\sqrt{V}t_0$, we get a solution $\widetilde{g}$ which has

$$\frac{1}{a}\widetilde{h} \le \widetilde{g}(t) \le a\widetilde{h}, \quad \sup_{x\in M} |^{\widetilde{h}}\nabla\widetilde{g}|^2(x,t) \le \frac{b}{t}, \quad |^{\widetilde{h}}\nabla^2\widetilde{g}|^2(x_0, \sqrt{N/V}) \ge V$$

for all $t \in [0, t_0 c] = [0, \sqrt{N/V}]$. This implies

$$|^{\widetilde{h}}\nabla\widetilde{g}|^2(t) \le \frac{b}{t} \le \frac{b}{\sqrt{N/V} - 10} \le \varepsilon$$

for $t \in (-10 + \sqrt{N/V}, \sqrt{N/V}]$ for any $\varepsilon > 0$ as long as $N = N(b, V, n, \varepsilon)$ is chosen large enough. In that which follows, we use once again g to denote the solution $\widetilde{g}$ and h to denote $\widetilde{h}$. That is, we have a smooth solution $g(t)_{t\in[0,\sqrt{N/V}]}$ of the Ricci–DeTurck flow with

$$\frac{1}{a}h \le g(t) \le ah, \quad \sup_{x\in M} |^h\nabla g|^2(x,t) \le \varepsilon, \quad |^h\nabla^2 g|^2(x_0, \sqrt{N/V}) \ge V$$

for all $t \in (-10 + \sqrt{N/V}, \sqrt{N/V}]$. As shown in [28], the evolution for $|^h\nabla^m g|^2$ is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}|^h\nabla^m g|^2(x,t) \\
&\quad = g^{ij}(x,t)^h\nabla^2_{ij}|^h\nabla^m g|^2(x,t) - 2g^{ij}(x,t)(^h\nabla_i{}^h\nabla^m g, {}^h\nabla_j{}^h\nabla^m g)_h(x,t) \\
&\qquad + \sum_{\substack{0\le k_1,k_2,\dots,k_{m+2}\le m+1,\\ k_1+\dots+k_{m+2}\le m+2}} \big({}^h\nabla^{k_1} g(x,t) * {}^h\nabla^{k_2} g(x,t) * \cdots \\
&\qquad\qquad \cdots * {}^h\nabla^{k_{m+2}} g(x,t) * {}^h\nabla^m g * P(h)_{k_1 k_2 \dots k_{m+2}}(x,t)\big), \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
&P(h)_{k_1,\dots,k_{m+2}}(x,t) \\
&\quad = P(h)_{k_1,\dots,k_{m+2}}(x,t)(g, g^{-1}, \mathrm{Rm}(h), {}^h\nabla\mathrm{Rm}(h), \dots, {}^h\nabla^m\mathrm{Rm}(h))
\end{aligned}$$

is a polynomial in the terms appearing in the brackets, and

$$g^{ij}({}^h\nabla_i T, {}^h\nabla_j T) = g^{ij} h^{s_1 r_1} h^{s_2 r_2} \dots h^{s_m r_m} {}^h\nabla_i T_{s_1 \dots s_m} {}^h\nabla_j T_{r_1 \dots r_m}$$

for a $(0\ \ m)$ tensor T. We have $|P(h)|^2(x,t) \le c(a,m,n)$ since without loss of generality, the norm of the curvature of h (after scaling) and all its derivatives up to order m are bounded by a constant (see (2.4)). In particular, for $m = 1$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}|{}^h\nabla g|^2(x,t) - g^{ij}(x,t)^h\nabla^2_{ij}|{}^h\nabla g|^2(x,t)\\
&\qquad \le -2g^{ij}(x,t)({}^h\nabla_i {}^h\nabla g, {}^h\nabla_j {}^h\nabla g)_h(x,t)\\
&+ \sum_{\substack{0\le k_1,k_2,k_3\le 2,\\ k_1+k_2+k_3\le 3}} {}^h\nabla^{k_1} g(x,t) * {}^h\nabla^{k_2} g(x,t) * {}^h\nabla^{k_3} g(x,t) * {}^h\nabla g * P(h)_{k_1k_2k_3}(x,t)\\
&\qquad \le -2g^{ij}(x,t)({}^h\nabla_i {}^h\nabla g, {}^h\nabla_j {}^h\nabla g)(x,t)\\
&+ c(n,a)|{}^h\nabla g|\big(|{}^h\nabla^2 g||{}^h\nabla g| + |{}^h\nabla^2 g| + |{}^h\nabla g|^3 + |{}^h\nabla g|^2 + |{}^h\nabla g| + c(n,a)\big).
\end{aligned} \tag{4.4}$$

Here $c(n,a)$ denotes a constant which may change from line to line but only depends on n and a. Combinations of constants involving b, a, n multiplied by ε shall sometimes be written as ε. In what follows, we restrict ourselves to the region

$$t \in (-10 + \sqrt{N/V}, \sqrt{N/V}].$$

Using

$$\sup_{x\in M} |{}^h\nabla g|(x,t) \le \varepsilon \le 1$$

for all $t \in (-10 + \sqrt{N/V}, \sqrt{N/V}]$ and $\frac{1}{a}h \le g \le ah$, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}|{}^h\nabla g|^2(x,t) - g^{ij}(x,t)^h\nabla^2_{ij}|{}^h\nabla g|^2(x,t)\\
&\qquad \le -\frac{2}{a}|{}^h\nabla^2 g|^2(x,t) + c(n,a)\varepsilon|\big(|{}^h\nabla^2 g|\varepsilon + |{}^h\nabla^2 g| + 3\varepsilon + c(n,a)\big)\\
&\qquad \le -\frac{2}{a}|{}^h\nabla^2 g|^2(x,t) + c(n,a)\varepsilon|{}^h\nabla^2 g| + c(n,a)\varepsilon\\
&\qquad \le -\frac{1}{a}|{}^h\nabla^2 g|^2(x,t) + c(n,a)\varepsilon
\end{aligned}$$

in view of Young's inequality. Similarly, we estimate

$$\begin{aligned}
&\frac{\partial}{\partial t}|{}^h\nabla^2 g|^2(x,t) - g^{ij}(x,t)^h\nabla^2_{ij}|{}^h\nabla^2 g|^2(x,t) \le -\frac{2}{a}|{}^h\nabla^3 g|^2\\
&\quad + \sum_{\substack{0\le k_1,k_2,k_3,k_4\le 3,\\ k_1+\dots+k_4\le 4}} {}^h\nabla^{k_1} g(x,t) * \dots * {}^h\nabla^{k_4} g(x,t) * {}^h\nabla^2 g * P(h)_{k_1k_2k_3k_4}(x,t)
\end{aligned}$$

$$
\begin{aligned}
&\le -\frac{2}{a}|{}^h\nabla^3 g|^2 + c(n,a)|{}^h\nabla^2 g|\Big(|{}^h\nabla^3 g|\big(|{}^h\nabla g| + c(n,a)\big) \\
&\quad + |{}^h\nabla^2 g|\big(|{}^h\nabla^2 g| + |{}^h\nabla g|^2 + |{}^h\nabla g| + c(n,a)\big) \\
&\quad + |{}^h\nabla g|\big(|{}^h\nabla g|^3 + |{}^h\nabla g|^2 + |{}^h\nabla g| + c(n,a)\big) + c(n,a)\Big) \\
&\le -\frac{2}{a}|{}^h\nabla^3 g|^2 + c(n,a)|{}^h\nabla^2 g|\big(|{}^h\nabla^3 g|c(n,a) + |{}^h\nabla^2 g|^2 + c(n,a)\big) \\
&\le -\frac{1}{a}|{}^h\nabla^3 g|^2 + c(n,a)|{}^h\nabla^2 g| + c(n,a)|{}^h\nabla^2 g|^3 \\
&\le -\frac{1}{a}|{}^h\nabla^3 g|^2 + c(n,a)|{}^h\nabla^2 g|^3 + c(n,a), \qquad (4.5)
\end{aligned}
$$

where we have used Young's inequality a number of times. Combining these two evolution inequalities, we see that $f = (|{}^h\nabla g|^2 + 1)(|{}^h\nabla^2 g|^2)$ satisfies

$$
\begin{aligned}
&\frac{\partial}{\partial t} f - g^{ij}\,{}^h\nabla^2_{ij} f \\
&\qquad \le -\frac{1}{a}|{}^h\nabla^2 g|^4 + \varepsilon|{}^h\nabla^2 g|^2 \\
&\qquad\quad + \big(|{}^h\nabla g|^2 + 1\big)\Big(-\frac{1}{a}|{}^h\nabla^3 g|^2 + c(n,a)|{}^h\nabla^2 g|^3 + c(n,a)\Big) \\
&\qquad\quad - 2g^{ij}\,{}^h\nabla_i\big(|{}^h\nabla g|^2 + 1\big)\,{}^h\nabla_j\big(|{}^h\nabla^2 g|^2\big) \\
&\qquad \le -\frac{1}{2a}|{}^h\nabla^2 g|^4 - \frac{1}{a}|{}^h\nabla^3 g|^2 + c(n,a)\big(1 + |{}^h\nabla g|^2\big)\big(1 + |{}^h\nabla^2 g|^3\big) \\
&\qquad\quad + c(n,a)|{}^h\nabla g||{}^h\nabla^2 g|^2|{}^h\nabla^3 g|
\end{aligned}
$$

for the t that we are considering.

Now using once again that $\sup_{x\in M}|{}^h\nabla g|(x,t) \le \varepsilon$, which is true by assumption, we see that

$$\frac{\partial}{\partial t} f - g^{ij}\,{}^h\nabla^2_{ij} f \le -(1/4a) f^2 + c(n,a).$$

Standard techniques (cut-off function and a Bernstein-type argument; see [29] or [30]) now show that $f \le c_1(n,a)$ at $t = \sqrt{N/V}$, which implies that

$$|{}^h\nabla^2 g|^2 \le c_1(n,a)$$

at $t = \sqrt{N/V}$ and this contradicts the estimate

$$|{}^h\nabla^2 g|^2(x_0, \sqrt{N/V}) \ge V$$

if $V = \max(100c_1(n,a), r(h,m))$, where $r(h,m)$ is chosen large so that the curvature of h and all of its covariant derivatives up to order $m = 2$ are bounded by one after scaling (which was used in the proof).

For the readers' convenience, we explain the Bernstein-type argument in more detail. By translating in time, we may assume that the time $\sqrt{N/V}$ corresponds to time 10 and the time $\sqrt{N/V} - 10$ corresponds to time zero. We multiply f by a cut-off function η in space (with support in a Ball $B_1(y_0)$ ball around any point y_0) and such that $|{}^h\nabla\eta|^2/\eta \leq C$. Next we consider a point $(x_0, t_0) \in B_1(0) \times [0, 10]$ where $t\eta f$ achieves its positive maximum (assuming f is not identically zero). The point x_0 must be in the interior of $B_1(0)$, since the support of η is contained in $B_1(0)$, and t_0 must be larger than zero, since $t\eta f = 0$ for $t = 0$, and hence, by calculating at (x_0, t_0), we obtain

$$
\begin{aligned}
0 &\leq \frac{\partial}{\partial t}(tf\eta) \\
&\leq g^{ij\,h}\nabla^2_{ij}(tf\eta) - \frac{1}{4a}tf^2\eta - 2g^{ij\,h}\nabla_i(tf)^h\nabla_j\eta \\
&\quad - (tf)g^{ij\,h}\nabla^2_{ij}\eta + tc(n,a)\eta + f\eta \\
&\leq -\frac{1}{4a}\frac{(t\eta f)^2}{\eta t} - \frac{2g^{ij\,h}\nabla_i(tf\eta)^h\nabla_j\eta}{\eta} + c(n,a)tf\frac{|{}^h\nabla\eta|^2}{\eta} \\
&\quad + c(n,a)\frac{ft\eta}{\eta} + c(n,a) + f\eta \\
&\leq -\frac{1}{4a}\frac{(\eta tf)^2}{\eta t} + c(n,a)\frac{ft\eta}{\eta t} + c(n,a),
\end{aligned}
$$

where we used that ${}^h\nabla_i(tf\eta)(x_0, t_0) = 0$ and Young's inequality. Multiplying by $t\eta$, we see that

$$
\frac{1}{4a}(\eta tf)^2(x_0, t_0) - c(n,a)(t\eta)f(x_0, t_0) \leq c(n,a),
$$

and hence

$$
f(x_0, t_0)\eta(x_0, t_0)t_0 \leq \hat{c}(n,a)\eta(x_0, t_0)t_0,
$$

which implies $f(x_0, t_0) \leq c(n,a)$.

Next we assume by induction that for $i \geq 2$,

$$
\sup_{x \in M} |{}^h\nabla^m g|^2(x,t) \leq \frac{N_m}{t^m}
$$

for all $m \leq i$, $t \in [0, T)$, and we want to show that there exists a constant N_{i+1}, so that

$$
\sup_{x \in M} |{}^h\nabla^{i+1} g|^2(x,t) \leq \frac{N_{i+1}}{t^{i+1}}
$$

for all $t \in [0, T)$. Again we argue by contradiction. For this we assume that there is a large constant N (to be determined later) and $x_0 \in M$, respectively $0 < t_0 < T$, so that

$$
|{}^h\nabla^{i+1} g|^2(x_0, t_0) > N/t_0^{i+1}.
$$

Using the same scaling argument as above, we can arrange that we obtain a solution g of the h-flow so that

$$\frac{1}{a}h \le g \le ah,$$
$$\sup_{x \in M} |{}^h\nabla^m g|^2(x,t) \le \varepsilon \quad \forall t \in (-10 + \sqrt[i+1]{N/V}, \sqrt[i+1]{N/V}],\ m \le i,$$
$$|{}^h\nabla^{i+1} g|^2(x_0, \sqrt[i+1]{N/V}) \ge V.$$

As before, respectively as in the paper of Shi (proof of [29, Lemma 4.2]), we now obtain

$$\frac{\partial}{\partial t}|{}^h\nabla^i g|^2 \le g^{jk\,h}\nabla^2_{jk}|{}^h\nabla^i g|^2 - \frac{1}{2a}|{}^h\nabla^{i+1}g|^2 + c(i,n,a)$$

and

$$\frac{\partial}{\partial t}|{}^h\nabla^{i+1} g|^2 \le g^{jk\,h}\nabla^2_{jk}|{}^h\nabla^{i+1} g|^2 - \frac{1}{2a}|{}^h\nabla^{i+2}g|^2 + c(i,n,a)|{}^h\nabla^{i+1}g|^2 + c(i,n,a)$$

for all $t \in (-10 + \sqrt[i+1]{N/V}, \sqrt[i+1]{N/V}]$. The first estimates

$$|{}^h\nabla g|^2(\cdot,t) + |{}^h\nabla^2 g|^2(\cdot,t) \le c/t$$

simplify the calculation for general $i \ge 2$; see (4.3). Calculating as before, we thus obtain for $f := |{}^h\nabla^{i+1}g|^2(1 + |{}^h\nabla^i g|^2)$ that

$$\partial_t f - g^{jk}({}^h\nabla^2_{jk} f) \le -\frac{1}{4a}f^2 + c(i,n,a)$$

for all $t \in (-10 + \sqrt[i+1]{N/V}, \sqrt[i+1]{N/V}]$, and we obtain the same contradiction as before. ∎

Now we show that if we have L^∞-control on our initial metric and the L^2-norm of the gradient and the second gradient of g remain locally, uniformly small, then we have an estimate on the L^∞-norm of the evolving metric (and its inverse) and a time dependent gradient estimate.

Theorem 4.2. *For every $1 \le a \in \mathbb{R}$, $n \in \mathbb{N}$, there exist (small) $\varepsilon_0(a,n), S_1(a,n) > 0$ such that the following holds. Let g_0 be smooth and satisfy*

$$\frac{1}{a}h \le g_0 \le ah, \tag{4.6}$$

where (M,h) is an n-dimensional manifold satisfying (2.4), and assume that we have a smooth solution g to (1.1) on $[0,T]$, which satisfies

$$\int_{B_1(x)} \left(|{}^h\nabla g|^{n/2} + |{}^h\nabla^2 g|^{n/2}\right)(t)\, dh \le \varepsilon_0 \tag{4.7}$$

for all $x \in M$ and for all $t \in [0, T]$, where $T < 1$. We also assume

$$\sup_{M\times[0,T]} |{}^h\nabla g|^2 + |{}^h\nabla^2 g|^2 + |{}^h\nabla^3 g|^2 + F + \varphi < \infty,$$

where $\varphi(x,t) := g^{ij}(x,t)h_{ij}(x)$ and $F(x,t) := g_{ij}(x,t)h^{ij}(x)$. Then

$$\frac{1}{20na} h < g(t) < 20nah, \tag{4.8}$$

$$\sup_{x\in M} |{}^h\nabla g|^2(x,t) < \frac{1}{t} \tag{4.9}$$

for all $t \leq S_1(n,a)$.

Remark 4.3. The functions $\varphi(x,t) := g^{ij}(x,t)h_{ij}(x)$ and $F(x,t) := g_{ij}(x,t)h^{ij}(x)$ are both well-defined smooth functions. The assumption that

$$\sup_{M\times[0,T]} |{}^h\nabla g|^2 + |{}^h\nabla^2 g|^2 + |{}^h\nabla^3 g|^2 + F + \varphi < \infty \tag{4.10}$$

is always satisfied on a compact manifold due to smoothness and compactness.

We will use this result in the proof of Theorem 6.1 and in that situation this condition is satisfied.

Proof. We may replace the condition (4.7), by the scale invariant condition

$$\int_{B_1(x)} \left(|{}^h\nabla g|^n + |{}^h\nabla^2 g|^{n/2}\right) dh \leq \varepsilon_0 \tag{4.11}$$

for all $x \in M$ and for all $t \in [0, T]$ in view of Lemma B.1 (v), after replacing $c(n)\varepsilon_0$ by ε_0. Let

$$\delta = \delta(n,a) = \frac{1}{(an)^{100}} \ll 1$$

(we are assuming $n \geq 2$). Let

$$S_1 = \sup\left\{s \in [0,T] \mid \frac{1}{20na} h \leq g(t) \leq 20nah,\ \sup_{x\in M} |{}^h\nabla g|^2(x,t) \leq \frac{\delta}{t} \text{ hold on } [0,s]\right\}.$$

We have $S_1 > 0$ due to the inequality (4.10) and the fact that g satisfies (1.1). Next we want to show that S_1 can be bounded from below by a constant depending only on n and a.

For this we argue by contradiction and we assume that S_1 is extremely small, so that if we rescale the background metric h by $1/S_1$, then the resulting Riemannian manifold is as close to the standard euclidean space $\mathbb{R}^n$ on balls as large as we like in the C^k-norm ($k \in \mathbb{N}$ chosen as we please) in geodesic coordinates, due to the conditions on h, as we explained at the beginning of the paper.

Let us now scale g and h via $\widetilde{g}(x,t) = cg(x,t/c)$, $\widetilde{h} = ch$ with $c = 1/S_1 \gg 1$. We denote $\widetilde{g}$ and $\widetilde{h}$ once again by g, respectively h. We have for the rescaled solution that

$$1 = S_1 = \sup\big\{s \in [0,T) \mid (4.8) \text{ and } (4.9) \text{ hold on } [0,s]\big\}$$

and (4.11) still holds, and hence, (4.7) holds, in view of Hölder's inequality, after replacing $c(n)\varepsilon_0$ by ε_0. Due to the definition of $S_1(=1)$, the smoothness of all metrics and (4.10), we see that

$$\frac{1}{20na}h \le g \le 20nah, \qquad \sup_{x\in M} |{}^h\nabla g|^2(x,t) \le \frac{\delta}{t}$$

for all $t \in [0,1]$, and there must exist a point $x_0 \in M$ with either

(a) $g(x_0,1)(v,v) < \frac{1}{10na}h(x_0)(v,v) = \frac{1}{10na}$ for some h length one vector v, or

(b) $g(x_0,1)(v,v) > 10nah(v,v) = 10na$ for some h length one vector v, or

(c) $|{}^h\nabla g|^2(x_0,1) > \frac{\delta}{2}$,

otherwise, using the smoothness of g and (4.10), we get a contradiction to the definition of $S_1 = 1$.

We rule out the case (c) first. We argue by contradiction and by the smoothness Lemma 4.1, we know that $|{}^h\nabla g|^2(\cdot,1) \ge \delta/4$ on a ball of radius $R(n,a,\delta) = R(n,a) > 0$ around x_0, and hence

$$\varepsilon_0 \ge \int_{B_1(x_0)} \big(|{}^h\nabla g|^2(y,1)\big)\, dh(y) \ge \frac{\delta(n,a)}{8}\omega_n (R(n,a,\delta))^n,$$

which leads to a contradiction if $\varepsilon_0 = \varepsilon_0(n,a)$ is chosen small enough. Note that here we used that the manifold is very close to the euclidean space. This contradiction shows that (c) does not occur.

Now we rule out (a) and (b). Note that in our case

$$\int_{B_1(x)} \varphi(y,0)\, dh(y) \le \int_{B_1(x)} na\, dh(y) \le \frac{3}{2}\omega_n na, \tag{4.12}$$

respectively,

$$\int_{B_1(x)} F(y,0)\, dh(y) \le \int_{B_1(x)} na\, dh(y) \le \frac{3}{2}\omega_n na, \tag{4.13}$$

where we have used the initial conditions (4.6). From the evolution equation for g, we have

$$\begin{aligned} \frac{\partial}{\partial t}\int_{B_1(x)} \varphi\, dh &\le c(n,a)\int_{B_1(x)} \big(|{}^h\nabla g|^2 + |{}^h\nabla^2 g| + |\mathrm{Rm}(h)|\big)\, dh \\ &\le C_S(n)c(n,a)(\varepsilon_0)^{2/n}, \end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}\int_{B_1(x)} F\,dh &\le c(n,a)\int_{B_1(x)} \big(|{}^h\nabla g|^2 + |{}^h\nabla^2 g| + |\mathrm{Rm}(h)|\big)\,dh\\ &\le C_S(n)c(n,a)(\varepsilon_0)^{2/n},\end{aligned}$$

and thus

$$\begin{aligned}\left(\int_{B_1(x_0)} \varphi(x,1)\,dh(x)\right) &\le \int_{B_1(x_0)} \varphi_0\,dh + C_S(n)c(n,a)(\varepsilon_0)^{2/n}\\ &\le (3/2)\omega_n nra + c(n,a)(\varepsilon_0)^{2/n}\\ &\le 2\omega_n na\left(\int_{B_1(x_0)} F(x,1)\,dh(x)\right)\\ &\le \int_{B_1(x_0)} F_0\,dh + c(n,a)(\varepsilon_0)^{2/n}\\ &\le (3/2)\omega_n na + c(n,a)(\varepsilon_0)^{2/n} \le 2\omega_n na\end{aligned}$$

if $\varepsilon_0(n,a)$ is sufficiently small. Here we used the initial conditions (4.12) and (4.13) freely, and the Hölder and Sobolev inequalities to obtain

$$\int_{B_1(x)} \big(|{}^h\nabla g|^2 + |{}^h\nabla^2 g|\big)\,dh \le C_S(n)c(n,a)(\varepsilon_0)^{2/n}.$$

In particular, there must be a point y_0 in $B_1(x_0)$ with $\varphi(y_0,1) \le 4na$ and a point y_1 in $B_1(x_0)$ with $F(y_1,1) \le 4na$. First, we consider φ. At y_0 we choose a basis so that $h_{ij}(y_0) = \delta_{ij}$ and $g_{ij}(y_0,1) = \lambda_i\delta_{ij}$ is diagonal. Then we see that $\varphi(y_0,1) \le 4na$ implies that

$$\lambda_i \ge \frac{1}{4na}$$

for each $i \in 1,\dots,n$, and hence

$$g(y_0,1) \ge \frac{1}{4na}h(y_0).$$

Using the fact that $|{}^h\nabla g| \le \delta$ and that (M,h) is very close to the standard $\mathbb{R}^n$, in particular, $|\Gamma(h)^i_{jk}(x)| \le \eta_0$ in geodesic coordinates on a ball of radius 10 centred at x_0, where η_0 is as small as we like, we get

$$|\partial_i g_{kl}| \le \delta + \sigma(\eta_0,a,n)$$

with $\sigma(\eta_0,a,n) \to 0$ as $\eta_0 \to 0$. Hence, without loss of generality, $\sigma(\eta_0,a,n) \le \delta$, that is

$$|\partial_i g_{kl}| \le 2\delta = \frac{2}{(an)^{100}}$$

on $B_1(x_0)$ in geodesic coordinates (for h). This combined with $g(y_0,1) \geq \frac{1}{4na}h(y_0)$, and the fact that $(1-\eta_0)\delta_{ij} \leq h_{ij} \leq (1+\eta_0)\delta_{ij}$ (with η_0 as small as we like) leads to

$$g_{ij}(y,1) \geq \left(\frac{1}{4na} - \frac{4}{(an)^{100}}\right)h_{ij}(y) \geq \frac{1}{8na}h_{ij}(y)$$

for all $y \in B_1(x_0)$, which contradicts the fact that

$$g(x_0,1)(v,v) < \frac{1}{10na}h(x_0)(v,v) = \frac{1}{10na}.$$

Hence, (a) does not occur. The argument to show that (b) does not occur is essentially the same. ∎

5. Preservation of smallness of the $W^{2,2}$-energy and $W^{2,2}$-continuity of g in time

In this section we consider smooth four-dimensional solutions to the Ricci–DeTurck flow which satisfy $\frac{1}{a}h \leq g(t) \leq ah$ for some uniform constant a and our fixed background metric h, and whose initial $W^{2,2}$-energy is locally small. Under these assumptions, we prove an estimate on the growth of the local $W^{2,2}$-energy, which shows that this smallness is preserved under the flow, if the time interval being considered is small enough. We see in Theorems 5.6 and 5.8, that these estimates also imply estimates on the modulus of continuity of the local L^2 and $W^{2,2}$-energy of a solution, respectively limits of smooth solutions to (1.1).

Theorem 5.1. *Let (M,h) be four-dimensional and satisfy (2.4). For all $0 < a \in \mathbb{R}$, there exists a $\delta = \delta(a) > 0$ such that for any smooth solution $g \in C^\infty(M \times [0,T))$ of the Ricci–DeTurck flow with*

$$\sup_{x\in M}\int_{B_1(x)} \left(|^h\nabla g(\cdot,t)|^2 + |^h\nabla^2 g(\cdot,t)|^2\right) dh \leq \delta, \quad \frac{1}{a}h(\cdot) \leq g(\cdot,t) \leq ah(\cdot)$$

for all $t \in [0,T)$, the following holds: For every $1/2 \leq R_0 < R_1 \leq 2$, there exists $V(R_0,R_1,a) > 0$ such that

$$\int_{B_{R_0}(x_0)} \left(|^h\nabla g(\cdot,t)|^2 + |^h\nabla^2 g(\cdot,t)|^2\right) dh$$
$$\leq \int_{B_{R_1}(x_0)} \left(|^h\nabla g(\cdot,0)|^2 + |^h\nabla^2 g(\cdot,0)|^2\right) dh + V(R_0,R_1,a)t$$

for any $x_0 \in M$ and for all $t \in [0,T)$.

Remark 5.2. The condition

$$\sup_{x\in M}\int_{B_1(x)}\left(|^h\nabla g(\cdot,t)|^2+|^h\nabla^2 g(\cdot,t)|^2\right)dh\le\delta$$

means we restrict to the class of solutions which stay locally small in $W^{2,2}$. Later we will see that this is not a restriction for the solutions that we construct, starting with initial data which is locally sufficiently small in $W^{2,2}$, as they do indeed satisfy this condition.

Remark 5.3. The constant $V(R_0,R_1,a)\to\infty$ for $R_0\nearrow R_1$.

Corollary 5.4. *Let (M,h) be four-dimensional and satisfy (2.4). For all $0<b\in\mathbb{R}$, there exists a $\delta=\delta(b)>0$ and universal constant $c_0>0$ such that the following holds. For every $\varepsilon>0$, there exists an $S_2=S_2(b,\varepsilon)>0$ such that if $g\in C^\infty(M\times[0,T))$ is a smooth solution to the Ricci–DeTurck flow with initial data g_0 which satisfies*

$$\sup_{x\in M}\int_{B_1(x)}\left(|^h\nabla g(\cdot,t)|^2+|^h\nabla^2 g(\cdot,t)|^2\right)dh\le\delta,\quad \frac{1}{b}h(\cdot)\le g(\cdot,t)\le bh(\cdot)$$

for all $t\in[0,T)$, and for some $x_0\in M$, we have

$$\int_{B_2(x_0)}\left(|^h\nabla g_0|^2+|^h\nabla^2 g_0|^2\right)dh\le\varepsilon.$$

Then

$$\int_{B_1(x_0)}\left(|^h\nabla g(\cdot,t)|^2+|^h\nabla^2 g(\cdot,t)|^2\right)dh\le\frac{3}{2}\varepsilon$$

for all $t\in[0,S_2)\cap[0,T)$.

Proof of Corollary 5.4. Theorem 5.1 implies

$$\begin{aligned}&\int_{B_1(x_0)}\left(|^h\nabla g(\cdot,t)|^2+|^h\nabla^2 g(\cdot,t)|^2\right)dh\\ &\qquad\le\int_{B_2(x_0)}\left(|^h\nabla g(\cdot,0)|^2+|^h\nabla^2 g(\cdot,0)|^2\right)dh+V(R_0,R_1,b)t\\ &\qquad\le\varepsilon+V(1,2,b)t\le\frac{3}{2}\varepsilon\end{aligned}$$

for $t\le\varepsilon/2V(1,2,b)=:S_2$. ■

Before proving Theorem 5.1, we need a version of the Gagliardo–Nirenberg inequality.

Lemma 5.5. *Let (M,h) be four-dimensional and satisfy (2.4), $1/2 \le R_0 < R_1 \le 2$, g be a smooth metric on M satisfying $\frac{1}{a} g \le h \le ag$, $x \in M$, and $\eta \in C_c^\infty(B_{R_1}(x_0))$ be a standard cut-off function which is equal to 1 on $B_{R_0}(x)$ and equal to zero outside of $B_{(R_1+R_0)/2}(x_0)$. We choose η so that $\sqrt{\eta} \in C^\infty(B_{R_1}(x))$ with $|{}^h\nabla \eta| \le c(R_1, R_0)$ for some constant $c(R_1,R_0)$. Then there exists a $C = C(a, R_0, R_1)$ and a $B = B(a)$ such that*

$$\begin{aligned}\int_M \eta^4 |{}^h\nabla^2 g|^3\, dh &\le \left(\int_{B_{R_1}(x)} |{}^h\nabla^2 g|^2\, dh\right)^{1/2} \\ &\quad \times \left(\int_{B_{R_1}(x)} (B\eta^4 |{}^h\nabla^3 g|^2 + C|{}^h\nabla^2 g|^2)\, dh\right) \\ \int_M \eta^4 |{}^h\nabla g|^6\, dh &\le B\left(\int_{B_{R_1}(x)} |{}^h\nabla^2 g|^2\, dh\right)^{1/2} \left(\int_{B_{R_1}(x)} \eta^4 |{}^h\nabla^3 g|^2\, dh\right) \\ &\quad + C\left(\int_{B_{R_1}(x)} |{}^h\nabla g|^2\, dh\right)^{1/2} \left(\int_{B_{R_1}(x)} (|{}^h\nabla^2 g|^2 + |{}^h\nabla g|^2)\, dh\right) \\ &\quad + C\left(\int_{B_{R_1}(x)} |{}^h\nabla^2 g|^2\, dh\right)^{3/2}.\end{aligned}$$

Proof of Lemma 5.5. In the following C refers to a constant which depends on a, R_0, R_1, and B refers to a constant which only depends on a. Both constants may vary from line to line, but continue to be denoted by C respectively B. Using Hölder's inequality and the Sobolev inequality applied to the function $f = \eta^2 |{}^h\nabla^2 g|$, we obtain

$$\begin{aligned}\int_M \eta^4 |{}^h\nabla^2 g|^3\, dh &\le \left(\int_{B_{R_1}(x)} |{}^h\nabla^2 g|^2\, dh\right)^{1/2} \left(\int_M \eta^8 |{}^h\nabla^2 g|^4\, dh\right)^{1/2} \\ &\le \left(\int_{B_{R_1}(x)} |{}^h\nabla^2 g|^2\, dh\right)^{1/2} \left(\int_{B_{R_1}(x)} (B\eta^4 |{}^h\nabla^3 g|^2 + C\eta^2 |{}^h\nabla^2 g|^2)\, dh\right), \quad (5.1)\end{aligned}$$

which is the first estimate. For the second estimate, we integrate by parts with respect to one of the covariant derivatives ${}^h\nabla$ and use Hölder's inequality, to get

$$\begin{aligned}\int_M \eta^4 |{}^h\nabla g|^6\, dh &\le B \int_M \left(\eta^4 |g| |{}^h\nabla g|^4 |{}^h\nabla^2 g| + C\eta^3 |g| |{}^h\nabla g|^5\right) dh \\ &\le B\left(\int_M \eta^4 |{}^h\nabla g|^6\, dh\right)^{2/3} \left(\left(\int_M \eta^4 |{}^h\nabla^2 g|^3\, dh\right)^{1/3} + C\left(\int_M \eta |{}^h\nabla g|^3\, dh\right)^{1/3}\right),\end{aligned}$$

which implies

$$\int_M \eta^4 |{}^h\nabla g|^6\, dh \le B \int_M \eta^4 |{}^h\nabla^2 g|^3\, dh + C \int_M \eta |{}^h\nabla g|^3\, dh$$
$$\le B \int_M \eta^4 |{}^h\nabla^2 g|^3\, dh + C\Big(\int_{B_{R_1}(x)} |{}^h\nabla g|^2 dh\Big)^{1/2} \Big(\int_M \big(\sqrt{\eta}|{}^h\nabla g|\big)^4\, dh\Big)^{1/2}.$$

Using (5.1) to estimate the first term of the right-hand side of this inequality, and the Sobolev inequality, applied to the function $\sqrt{\eta}|{}^h\nabla g|$ to estimate the second term, we conclude

$$\int_M \eta^4 |{}^h\nabla g|^6\, dh$$
$$\le B \int_M \eta^4 |{}^h\nabla^2 g|^3\, dh + C\Big(\int_{B_{R_1}(x)} |{}^h\nabla g|^2\, dh\Big)^{1/2} \Big(\int_M \big(\sqrt{\eta}|{}^h\nabla g|\big)^4\, dh\Big)^{1/2}$$
$$\le B\Big(\int_{B_{R_1}(x)} |{}^h\nabla^2 g|^2\, dh\Big)^{1/2} \Big(\int_{B_{R_1}(x)} \big(\eta^4 |{}^h\nabla^3 g|^2 + C|{}^h\nabla^2 g|^2\big)\, dh\Big)$$
$$+ C\Big(\int_{B_{R_1}(x)} |{}^h\nabla g|^2\, dh\Big)^{1/2} \Big(\int_{B_{R_1}(x)} \big(|{}^h\nabla^2 g|^2 + |{}^h\nabla g|^2\big)\, dh\Big),$$

as required. Note, without loss of generality, we have $|{}^h\nabla\sqrt{\eta}| \le C_1$ $(n = 4)$; if not, replace η by η^2. ■

In the following proof, C will once again be a constant which may change from line to line and depends on a, R_0, R_1, and B denotes a constant which can change from line to line but only depends on a.

Proof of Theorem 5.1. Using equation (4.4) and Young's inequality, we see that

$$\frac{\partial}{\partial t}|{}^h\nabla g|^2 - g^{ij}\,{}^h\nabla^2_{ij}|{}^h\nabla g|^2 + \frac{2}{a}|{}^h\nabla^2 g|^2$$
$$\le B|{}^h\nabla g|\big(|{}^h\nabla^2 g||{}^h\nabla g| + |{}^h\nabla^2 g| + |{}^h\nabla g|^3 + |{}^h\nabla g|^2 + |{}^h\nabla g| + 1\big)$$
$$\le \frac{1}{2a}|{}^h\nabla^2 g|^2 + B\big(|{}^h\nabla g|^4 + 1\big). \tag{5.2}$$

Integration by parts (once) and Young's inequality yields

$$\Big|\int_M \eta^4 g^{ij}\,{}^h\nabla^2_{ij}|{}^h\nabla g|^2\, dh\Big|$$
$$\le B \int_M \eta^4 |{}^h\nabla g|^2 |{}^h\nabla^2 g|\, dh + C \int_M \eta^3 |{}^h\nabla g||{}^h\nabla^2 g|\, dh,$$
$$\le \frac{1}{2a}\int_M \eta^4 |{}^h\nabla^2 g|^2 + C \int_M \eta^2 (|{}^h\nabla g|^4 + 1)\, dh, \tag{5.3}$$

for η a standard cut-off function as in Lemma 5.5. Multiplying the above differential inequality (5.2) with η^4, integrating and using the inequality (5.3), we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_M \eta^4 |{}^h\nabla g|^2\, dh\right) + \frac{2}{a}\int_M \eta^4 |{}^h\nabla^2 g|^2\, dh \\
&\quad\le \frac{1}{2a}\int_M \eta^4 |{}^h\nabla^2 g|^2\, dh + C\int_M \eta^2\big(|{}^h\nabla g|^4 + 1\big)\, dh \\
&\quad\le \frac{1}{2a}\int_M \eta^4 |{}^h\nabla^2 g|^2\, dh + C\left(\int_{B_{R_1}(x_0)} \big(|{}^h\nabla^2 g|^2 + |{}^h\nabla g|^2 + 1\big)\, dh\right)^2,
\end{aligned}$$

where we used the Sobolev inequality applied to the function $f = \sqrt{\eta}|{}^h\nabla g|$, and $|{}^h\nabla\sqrt{\eta}|^2 = |{}^h\nabla\eta|^2/4\eta \le C$ in the last step. Absorbing the first term on the right-hand side into the left-hand side and integrating from 0 to S, we conclude

$$\int_M \eta^4 |{}^h\nabla g|^2(\cdot, S)\, dh \le \int_M \eta^4 |{}^h\nabla g_0|^2\, dh + CS(\delta + 1)^2$$

for all $S \in [0, T]$.

Now we turn to the corresponding estimate for the second derivatives. Recalling (4.5) and using Young's inequality, we see that

$$\begin{aligned}
&\frac{\partial}{\partial t}|{}^h\nabla^2 g|^2(x,t) - g^{ij}(x,t)\,{}^h\nabla^2_{ij}|{}^h\nabla^2 g|^2 + \frac{2}{a}|{}^h\nabla^3 g|^2(x,t) \\
&\quad\le B|{}^h\nabla^2 g|\Big(|{}^h\nabla^3 g|\big(|{}^h\nabla g| + 1\big) + |{}^h\nabla^2 g|\big(|{}^h\nabla^2 g| + |{}^h\nabla g|^2 + |{}^h\nabla g| + 1\big) \\
&\qquad\quad + |{}^h\nabla g|\big(|{}^h\nabla g|^3 + |{}^h\nabla g|^2 + |{}^h\nabla g| + 1\big) + 1\Big) \\
&\quad\le \frac{1}{2a}|{}^h\nabla^3 g|^2 + B\big(|{}^h\nabla^2 g|^3 + |{}^h\nabla^2 g|^2 + |{}^h\nabla g|^6 + |{}^h\nabla g|^2 + 1\big)
\end{aligned}$$

holds. As above, we note that integration by parts (once) followed by applications of Young's inequality yields

$$\begin{aligned}
&\left|\int_M \eta^4 g^{ij}(x,t)\,{}^h\nabla^2_{ij}|{}^h\nabla^2 g|^2(x,t)\, dh\right| \\
&\quad\le B\int_M \eta^4 |{}^h\nabla g||{}^h\nabla^2 g||{}^h\nabla^3 g|\, dh + C\int_M \eta^3 |{}^h\nabla^2 g||{}^h\nabla^3 g|\, dh \\
&\quad\le \int_M \frac{\eta^4}{4a}|{}^h\nabla^3 g|^2 + B\eta^4|{}^h\nabla g|^2|{}^h\nabla^2 g|^2\, dh + C\int_M \eta^2|{}^h\nabla^2 g|^2\, dh \\
&\quad\le \int_M \frac{\eta^4}{4a}|{}^h\nabla^3 g|^2 + B\eta^4|{}^h\nabla^2 g|^3 + B\eta^4|{}^h\nabla g|^6 + C\eta^2|{}^h\nabla^2 g|^2\, dh.
\end{aligned}$$

Multiplying the differential inequality for $|{}^h\nabla^2 g|^2$ again with η^4, integrating and using the above two estimates, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_M \eta^4|{}^h\nabla^2 g|^2\,dh\right)+\frac{1}{a}\int_M \eta^4|{}^h\nabla^3 g|^2\,dh\\&\quad\le\int_{B_{R_1}} B\eta^4\big(|{}^h\nabla^2 g|^3+|{}^h\nabla g|^6\big)\,dh+C\big(|{}^h\nabla^2 g|^2+|{}^h\nabla g|^2+1\big)\,dh.\end{aligned}$$

Using the estimates from Lemma 5.5, and the assumption, with this estimate, we see that this implies

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_M \eta^4|{}^h\nabla^2 g|^2\right)+\frac{1}{4a}\int_M \eta^4|{}^h\nabla^3 g|^2\,dh\\&\quad\le C\int_{B_{R_1}}\Big(|{}^h\nabla^2 g|^2+|{}^h\nabla g|^2+1\Big)+C\left(\int_{B_{R_1}(x)}|{}^h\nabla^2 g|^2\,dh\right)^{3/2}\\&\qquad+C\left(\int_{B_{R_1}}|{}^h\nabla g|^2\,dh\right)^{1/2}\left(\int_{B_{R_1}}|{}^h\nabla^2 g|^2+|{}^h\nabla g|^2\,dh\right)\\&\qquad+B\left(\int_{B_{R_1}(x)}|{}^h\nabla^2 g|^2\,dh\right)^{1/2}\left(\int_{B_{R_1}}\eta^4|{}^h\nabla^3 g|^2\,dh\right)\\&\quad\le C\left(\int_{B_{R_1}}|{}^h\nabla^2 g|^2+|{}^h\nabla g|^2+1\,dh\right)\\&\qquad+C\left(\int_{B_{R_1}}|{}^h\nabla^2 g|^2+|{}^h\nabla g|^2+1\,dh\right)^2+B\delta\int_{B_{R_1}}\eta^4|{}^h\nabla^3 g|^2\,dh,\end{aligned}$$

and hence

$$\frac{\partial}{\partial t}\left(\int_M \eta^4|{}^h\nabla^2 g|^2\,dh\right)+\frac{1}{8a}\int_M \eta^4|{}^h\nabla^3 g|^2\,dh\le C(\delta+1)^2$$

if $B(a)\delta\le 1/10a$. Integration in time from 0 to S gives

$$\int_M \eta^4|{}^h\nabla^2 g|^2(\cdot,S)\,dh\le\int_M \eta^4|{}^h\nabla^2 g_0|^2\,dh+CS(\delta+1)^2,$$

as required. ∎

Lemma 5.6. *Let (M,h) be n-dimensional and satisfy (2.4), g be a smooth solution to (1.1) on $M\times(0,T]$, $T\le 1$ and assume that there exist $0<a\in\mathbb{R}$, $\delta\in\mathbb{R}$, so that*

$$\frac{1}{a}h\le g(\cdot,t)\le ah,$$

$$\sup_{x\in M}\int_{B_1}|{}^h\nabla g(\cdot,t)|^2\,dh\le\delta\quad\forall\, t\in(0,T],$$

$$K_0:=\sup_{x\in M}|\mathrm{Rm}(h)|\le 1.$$

Then there exists a $B = B(n, a)$ such that

$$\int_{B_1(x)} |g(\cdot,t) - g(\cdot,s)|^2\, dh \le B(\delta + K_0)|t-s|,$$

$$\int_{B_1(x)} |g^{-1}(\cdot,t) - g^{-1}(\cdot,s)|^2\, dh \le B(\delta + K_0)|t-s|,$$

$$\left|\int_{B_1(x)} |g(\cdot,t)|^2 - |g(\cdot,s)|^2\, dh\right| \le B(\delta + K_0)|t-s|,$$

$$\left|\int_{B_1(x)} |g^{-1}(\cdot,t)|^2 - |g^{-1}(\cdot,s)|^2\, dh\right| \le B(\delta + K_0)|t-s|$$

for all $x \in M$, for all $t, s \in (0, T]$, and for all $x \in M$.

Corollary 5.7. *Let (M, h) be n-dimensional and satisfy (2.4), g be a smooth solution to (1.1) on $M \times [0, T]$, $T \le 1$, and assume that there exist $0 < a \in \mathbb{R}$, $\delta \in \mathbb{R}$, so that*

$$\frac{1}{a}h \le g(\cdot,t) \le ah,$$

$$\sup_{x\in M}\int_{B_1} |{}^h\nabla g(\cdot,t)|^2\, dh \le \delta \quad \forall\, t \in [0,T],$$

$$K_0 := \sup_{x\in M} |\mathrm{Rm}(h)| \le 1.$$

Then there exists a $B = B(n, a)$ such that

$$\int_{B_1(x)} |g(\cdot,t) - g(\cdot,0)|^2\, dh \le B(\delta + K_0)|t,$$

$$\int_{B_1(x)} |g^{-1}(\cdot,t) - g^{-1}(\cdot,0)|^2\, dh \le B(\delta + K_0)|t,$$

$$\left|\int_{B_1(x)} |g(\cdot,t)|^2 - |g(\cdot,0)|^2\, dh\right| \le B(\delta + K_0)t,$$

$$\left|\int_{B_1(x)} |g^{-1}(\cdot,t)|^2 - |g^{-1}(\cdot,0)|^2\, dh\right| \le B(\delta + K_0)t$$

for all $t \in [0, T]$ and for all $x \in M$.

Proof of Corollary 5.7. For any sequence $t_i \to 0$ and any $x \in M$, we have

$$\int_{B_1(x)} |g(\cdot,t) - g(\cdot,t_i)|^2\, dh \le B(\delta + 1)|t - t_i|.$$

Letting $i \to \infty$ implies the first estimate view of the smoothness of the solution. The other estimates follow with an almost identical argument. ∎

Proof of Lemma 5.6. We calculate for a standard cut-off function η with $\eta = 1$ on $B_1(x)$, and $\eta = 0$ on $(B_2(x))^c$, and $|{}^h\nabla\eta|^2 \leq c(n)|\eta|$ that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(e^{-B(n,a)t}\int_M \eta|g(\cdot,t)-g(\cdot,s)|^2\,dh\right)\\
&\quad= e^{-B(n,a)t}\int_M \eta\frac{\partial}{\partial t}h^{ik}h^{jl}(g(t)_{ij}-g(s)_{ij})(g(t)_{kl}-g(s)_{kl})\,dh\\
&\qquad - B(n,a)e^{-B(n,a)t}\int_M \eta|g(\cdot,t)-g(\cdot,s)|^2\,dh\\
&\quad= e^{-B(n,a)t}\int_M 2\eta h^{ik}h^{jl}\frac{\partial}{\partial t}g(t)_{ij}(g(t)_{kl}-g(s)_{kl})\,dh\\
&\qquad - B(n,a)e^{-B(n,a)t}\int_M \eta|g(\cdot,t)-g(\cdot,s)|^2\,dh\\
&\quad= e^{-B(n,a)t}\int_M 2\eta h^{ik}h^{jl}\mathcal{L}(g(t),h)_{ij}(g(t)_{kl}-g(s)_{kl}\,dh)\\
&\qquad - B(n,a)e^{-B(n,a)t}\int_M \eta|g(\cdot,t)-g(\cdot,s)|^2\,dh\\
&\quad\leq e^{-B(n,a)t}B(n,a)\int_{B_2(x)}\left(|{}^h\nabla g(t)|^2+K_0+|{}^h\nabla g(s)|^2\right)dh\\
&\qquad + e^{-B(n,a)t}B(n,a)\int_M \eta|g(\cdot,t)-g(\cdot,s)|^2\,dh\\
&\qquad - B(n,a)e^{-B(n,a)t}\int_M \eta|g(\cdot,t)-g(\cdot,s)|^2\,dh\\
&\quad\leq e^{-B(n,a)t}B(n,a)(\delta+K_0),
\end{aligned}$$

where $\mathcal{L}(g(t),h)_{ij}$ is the right-hand side of the equation (1.1), and we used integration by parts, with respect to ${}^h\nabla$, in the second to last step, and a covering of $B_2(x)$ by $c(n)$ balls of radius one in the last step. Integrating from s to t implies the first estimate. Also,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(e^{-B(n,a)t}\int_M \eta|g^{-1}(\cdot,t)-g^{-1}(\cdot,s)|^2\,dh\right)\\
&\quad= e^{-B(n,a)t}\int_M \eta\frac{\partial}{\partial t}h_{ik}h_{jl}(g(t)^{ij}-g(s)^{ij})(g(t)^{kl}-g(s)^{kl})\,dh\\
&\qquad - B(n,a)e^{-B(n,a)t}\int_M \eta|g^{-1}(\cdot,t)-g^{-1}(\cdot,s)|^2\,dh\\
&\quad= e^{-B(n,a)t}\int_M 2\eta h^{ik}h^{jl}g^{iv}g^{jw}\frac{\partial}{\partial t}g(t)_{vw}(g(t)^{kl}-g(s)^{kl})\,dh\\
&\qquad - B(n,a)e^{-B(n,a)t}\int_M \eta|g^{-1}(\cdot,t)-g^{-1}(\cdot,s)|^2\,dh
\end{aligned}$$

$$
\begin{aligned}
&= e^{-B(n,a)t} \int_M 2\eta h^{ik} h^{jl} g^{iv} g^{jw} \mathcal{L}(g(t), h)_{vw} (g(t)^{kl} - g(s)^{kl})\, dh \\
&\quad - B(n,a) e^{-B(n,a)t} \int_M \eta |g^{-1}(\cdot, t) - g^{-1}(\cdot, s)|^2\, dh \\
&\le e^{-B(n,a)t} B(n,a) \int_{B_2(x)} \left(|{}^h\nabla g(t)|^2 + K_0 + |{}^h\nabla g(s)|^2 \right) dh \\
&\quad + e^{-B(n,a)t} B(n,a) \int_M \eta |g^{-1}(\cdot, t) - g^{-1}(\cdot, s)|^2\, dh \\
&\quad - B(n,a) e^{-B(n,a)t} \int_M \eta |g^{-1}(\cdot, t) - g^{-1}(\cdot, s)|^2\, dh \\
&\le e^{-B(n,a)t} B(n,a)(\delta + K_0).
\end{aligned}
$$

Integrating from s to t implies the second estimate. Also,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(e^{-B(n,a)t} \int_M \eta |g(\cdot, t)|^2\, dh \right) \\
&= e^{-B(n,a)t} \int_M \eta \frac{\partial}{\partial t} h^{ik} h^{jl} g(t)_{ij} g(t)_{kl}\, dh - B e^{-B(n,a)t} \int_M \eta |g(\cdot, t)|^2\, dh \\
&= e^{-B(n,a)t} \int_M 2\eta h^{ik} h^{jl} \frac{\partial}{\partial t} g(t)_{ij} g(t)_{kl}\, dh - B e^{-B(n,a)t} \int_M \eta |g(\cdot, t)|^2\, dh \\
&= e^{-B(n,a)t} \int_M 2\eta h^{ik} h^{jl} \mathcal{L}(g(t), h)_{ij} g(t)_{kl}\, dh \\
&\quad - B e^{-B(n,a)t} \int_M \eta |g(\cdot, t)|^2\, dh \\
&\le e^{-B(n,a)t} B(n,a) \int_{B_2(x)} \left(|{}^h\nabla g(t)|^2 + K_0 \right) dh \\
&\quad + B e^{-B(n,a)t} B(n,a) \int_M \eta |g(\cdot, t)|^2\, dh - B e^{-B(n,a)t} \int_M \eta |g(\cdot, t)|^2\, dh \\
&\le e^{-B(n,a)t} B(n,a)(\delta + K_0),
\end{aligned}
$$

where we used integration by parts in the second to last step. Integrating from s to t implies the third estimate. The fourth estimate follows similarly. ∎

The previous lemma showed us that solutions which are smooth on $M \times [0, T]$ and whose $W^{1,2}$-energy is bounded on balls of radius one by δ, and which are uniformly (independent of time) equivalent to h, $\frac{1}{a}h \le g(t) \le ah$, converge strongly in the L^2-norm back to g_0 as time goes to zero, and the rate of convergence depends only on a, n and δ. If we only assume that the solution is smooth on $M \times (0, T)$, then the previous lemma shows us that the solution is Cauchy in the $L^2(B_1(x))$-norm in time, and hence there exists a well-defined L^2_{loc} limit, g_0 at time $t = 0$ on M.

In the four-dimensional setting, the assumption that the solution is bounded in $W^{2,2}(B_1(x))$ for all $x \in M$ means that there must be a sequence of times t_i, such that $g(t_i)$ converge weakly in $H := W^{2,2}(B_1(x))$ back to $g_0 \in H$ (the details are given in the proof of Theorem 5.8). The estimates of Lemma 5.5, show us that in fact the convergence is also strong in $W^{2,2}_{\mathrm{loc}}$, if the solution is a limit of smooth solutions, whose initial data converge locally strongly in $W^{2,2}$, as is explained in the following theorem. Note that this is precisely the situation which we study in the next section.

Theorem 5.8. *For all $0 < a \in \mathbb{R}$, there exists $\delta = \delta(a)$ so that the following holds. Let (M, h) be four-dimensional and satisfy (2.4) and $(M, g(t), p)_{t\in(0,T]}$ be the smooth limit, on compact subsets of $M \times (0, T]$, of $(M, g(i)(t), p_i)|_{t\in(0,T]}$ as $i \to \infty$ of a sequence of smooth solutions $g(i)$ to (1.1) defined on $M \times [0, T]$ which satisfy*

$$\frac{1}{a}h \leq g(i)(\cdot,t) \leq ah, \quad \sup_{x\in M} \int_{B_1(x)} \left(|^h\nabla g(i)(\cdot,t)|^2 + |^h\nabla^2 g(i)(\cdot,t)|^2\right) dh \leq \delta$$

for all $t \in [0, T]$. Assume further that the initial data $g(i)(\cdot, 0)$ converge strongly locally in $W^{2,2}_{\mathrm{loc}}$ to some $g_0 \in W^{2,2}_{\mathrm{loc}}$ as $i \to \infty$, that is

$$\|g(i)(0) - g_0\|_{W^{2,2}(K)} \to 0$$

for all compact sets $K \subseteq M$ as $i \to \infty$. Then, $g(t) \to g_0$ as $t \searrow 0$ locally strongly in the $W^{2,2}$-norm, that is

$$\int_{B_1(x)} |g(\cdot,t) - g_0(\cdot)|^2\, dh + \int_{B_1(x)} |^h\nabla(g(\cdot,t) - g_0(\cdot))|^2(\cdot,t)\, dh \\ + \int_{B_1(x)} |^h\nabla^2(g(\cdot,t) - g_0(\cdot))|^2(\cdot,t)\, dh \to 0$$

as $t \searrow 0$ for any $x \in M$.

Proof. The solutions $g(i)$ defined on $M \times [0, T]$ are smooth and satisfy the hypotheses of Theorem 5.1. Without loss of generality $T \leq 1$. Hence, the conclusions of that theorem hold and we get

$$\int_{B_{R_0}(x_0)} \left(|^h\nabla g(i)(\cdot,t)|^2 + |^h\nabla^2 g(i)(\cdot,t)|^2\right) dh \\ \leq \int_{B_{R_1}(x_0)} \left(|^h\nabla g(i)(\cdot,0)|^2 + |^h\nabla^2 g(i)(\cdot,0)|^2\right) dh + V(R_0, R_1, a)t$$

for any $x_0 \in M$ and for all $t \in [0, T)$. The third estimate of Corollary 5.7 implies additionally for any $x_0 \in M$ and all $t \in [0, T)$ (remembering $T \leq 1$),

$$\int_{B_{R_0}(x_0)} |g(i)(\cdot,t)|^2\, dh \leq \int_{B_{R_1}(x_0)} |g(i)(\cdot,0)|^2\, dh + V(R_0, R_1, a)t.$$

Letting $i \to \infty$ for fixed $t \in (0, T)$, and using that the solution converges smoothly locally away from $t = 0$ and in the $W^{2,2}_{\text{loc}}$-norm at time zero, we see that the limit solution also satisfies

$$\begin{aligned}\int_{B_{R_0}(x_0)} & \left(|{}^h\nabla g(\cdot,t)|^2 + |{}^h\nabla^2 g(\cdot,t)|^2 + |g(t)|^2\right) dh \\ & \leq \int_{B_{R_1}(x_0)} \left(|{}^h\nabla g_0|^2 + |{}^h\nabla^2 g_0|^2 + |g_0|^2\right) dh + V(R_0, R_1, a)t \qquad (5.4)\end{aligned}$$

for any $x_0 \in M$, for all $t \in (0, T)$, that is

$$\|g(t)\|^2_{W^{2,2}(B_{R_0}(x_0))} \leq \|g_0\|^2_{W^{2,2}(B_{R_1}(x_0))} + V(R_0, R_1, a)t$$

for any $x_0 \in M$, for all $t \in (0, T)$, where

$$\|T\|^2_{W^{2,2}(B_{R_0}(x_0))} := \int_{B_{R_0}(x_0)} \left(|T|^2_h + |{}^h\nabla T|^2 + |{}^h\nabla^2 T|^2\right) dh$$

for any zero-two tensor defined on $B_{R_0}(x_0)$ whose components are in $W^{2,2}$. Furthermore, we have

$$\int_{B_1(x_0)} |g(t) - g_0|^2 \, dh \to 0$$

for $t \searrow 0$ in view of Corollary 5.7, and the fact that $g(i)(0) \to g_0$ as $i \to \infty$ in the $W^{2,2}_{\text{loc}}$-norm.

Fixing $x_0 \in M$ and $R_0 := 1$, we define the Hilbert space $H := W^{2,2}(B_1(x_0))$ to be the space of zero-two tensors whose components are in $W^{2,2}(B_1(x_0))$ and whose scalar product is defined by

$$(T, S)_H := \int_{B_1(x_0)} (T, S)_h + ({}^h\nabla T, {}^h\nabla S)_h + ({}^h\nabla^2 T, {}^h\nabla^2 S)_h \, dh.$$

Using this notation, we may write (5.4) as

$$(g(t), g(t))_H \leq \|g_0\|^2_{W^{2,2}(B_{R_1}(x_0))} + V(R_0, R_1, a)t \qquad (5.5)$$

for all $t \in (0, T)$ and for any $2 > R_1 > R_0 = 1$. We are going to show that every sequence $(g(t_i))_{i\in\mathbb{N}}$ with $t_i \searrow 0$ contains a strongly converging subsequence with limit g_0 in H. This then clearly implies that $g(s) \to g_0$ in H as $s \searrow 0$ since otherwise we can find a sequence $t_i \to 0$, and a $\delta > 0$ such that

$$\|g(t_i) - g_0\|_H > \delta,$$

and hence no subsequence of $g(t_i)$ will converge to g_0 in H, which would be a contradiction.

Now to the details. For any sequence of times $0 < t_i \to 0$ as $i \to \infty$, there exists a subsequence, $g(t_{i_j}) =: g_{i_j}$ of $g(t_i)$ such that $g_{i_j} \rightharpoonup z$ as $j \to \infty$ for some $z \in H$, in view of the definition of a Hilbert space and weak convergence.

But g_{i_j} must then converge strongly to z in $L^2(B_1(x_0))$, and hence $z = g_0$. Setting $r_j := t_{i_j}$ this means we have $g(r_j) \to g_0$ strongly in $L^2(B_1(x_0))$ and $g(r_j) \rightharpoonup g_0$ weakly in H. It remains to show that $g(r_j) \to g_0$ strongly in $H = W^{2,2}(B_1(x_0))$ for all $x_0 \in M$.

Assume that this is not true for some $x_0 \in M$. Then we can find a subsequence $s_k := r_{i_k}$, $k \in \mathbb{N}$ of $(r_k)_{k\in\mathbb{N}}$ and a $\delta > 0$ such that

$$\|g(s_k) - g_0\|_H^2 \geq \delta > 0$$

for all $k \in \mathbb{N}$. But then

$$\begin{aligned}\delta &\leq (g(s_k) - g_0, g(s_k) - g_0)_H \\ &= (g(s_k), g(s_k))_H + (g_0, g_0)_H - 2(g(s_k), g_0)_H \\ &= \|g(s_k)\|_H^2 + \|g_0\|_H^2 - 2(g(s_k), g_0)_H\end{aligned}$$

for all $k \in \mathbb{N}$, and hence, using (5.5), we have

$$\delta \leq \|g_0\|^2_{W^{2,2}(B_{R_1}(x_0))} + \|g_0\|_H^2 - 2(g(s_k), g_0)_H + V(1, R_1, a)s_k.$$

Since g_0 in $W^{2,2}(B_2(x_0))$ (here we use the covering argument from Lemma B.1), there must exist a $1 < R_1 < 2$ such that

$$\|g_0\|^2_{W^{2,2}(B_{R_1}(x_0))} \leq \|g_0\|_H^2 + \frac{\delta}{8},$$

and hence we obtain

$$\delta \leq 2\|g_0\|_H^2 - 2(g(s_k), g_0)_H + \frac{\delta}{8} + V(1, R_1, a)s_k$$

for this choice of R_1 independent of $k \in \mathbb{N}$. Letting $k \to \infty$, we obtain a contradiction, since

$$2\|g_0\|_H^2 - 2(g(s_k), g_0)_H \to 0$$

as $k \to \infty$. ∎

6. Existence and regularity

In this section we prove the main results for the Ricci–DeTurck flow of data which is initially $W^{2,2}$.

Theorem 6.1. *Let $1 < a < \infty$ and (M, h) be four-dimensional and satisfy (2.4). Then there exists a constant $\varepsilon_1 = \varepsilon_1(a) > 0$ with the following properties. Assume g_0 is a smooth Riemannian metric on M which is uniformly bounded in $W^{2,2}_{\mathrm{loc}} \cap L^\infty$ in the following sense:*

$$\frac{1}{a} h \leq g_0 \leq ah, \tag{a}$$

$$\int_{B_2(x)} \left(|{}^h\nabla g_0|^2 + |{}^h\nabla^2 g_0|^2 \right) dh \leq \varepsilon \quad \text{for all } x \in M, \tag{b}$$

where $\varepsilon \leq \varepsilon_1(a)$, and g_0 satisfies

$$\sup_M |{}^h\nabla^i g_0|^2 < \infty$$

for all $i \in N$. Then there exists constants $T = T(a, \varepsilon) > 0$ and $c_j = c_j(a, h) > 0$, and a smooth solution $(g(t))_{t \in [0,T]}$ to (1.1) with $g(\cdot, 0) = g_0(\cdot)$ such that

$$\frac{1}{400a} h \leq g(t) \leq 400ah, \tag{a$_t$}$$

$$\int_{B_1(x)} \left(|{}^h\nabla g(\cdot, t)|^2 + |{}^h\nabla^2 g(\cdot, t)|^2 \right) dh < 2\varepsilon \quad \text{for all } x \in M,\ t \in [0, T], \tag{b$_t$}$$

$$|{}^h\nabla^j g(\cdot, t)|^2 \leq \frac{c_j(a, h)}{t^j} \tag{c$_t$}$$

and

$$\sup_M |{}^h\nabla^j g(t)|^2 < \infty$$

for all $j \in N$ and for all $t \in [0, T]$.

Proof. Using the existence theory for parabolic equations, for example the method of Shi (see [29, Sections 3 and 4], which in turn uses [19, Theorem 7.1, Section VII]), we see that we have a solution to (1.1) for a short time $[0, V]$ for some $V > 0$ and $\sup_{M \times [0,V]} |{}^h\nabla^j g|^2 < \infty$ for all $j \in \mathbb{N}$; see Theorem A.1.

We assume, without loss of generality, that

$$\varepsilon_1(a) \leq \frac{\varepsilon_0}{4} = \frac{\varepsilon_0(a, 4)}{4},$$

where ε_0 is the constant from Theorem 4.2. Let

$$\hat{S} := \sup\{t \in [0, V] \mid (\mathrm{b}_t) \text{ holds for } t \leq \hat{S}\},$$

where $\hat{S} > 0$ due to smoothness (and boundedness of covariant derivatives of g). We have

$$|{}^h\nabla g(\cdot, t)| \leq \frac{1}{t} \quad \text{and} \quad \frac{1}{80a} h \leq g(t) \leq 80ah,$$

for $t \le \min(\widehat{S}, S_1(4,a))$, in view of Theorem 4.2, where $S_1(4,a)$ is the constant from that theorem. Hence, for such $t \le \min(S_1(4,a), \widehat{S})$, we have

$$|^h\nabla^i g(\cdot,t)|^2 \le \frac{N_i(80a,1,4,h)}{t^i}$$

in view of Lemma 4.1.

Also, using Corollary 5.4 with $b = 80a$, we can improve the estimate (b_t) to

$$\int_{B_1(x)} \left(|^h\nabla g|^2 + |^h\nabla g|^2\right) dh \le \frac{3}{2}\varepsilon < 2\varepsilon,$$

for $t \le \min(S_1(4,a), \widehat{S}, S_2(b = 80a, \varepsilon))$, where $S_2(b,\varepsilon) > 0$ is the constant from that corollary, since $2\varepsilon \le 2\varepsilon_1(a) < \varepsilon_0$, and without loss of generality, $\varepsilon_0 \le \delta = \delta(80a)$, where δ is the constant appearing in that corollary. Hence,

$$\widehat{S} \ge \min(S_1(4,a), V, S_2(b = 80a, \varepsilon)),$$

and (a_t), (b_t), and (c_t) hold for $t \le \min(S_1(4,a), V, S_2(b = 80a, \varepsilon))$, with $c_1 = 2$, $c_i = N_i(80a,1,4,h)$ for all $i \in \mathbb{N}, i \ge 2$. Applying Theorem A.1 and repeating this argument as often as necessary, we may extend this solution to a smooth solution $g(t)_{t\in[0,T]}$ satisfying (a_t), (b_t), and (c_t) for $t \le T := \min(S_1(4,a), S_2(b = 80a, \varepsilon))$. The estimates

$$\sup_{M\times[0,V]} |^h\nabla^j g|^2 < \infty$$

for all $j \in \mathbb{N}$ and (c_t) guarantee that

$$\sup_{M\times[0,T]} |^h\nabla^j g|^2 < \infty$$

for all $j \in \mathbb{N}$. ∎

Remark 6.2. In the proof of Theorem 6.1, we obtained

$$\int_{B_1(x)} \left(|^h\nabla g(\cdot,t)|^2 + |^h\nabla^2 g(\cdot,t)|^2\right) dh < \frac{3}{2}\varepsilon$$

for all $x \in M$, $t \in [0,T]$, and

$$\frac{1}{80a} hg(t) \le 80ah$$

for $t \in [0,T]$. We will use these facts in the proof of the next theorem.

Theorem 6.3. *Let (M, h) be four-dimensional and satisfy (2.4) and $\infty > a > 1$. Assume g_0 is a $W^{2,2}_{\mathrm{loc}} \cap L^\infty$ Riemannian metric, not necessarily smooth, which satisfies (a) and (b), where $\varepsilon \leq \varepsilon_1(2a)/2$ and ε_1 is the constant from Theorem 6.1.*

Then there exists a constant $S = S(a, \varepsilon) > 0$ and a smooth solution $(g(t))_{t\in(0,S]}$ to (1.1), where $g(t)$ satisfies (a_t), (b_t), and (c_t) for all $x \in M$, for all $t \in (0, S]$, and

$$\int_{B_1(x)} \left(|g_0 - g(t)|^2 + |{}^h\nabla(g_0 - g(t))|^2 + |{}^h\nabla^2(g_0 - g(t))|^2\right) dh \to 0 \tag{d_t}$$

as $t \searrow 0$ for all $x \in M$. The solution is unique in the class of solutions satisfying (a_t), (b_t), (c_t), and (d_t). The solution also satisfies the local estimates

$$\sup_{x\in B_1(x_0)} |{}^h\nabla^j g(\cdot, t)|^2 t^j \to 0 \quad \text{for } t \to 0 \tag{e_t}$$

$$\begin{aligned}&\int_{B_1(x_0)} \left(|{}^h\nabla g(\cdot, t)|^2 + |{}^h\nabla^2 g(\cdot, t)|^2\right) dh \\ &\quad\leq \int_{B_R(x_0)} \left(|{}^h\nabla g_0(\cdot)|^2 + |{}^h\nabla^2 g_0(\cdot)|^2\right) dh + V(a, R)t\end{aligned} \tag{f_t}$$

for all $x_0 \in M$, $2 > R > 1$, for all $t \leq T$, and for some constant $0 < V(a, R) < \infty$.

Proof. Let $R > 0$ be given, and $\eta\colon M \to [0, 1] \subseteq \mathbb{R}$ be a smooth cut-off function as in Lemma B.1 (iv): $\eta = 1$ on $B_R(x_0)$, $\eta = 0$ on $M \setminus (B_{CR}(x_0))$, $|{}^h\nabla^2\eta| + |{}^h\nabla\eta|^2/\eta \leq C/R^2$ on M (here $n = 4$), $|{}^h\nabla^i\eta|^2 \leq c_i(h)$ for all $i \in \mathbb{N}$. We mollify the metric g_0 everywhere locally, to obtain a metric $\hat{g}_{0,R}$, which is smooth, and then define

$$g_{0,R}(\cdot) := \eta(\cdot)\hat{g}_{0,R}(\cdot) + (1 - \eta(\cdot))h(\cdot).$$

We choose the mollification fine enough to ensure that $g_{0,R}(\cdot) \to g_0$ in $W^{2,2}(B_r(0))$ for all $r > 0$ fixed as $R \to \infty$ and so that (a) and (b) still hold for $g_{0,R}$ up to a factor $10/9$, for all $R > 0$ sufficiently large. That is, we have

$$\frac{9}{10a}h \leq g_{0,R} \leq \frac{10}{9}ah, \tag{$\tilde{a}$}$$

$$\int_{B_2(x)} \left(|{}^h\nabla g_{0,R}|^2 + |{}^h\nabla^2 g_{0,R}|^2\right) dh \leq \frac{10}{9}\varepsilon \quad \text{for all } x \in M. \tag{$\tilde{b}$}$$

Furthermore, $g_{0,R} = h$ outside of $B_{CR}(x_0)$, and so

$$\sup_M |{}^h\nabla^j g_{0,R}|^2 < \infty$$

for all $j \in \mathbb{N}$.

Theorem 6.1, with a replaced by $(10/9)a$ and ε by $10/9$, and Remark 6.2, guarantee the existence of a solution $(g_R(t))_{t\in[0,T]}$ with $T = T(a, \varepsilon) > 0$ to (1.1) satisfying (a_t), (b_t), and (c_t) for all $x \in M$, for all $t \in [0, T]$, with $g_R(0) = g_{0,R}$. Note that the constants $c_j((10/9)a, h)$ of (c_t) do not depend on R. Hence, there exists a limit solution $(M, g(t)_{t\in(0,T]}$ (in the C^∞_{loc}-sense on $M \times (0, T)$), by taking the sequence of radii $R(i) = i \to \infty$, which satisfies (a_t), (b_t) and (c_t) for all $x \in M$, for all $t \in (0, T]$. Theorem 5.1 applied to each $g_{R(i)}$ implies (f_t), and Theorem 5.8 implies (d_t), for the limit solution $g(t)_{t\in(0,T]}$. Note without loss of generality, that $\varepsilon_1(2a)$ from Theorem 6.1 is less than $\delta(400a)/c(a, n)$, where δ is the constant from Theorem 5.1, and $c(a, n)$ is a large constant of our choice. Hence, without loss of generality, the scale invariant condition

$$\left(\int_{B_1(x)} |^h\nabla g(\cdot, t)|^4\, dh\right)^{1/2} + \int_{B_1(x)} |^h\nabla^2 g(\cdot, t)|^2\, dh \le \frac{\delta(400a)}{c(a, n)} \tag{6.1}$$

holds for all $x \in M$, for all $t \in (0, T]$, in view of Lemma B.1 (v). For any $x \in M$, we claim that

$$\sup_{B_1(x)} \left(t|^h\nabla g(\cdot, t)|^2\right) \to 0 \quad \text{as } t \searrow 0.$$

Assume that this is not the case for some $x \in M$. Then we obtain a sequence of points $y_j \in B_1(x) \subseteq M$ and $0 < t_j \to 0$ for $j \in \mathbb{N}$, $j \to \infty$ and an $r > 0$ such that

$$t_j|^h\nabla g(y_j, t_j)|^2 \ge r > 0.$$

Taking a subsequence we see that $y_j \to y \in M$, and hence

$$\int_{B_{2\sqrt{t_j}}(y_j)} \left(|^h\nabla g_0|^4 + |^h\nabla^2 g_0|^2\right) dh \to 0$$

as $j \to \infty$. Scale the solution to time 1, that is, we define

$$g_j(\cdot, \tilde{t}) = \frac{1}{t_j} g(\cdot, t_j\tilde{t}), \quad h_j(\cdot) = \frac{1}{t_j} h(\cdot), \quad g_{j,0} = \frac{1}{t_j} g_0.$$

Now, we have

$$|^{h_j}\nabla g_j(y_j, 1)|^2 \ge r > 0 \quad \text{and} \quad \int_{B_2(y_j)} \left(|^h\nabla g_{j,0}|^4 + |^h\nabla^2 g_{j,0}|^2\right) dh \to 0$$

as $j \to \infty$, and

$$\frac{1}{400a} h_j(\cdot) \le g_j(\cdot, t) \le 400a h_j(\cdot)$$

for all t where the solution is defined. Theorem 5.1, and the fact that (6.1) also holds for the scaled solutions, now implies that

$$\int_{B_1(y_j)} |^{h_j}\nabla g_j(\cdot, 1)|^2 + |^{h_j}\nabla^2 g_j(\cdot, 1)|^2\, dh_j \to 0$$

as $j \to \infty$, and

$$\frac{1}{400a} h_j(\cdot) \leq g_j(\cdot, t) \leq 400a h_j(\cdot)$$

for all t where the solution is defined.

But these estimates combined with (c_t) then imply that

$$|^{h_j}\nabla^k g_j(y_j, 1)|^2 \to 0$$

for all $k \in \mathbb{N}$ as $j \to \infty$, which is a contradiction. To see this, one can write all quantities in geodesic coordinates with respect to the metric h_j centred at y_j. The estimate

$$\sup_{B_1(x)} \left(t^j |^h\nabla^j g(\cdot, t)|^2\right) \to 0 \quad \text{as } t \searrow 0$$

for the other $j \in \mathbb{N}$ follow from an almost identical argument. That is (e_t) also holds. The uniqueness of the solution in the class of solutions satisfying (a_t), (b_t), (c_t), and (d_t) follows immediately from Theorem 11.2. ∎

Remark 6.4. In fact, the constants $c_j(h, a)$ in (c_t) of Theorem 6.3 can be replaced by $c_j(h, a, \varepsilon)$ where $c_j(h, a, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Assume $g_i(t)_{t \in (0, T(1/i, a)]}$ are solutions obtained in Theorem 6.3 with ε in (b_t) given by $\varepsilon = 1/i$, and assume

$$|^h\nabla g_i(x_i, t_i)|^2 \geq \frac{\alpha}{t_i} > 0$$

for a $t_i \in (0, T(1/i, a)]$ for some $\alpha > 0$. From Theorem B.1 (v), we have

$$\int_{B_1(x)} |^h\nabla g_i(\cdot, t)|^4 + |^h\nabla g_i^2(\cdot, t)|^2 \leq 2\frac{1}{i}$$

for all $t \in (0, T(1/i, a)]$. Scaling the solutions by $\hat{g}_i(t) := \frac{1}{t_i} g_i(t t_i)$, we obtain a smooth solution defined on $(0, 1]$ which satisfies

$$|^{h_i}\nabla^j \hat{g}_i(\cdot, 1)|^2 \leq c_j$$

for all $j \in \mathbb{N}$ in view of (c_t), and

$$|^{h_i}\nabla \hat{g}_i(x_i, 1)|^2 \geq \alpha \quad \text{and} \quad \int_{B_1(x)} \left(|^{h_i}\nabla \hat{g}_i(\cdot, 1)|^4 + |^{h_i}\nabla \hat{g}_i^2(\cdot, 1)|^2\right) d_{h_i}(x) \leq 2\frac{1}{i}$$

for all $x \in M$, where we denote the scaled metric $\frac{1}{t_i} h$ by h_i. But this means

$$|^{h_i}\nabla^j \hat{g}_i(\cdot, 1)|^2 \to 0$$

for all $j \in \mathbb{N}$ as $i \to 0$, as can be seen by writing all quantities in geodesic coordinates with respect to h_i at x_i. This contradicts $|{}^{h_i}\nabla \hat{g}_i(x_i, 1)|^2 \geq \alpha > 0$ for all $i \in \mathbb{N}$. Hence,

$$c_1(a, h) = c_1(h, a, \varepsilon) \to 0$$

as $\varepsilon \to 0$. An almost identical argument shows that $c_j(h, a)$ in (c_t) of Theorem 6.3 can be replaced by $c_j(h, a, \varepsilon)$, where $c_j(h, a, \varepsilon) \to 0$ as $\varepsilon \to 0$.

In the case that the energy is bounded uniformly, then a scaling argument leads to the setting of the previous theorem, and hence we may find a solution to equation (1.1) for a short time, which satisfies the conclusions of the previous theorems, for any $\varepsilon > 0$, if we shorten the length of the time interval.

Theorem 6.5. *Let (M, h) be four-dimensional and satisfy (2.1), and $1 < a < \infty$, and g_0 be a $W^{2,2} \cap L^\infty$ Riemannian metric, not necessarily smooth, which satisfies (a) and*

$$\int_M \left(|{}^h\nabla g_0|^2 + |{}^h\nabla^2 g_0|^2\right) dh < \infty.$$

Then for any $\varepsilon > 0$, there exists a constant $T = T(g_0, a, \varepsilon) > 0$, a $C = C(g_0, a, \varepsilon)$ and a smooth solution $(g(t))_{t\in(0,T]}$ to (1.1), such that after scaling the solution and the background metric h once, (a_t), (b_t), (c_t), (d_t), (e_t), and (f_t) hold, and $(g(t))_{t\in(0,T]}$ is the unique solution in the class of solutions satisfying the conditions (a_t), (b_t), (c_t), (d_t). The constants $c_j(h, a)$ in (c_t) can be replaced by $c_j(h, a, \varepsilon)$ where $c_j(h, a, \varepsilon) \to 0$ as $\varepsilon \to 0$

Remark 6.6. In this setting, we cannot expect

$$\int_M \left(|{}^h\nabla g(t)|^2 + |{}^h\nabla^2 g(t)|^2\right) dh < \infty$$

for any $t > 0$ as Example 6.7 below shows.

Proof. As explained in the introduction, by scaling the initial data g_0 and the background metric h once, we can guarantee that the new initial data and background metric, which we also call g_0 and h, satisfy (a) and (b), where $\varepsilon \leq \varepsilon_1(2a)/2$ and ε_1 is the constant from Theorem 6.1, and that h satisfies (2.4). Using Theorem 6.3, we obtain a solution $g(t)$, $t > 0$, $t \in [0, T]$ which satisfies (a_t), (b_t), (c_t), (d_t), (e_t), and (f_t). The uniqueness of the solution in the class of solutions satisfying (a_t), (b_t), (c_t), and (d_t) follows immediately from Theorem 11.2. The fact that the $c_j(h, a)$ in (c_t) can be replaced by $c_j(h, a, \varepsilon)$, where $c_j(h, a, \varepsilon) \to 0$ as $\varepsilon \to 0$, follows from Remark 6.4. ∎

Example 6.7. Let $n = 2$ and g_0 be a smooth metric on

$$C_1(0) := \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \max\left(|x_1|, |x_2|\right) < 1\right\}$$

such that

$$(1-\varepsilon)\delta \le g_0 \le (1+\varepsilon)\delta, \quad 0<\varepsilon \ll 1, \quad \text{and} \quad g_0 = \delta \text{ on } C_1(0) \setminus C_{1/2}(0),$$

and so that the curvature of g_0 is a constant $\sigma_1 > 0$ on $C_{\sigma_2}(0)$, for two small constants $\sigma_1, \sigma_2 > 0$, where δ is the standard metric on $\mathbb{R}^2$. We extend this metric to all of $\mathbb{R}^2$ through symmetry,

$$g_0(x) = g_0(x+p) \quad \text{for all } p \in \mathbb{Z}^2 = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{Z}\},$$

and in doing so obtain a smooth Riemannian metric g_0 on $\mathbb{R}^2$ satisfying

$$(1-\varepsilon)\delta \le g_0 \le (1+\varepsilon)\delta,$$

with $\psi_p^* g_0 = g_0$ for any $p \in \mathbb{Z}^2$, where $\psi_p(x) = x+p$.

Let T_i^2 refer to the standard 2-torus whose circles have radius $i \in \mathbb{N}$. Then

$$T_i^2 := \mathbb{R}^2/\Gamma(i),$$

where

$$\Gamma(i) := \left\{T_y\colon \mathbb{R}^2 \to \mathbb{R}^2 \mid y \in \mathbb{Z}^2, \text{ where } T_y(x) = x + iy \text{ for all } x \in \mathbb{R}^2\right\}.$$

That is,

$$T_i^2 = \left\{[x] \mid x \in \mathbb{R}^2\right\},$$

where $[x] = [z]$ if and only if $T(x) = z$ for some $T \in \Gamma(i)$. We give T_i^2 the unique metric $g_0(i)$ such that $\pi^*(g_0(i)) = g_0$, where $\pi\colon \mathbb{R}^2 \to T_i^2$ is the standard projection $\pi(x) = [x]$.

From the work of Shi [29], there exists a unique smooth solution

$$(T_i^2, g(i)(t))_{t\in[0,T]}$$

to (1.1) with $h = g_0(i)$, $g(i)(0) = g_0(i)$,

$$\sup_{t\in[0,T]} |{}^h\nabla^j g(i)(t)|^2 < c_j(g_0)$$

for all $j \in \mathbb{N}$, and all $t \in [0,T]$, and

$$(1-2\varepsilon)g_0(i) \le g(i)(t) \le (1+2\varepsilon)g_0(i)$$

for all $t \in [0,T]$.

Defining $\varphi_p\colon T_i^2 \to T_i^2$ by $\varphi_p([x]) = [p+x]$, where $p \in \mathbb{Z}^2$, we see that

$$\varphi_p^*(g_0(i)) = g_0(i)$$

by construction, and hence it is an isometry with respect to $g_0(i)$. Setting

$$\tilde{g}(i)(t) := (\varphi_p)^*(g(i)(t)),$$

we see that it is also a solution to (1.1), with $\tilde{g}(i)(0) = g_0(i)$ and $h = g_0(i)$. Uniqueness of such solutions, which can be seen by applying the maximum principle to the function

$$f(\cdot,t) := |g_1(\cdot,t) - g_2(\cdot,t)|^2,$$

shows us that $(\varphi_p)^* g(i)(t) = g(i)(t)$ for all $t \in [0,T]$.

Taking a subsequence and then a limit $i \to \infty$, we obtain a solution $g(t)$ to (1.1) with $g(0) = g_0$,

$$\sup_{t\in[0,T]} |{}^h\nabla^j g(t)|^2 < c_j(g_0) < \infty$$

for all $j \in \mathbb{N}$ and all $t \in [0,T]$, and such that

$$(\psi_p)^* g(t) = g(t), \quad \psi_p^* h = h,$$

where $\psi_p(x) = x + p$ and $p \in \mathbb{Z}^2$. Furthermore, $|{}^h\nabla^j g_0| = 0$ for all $j \in \mathbb{N}$, since $g_0 = h$.

Using a Taylor expansion in time for $g(t)$, and the fact that $g(t)$ is a smooth solution to (1.1) with $g(0) = h$, we see that

$$g(x,t) = h(x) + \frac{\partial}{\partial t} g(x,0)\cdot t + \Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t}\Big) g(x,s)\cdot t^2$$

for some $s \in (0,t)$. Notice that $\frac{\partial}{\partial t} g(x,0) = 0$ for $x \in C_{4/5}(0) \setminus C_{3/4}(0)$ because there, $g(x,0) = h(x) = \delta$ and δ has zero curvature. Hence,

$$g(x,t) - \delta = O(t^2)$$

for small t for $x \in C_{4/5}(0) \setminus C_{3/4}(0)$, and

$$g_{ii}(y,t) - h_{ii}(y) = -2\sigma_{ii} t + O(t^2)$$

for $y = 0$, for where $\sigma_{ii} = c(n)\sigma_1 > 0$.

If ${}^h\nabla g(t) = 0$ holds for all $t \in [0,P)$ then, for all $t \in (0,P)$ with $P > 0$ small enough, we would have $g(t) = h$ on $C_{4/5}(0) \setminus C_{3/4}(0)$ and $g(y,t) - h(y) \neq 0$ at $y = 0$. But taking any two vectors v, w at a point p in $C_{4/5}(0) \setminus C_{3/4}(0)$, we have

$$g(p,t)(v,w) = h(p)(v,w).$$

Parallel transporting the vectors along a geodesic (with respect to h) from p to 0, we would have

$$\frac{\partial}{\partial r} g(\gamma(r),t)(v(r),w(r)) = 0 = \frac{\partial}{\partial r} h(\gamma(r))(v(r),w(r)),$$

and hence

$$g(0,t)(v(r), w(r)) = h(0)(v(r), w(r)).$$

Since the vectors were arbitrary, we see $g(0,t) = h(0)$ for all $t \in [0, P)$; a contradiction.

By smoothness of the solution, there exists an $s_t > 0$ such that

$$\int_{C_1(0)} \left(|^h\nabla g(t)|^2 + |^h\nabla^2 g(t)|^2\right) dh \geq s_t > 0$$

for some $t > 0$. Using the isometry ψ_p, we see that this means

$$\int_{C_1(p)} \left(|^h\nabla g(t)|^2 + |^h\nabla^2 g(t)|^2\right) dh \geq s_t > 0$$

for all $p \in \mathbb{Z}^2$, and hence

$$\int_M \left(|^h\nabla g(t)|^2 + |^h\nabla^2 g(t)|^2\right) dh = \infty$$

for this t. To obtain an example in $\mathbb{R}^n$ with $n \in \mathbb{N}$, $n > 2$, we simply take

$$\widehat{g}_0 = \widehat{h} = g_0 \oplus \delta_{\mathbb{R}^k}$$

on $\mathbb{R}^{k+2} = \mathbb{R}^n$, where $\delta_{\mathbb{R}^k}$ is the standard metric on $\mathbb{R}^k$.

7. Ricci flow estimates

The results of the previous sections are in the setting of the Ricci–DeTurck flow. As mentioned at the beginning of the paper, in certain settings, for example the closed smooth setting, there is a Ricci flow related solution, which can be written as

$$\ell(t) := (\varphi_t)^* g(t),$$

where $\varphi_t \colon M \to M$, $t \in [0, T]$, or possibly only $t \in (0, T]$, is a smooth family of diffeomorphisms, and $g(t)_{t\in[0,T)}$ is a solution to Ricci–DeTurck flow. We say in this case, that the Ricci flow solution $\ell(t)$ *comes from* the Ricci–DeTurck solution $g(t)$, and we call $\ell(t)$ a *Ricci flow related solution* to $g(t)$.

In the next section we construct and analyse the behaviour of solutions $\ell(t)$ *coming from* a Ricci–DeTurck solution $g(t)$ constructed in the previous sections. In order to do this, we require various estimates for solutions to local Ricci flows. The statements and proofs thereof are contained in this section. The results of this section are written in a local setting assuming various geometric bounds, which we know will hold *if* the local Ricci flow solution comes from a Ricci–DeTurck flow solution constructed in the previous sections of this paper. However, it is not necessary to assume that the local Ricci flow solutions we consider are constructed in this manner.

Theorem 7.1. *For all $p \in [2, \infty)$, there exists $\alpha_0 = \alpha_0(p, n) > 0$ such that the following holds. Let Ω be a smooth n-dimensional manifold and $(\Omega^n, \ell(t))_{t \in (0,T]}$ be a smooth solution to Ricci flow satisfying*

$$\int_\Omega |\mathrm{Rc}(\ell(t))| d\ell(t) \le \varepsilon, \quad |\mathrm{Rc}(\ell(t))| \le \frac{\varepsilon}{t} \text{ on } \Omega \tag{7.1}$$

for all $t \in (0, T]$, where $\varepsilon \le \alpha_0$. Then there exist $\delta(n, p, \varepsilon), c(n, p) > 0$, with the property that $\delta(n, p, \varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\int_\Omega |\ell(t) - \ell(s)|^p_{\ell(t)} d\ell(t) \le \delta(n, p, \varepsilon)|t - s|, \tag{7.2}$$

$$\int_\Omega |(\ell(t))^{-1} - (\ell(s))^{-1}|^p_{\ell(t)} d\ell(t) \le \delta(n, p, \varepsilon)|t - s| \tag{7.3}$$

for all $t, s \in (0, T]$ with $s < t$. Furthermore,

$$\int_\Omega |\ell(s)|^p_{\ell(t)}\, d\ell(t) \le c(n, p)\big(\mathrm{Vol}(\Omega, \ell(t)) + |t - s|\big), \tag{7.4}$$

$$\int_\Omega |\ell^{-1}(s)|^p_{\ell(t)}\, d\ell(t) \le c(n, p)\big(\mathrm{Vol}(\Omega, \ell(t)) + |t - s|\big), \tag{7.5}$$

$$\int_\Omega |\ell(r) - \ell(s)|^p_{\ell(t)}\, d\ell(t) \le c(n, p)\big(\mathrm{Vol}(\Omega, \ell(t)) + t\big)^{3/4}|r - s|^{1/4}, \tag{7.6}$$

$$\int_\Omega |(\ell(r))^{-1} - (\ell(s))^{-1}|^p_{\ell(t)}\, d\ell(t) \le c(n, p)\big(\mathrm{Vol}(\Omega, \ell(t)) + t\big)^{3/4}|r - s|^{1/4} \tag{7.7}$$

for all $r, s, t \in (0, T]$ with $r, s < t$.

Remark 7.2. This theorem is true for *any smooth Ricci flow* satisfying (7.1). Completeness, compactness, volume bounds, Sobolev inequalities, are not assumed.

Proof. We write $v := \ell(s)$, and ℓ for $\ell(t)$, for $t \in (s, T]$, and Rc for $\mathrm{Rc}(\ell(t))$, and for

$$h(t) := \int_\Omega |\ell(t) - \ell(s)|^p_{\ell(t)}\, d\ell(t),$$

we calculate

$$\begin{aligned}\frac{\partial}{\partial t} h(t) &= \frac{\partial}{\partial t} \int_\Omega |\ell(t) - v|^p_{\ell(t)}\, d\ell(t) \\ &= \frac{\partial}{\partial t} \int_\Omega \big(\ell^{ik}\ell^{jm}(\ell_{ij} - v_{ij})(\ell_{km} - v_{km})\big)^{p/2}\, d\ell\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega -R(\ell)|\ell - v|_\ell^p \\
&\quad + \frac{p}{2}|\ell - v|_\ell^{p-2}\big(2\ell^{is}\ell^{kz}\mathrm{Rc}(\ell)_{sz}\ell^{jm}(\ell_{ij} - v_{ij})(\ell_{km} - v_{km}) \\
&\qquad + 2\ell^{ik}\ell^{js}\ell^{mz}\mathrm{Rc}(\ell)_{sz}(\ell_{ij} - v_{ij})(\ell_{km} - v_{km}) \\
&\qquad - 2\ell^{ik}\ell^{jm}\mathrm{Rc}(\ell)_{ij}(\ell_{km} - v_{km}) - 2\ell^{ik}\ell^{jm}(\ell_{ij} - v_{ij})\mathrm{Rc}_{km}(\ell)\big) \\
&\le c(n)p\int_\Omega \big(|\mathrm{Rc}|_\ell|\ell - v|_\ell^p + |\mathrm{Rc}|_\ell|\ell - v|_\ell^{p-1}\big)\,d\ell \\
&\le \frac{c(n)p\varepsilon}{t}h(t) + c(n)p\int_\Omega |\mathrm{Rc}|_\ell^{1/p}\big(|\mathrm{Rc}|_\ell^{(p-1)/p}|\ell - v|_\ell^{p-1}\big)\,d\ell \\
&\le \frac{c(n)p\varepsilon}{t}h(t) + c(n)p\int_\Omega \big(|\mathrm{Rc}|_\ell + |\mathrm{Rc}|_\ell|\ell - v|_\ell^p\big)\,d\ell \\
&\le \frac{c(n)p\varepsilon}{t}h(t) + c(n)p\varepsilon
\end{aligned}
$$

for some $c(n) \ge 1$ depending only on n. Hence, the function $f(t) = h(s + t)$ also satisfies

$$
\frac{\partial}{\partial t}f(t) \le \frac{\beta_0}{s + t}f(t) + \beta_0 \le \frac{\beta_0}{t}f(t) + \beta_0
$$

for all $t \in (0, T - s]$, where $\beta_0 := c(n)p\varepsilon$. The assumptions on ε guarantee that

$$
\beta_0 = c(n)p\varepsilon \le \frac{1}{2},
$$

and hence, using the ODE Lemma C.2, we see that $f(t) \le 2\beta_0 t$ for all $t \in (0, T - s]$. In particular, we obtain

$$
h(t) = f(t - s) \le 2\beta_0(t - s) = c(n)p\varepsilon(t - s)
$$

for all $t \in (s, T]$, that is (7.2) holds.

The estimate (7.3) is proved in an almost identical way. For

$$
y(t) := \int_\Omega |(\ell^{-1} - v^{-1})|_{\ell(t)}^p\,d\ell(t),
$$

we calculate

$$
\begin{aligned}
\frac{\partial}{\partial t}y(t) &= \frac{\partial}{\partial t}\int_\Omega |\ell^{-1} - v^{-1}|_\ell^p\,d\ell \\
&= \int_\Omega -R(\ell)|\ell^{-1} - v^{-1}|_\ell^p \\
&\quad + \frac{p}{2}|\ell^{-1} - v^{-1}|_\ell^{p-2}\big(-2\mathrm{Rc}(\ell)_{ik}\ell_{jm}(\ell^{ij} - v^{ij})(\ell^{km} - v^{km}) \\
&\qquad - 2\ell_{ik}\mathrm{Rc}(\ell)_{jm}(\ell^{ij} - v^{ij})(\ell^{km} - v^{km}) \\
&\qquad + 2\ell_{ik}\ell_{jm}\ell^{ip}\ell^{jq}\mathrm{Rc}(\ell)_{pq}(\ell^{km} - v^{km}) \\
&\qquad + 2\ell_{ik}\ell_{jm}(\ell^{ij} - v^{ij})\mathrm{Rc}(\ell)_{pq}\ell^{kp}\ell^{mq}\big)\,d\ell
\end{aligned}
$$

$$
\begin{aligned}
&\le c(n)p\Big(\int_\Omega |\mathrm{Rc}(\ell)|_\ell |\ell^{-1}-v^{-1}|_\ell^p + |\mathrm{Rc}(\ell)|_\ell |\ell^{-1}-v^{-1}|_\ell^{p-1}\, d\ell\Big)\\
&\le \frac{c(n)p\alpha_0}{t}\int_\Omega |\ell^{-1}-v^{-1}|_\ell^p\, d\ell + c(n)p\int_\Omega |\mathrm{Rc}(\ell)|_\ell |\ell^{-1}-v^{-1}|_\ell^{p-1}\, d\ell\\
&\le \frac{2c(n)p\alpha_0}{t}\ell(t) + c(n)p\alpha_0\\
&\le \frac{\beta_0 y(t)}{t} + \beta_0.
\end{aligned}
$$

The inequality (7.3) now follows as before from Lemma C.2. The inequalities (7.4), (7.5), (7.6) and (7.7) follow from the Hölder and triangle inequalities, as we now show. First, we show (7.4):

$$
\begin{aligned}
\int_\Omega |\ell(s)|_{\ell(t)}^p\, d\ell(t) &= \int_\Omega |\ell(s)-\ell(t)+\ell(t)|_{\ell(t)}^p\, d\ell(t)\\
&\le c(n,p)\int_\Omega |\ell(s)-\ell(t)|_{\ell(t)}^p\, d\ell(t) + c(n,p)\int_\Omega |\ell(t)|_{\ell(t)}^p\, d\ell(t)\\
&\le c(n,p)|t-s| + c(n,p)\,\mathrm{Vol}(\Omega,\ell(t)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_\Omega |\ell^{-1}(s)|_{\ell(t)}^p\, d\ell(t) = \int_\Omega |\ell^{-1}(s)-\ell^{-1}(t)+\ell^{-1}(t)|_{\ell(t)}^p\, d\ell(t)\\
&\quad\le c(n,p)\int_\Omega |\ell^{-1}(s)-\ell^{-1}(t)|_{\ell(t)}^p\, d\ell(t) + c(n,p)\int_\Omega |\ell^{-1}(t)|_{\ell(t)}^p\, d\ell(t)\\
&\quad\le c(n,p)|t-s| + c(n,p)\,\mathrm{Vol}(\Omega,\ell(t)).
\end{aligned}
$$

Thus, we see that (7.4) and (7.5) hold. To show (7.6) and (7.7) hold, we will use the estimates of Section D, which show that certain general inequalities hold which relate the L^p-norms of a tensor taken with respect to different metrics.

For any two tensor $T = T_{ij}$, using Corollary D.2, we have

$$
\begin{aligned}
\int_\Omega |T|_{\ell(t)}^p\, d\ell(t) &\le c(n,p)\Big(\int_\Omega |\ell(s)|_{\ell(t)}^{2p}\, d\ell(t)\Big)^{1/2}\\
&\qquad\times\Big(\int_\Omega |T|_{\ell(s)}^{4p}\, d\ell(s)\Big)^{1/4}\Big(\int_\Omega |\ell(t)|_{\ell(s)}^{n/2}\, d\ell(t)\Big)^{1/4}\\
&= c(n,p)\Big(\int_\Omega |\ell(s)|_{\ell(t)}^{2p}\, d\ell(t)\Big)^{1/2}\\
&\qquad\times\Big(\int_\Omega |T|_{\ell(s)}^{4p}\, d\ell(s)\Big)^{1/4}\Big(\int_\Omega |\ell^{-1}(s)|_{\ell(t)}^{n/2}\, d\ell(t)\Big)^{1/4}.
\end{aligned}
\tag{7.8}
$$

For $T = (\ell(r) - \ell(s))$ in (7.8), we obtain

$$\begin{aligned}\int_\Omega |\ell(r) - \ell(s)|^p_{\ell(t)} \, d\ell(t) &\le c(n,p)\Big(\int_\Omega |\ell(s)|^{2p}_{\ell(t)} \, d\ell(t)\Big)^{1/2} \\ &\quad \times \Big(\int_\Omega |\ell(s) - \ell(r)|^{4p}_{\ell(s)} \, d\ell(s)\Big)^{1/4}\Big(\int_\Omega |\ell^{-1}(s)|^{n/2}_{\ell(t)} \, d\ell(t)\Big)^{1/4} \\ &\le c(n,p)\big(\mathrm{Vol}(\Omega,\ell(t)) + t\big)^{3/4}|r-s|^{1/4}\end{aligned}$$

in view of the estimates (7.2), (7.4), and (7.5). For any two tensor $N = N^{ij}$, using Corollary D.2, we have

$$\begin{aligned}\int_\Omega |N|^p_{\ell(t)} \, d\ell(t) &\le c(n,p)\Big(\int_\Omega |\ell(t)|^{2p}_{\ell(s)} \, d\ell(t)\Big)^{1/2} \\ &\quad \times \Big(\int_\Omega |N|^{4p}_{\ell(s)} \, d\ell(s)\Big)^{1/4}\Big(\int_\Omega |\ell(t)|^{n/2}_{\ell(s)} \, d\ell(t)\Big)^{1/4} \\ &= c(n,p)\Big(\int_\Omega |\ell^{-1}(s)|^{2p}_{\ell(t)} \, d\ell(t)\Big)^{1/2} \\ &\quad \times \Big(\int_\Omega |N|^{4p}_{\ell(s)} \, d\ell(s)\Big)^{1/4}\Big(\int_\Omega |\ell^{-1}(s)|^{n/2}_{\ell(t)} \, d\ell(t)\Big)^{1/4}. \end{aligned} \tag{7.9}$$

For $N = (\ell^{-1}(r) - \ell^{-1}(s))$ in (7.9), we obtain

$$\begin{aligned}\int_\Omega |\ell^{-1}(r) - \ell^{-1}(s)|^p_{\ell(t)} \, d\ell(t) &\le c(n,p)\Big(\int_\Omega |\ell^{-1}(s)|^{2p}_{\ell(t)} \, d\ell(t)\Big)^{1/2} \\ &\quad \times \Big(\int_\Omega |\ell^{-1}(s) - \ell^{-1}(r)|^{4p}_{\ell(s)} \, d\ell(s)\Big)^{1/4}\Big(\int_\Omega |\ell^{-1}(s)|^{n/2}_{\ell(t)} \, d\ell(t)\Big)^{1/4} \\ &\le c(n,p)\big(\mathrm{Vol}(\Omega,\ell(t)) + t\big)^{3/4}|r-s|^{1/4}\end{aligned}$$

in view of the estimates (7.3), (7.4), and (7.5). That is, the inequalities (7.6) and (7.7) hold. ∎

The previous theorems show that, for $p \in [2,\infty)$ and $n \in \mathbb{N}$, a solution

$$(\Omega, \ell(t))_{t\in(0,T]}$$

which satisfies the conditions of the lemma, that is

$$\int_\Omega |\mathrm{Rc}(\ell(t))| \, d\ell(t) \le \varepsilon \quad \text{and} \quad |\mathrm{Rc}(\ell(t))| \le \frac{\varepsilon}{t} \text{ on } \Omega$$

for all $t \in (0,T]$, where $\varepsilon \le \alpha_0 = \alpha_0(n,p)$, must have a uniquely well-defined starting value $\ell_0 \in L^p(\Omega)$ which is a symmetric two tensor, whose inverse exists almost everywhere.

Corollary 7.3. *For all $p \in [2, \infty)$ and $n \in \mathbb{N}$, there exists $\alpha_0(n, p) > 0$ such that the following holds. Let Ω be a smooth n-dimensional manifold and $(\Omega^n, \ell(t))_{t\in(0,T]}$ be a smooth solution to Ricci flow satisfying*

$$\int_\Omega |\mathrm{Rc}(\ell(t))| d\ell(t) \le \varepsilon, \quad |\mathrm{Rc}(\ell(t))| \le \frac{\varepsilon}{t} \text{ on } \Omega$$

for all $t \in (0, T]$, where $\varepsilon \le \alpha_0$. Then there exists a unique two tensor $\ell_0 \in L^p$ such that $\ell(s) \to \ell_0$ in $L^p(\Omega)$ as $s \searrow 0$, where ℓ_0 is positive definite (except for a measure zero set), and $\ell^{-1}(s) \to (\ell_0)^{-1}$ in $L^p(\Omega)$ as $s \searrow 0$.

Proof. From (7.6) and (7.7), we see that $\ell(s)_{s\in(0,T]}$ and $\ell^{-1}(s)_{s\in(0,T]}$ are Cauchy with respect to s in $L^{2p}(\Omega, \ell(t))$ for fixed $t > 0$, if $\varepsilon \le \alpha_0(2p, n)$ is chosen small enough, where ℓ_0 is as in the statement of Theorem 7.1, so there exists $\ell_0, r_0 \in L^{2p}(\Omega, \ell(t))$ such that $\ell(s) \to \ell_0$ and $\ell^{-1}(s) \to r_0$ as $s \searrow 0$ in the L^{2p}-norm. Furthermore,

$$\delta^i_j = \ell^{ik}(s)\ell_{jk}(s),$$

and so for $\|\cdot\|_{L^q} = \|\cdot\|_{L^q(\Omega,\ell(t))}$ for some fixed $t > 0$, we have

$$\begin{aligned}
\|\delta^i{}_j - (\ell_0)_{jk} r_0^{ik}\|_{L^p} &= \|\ell^{ik}(s)\ell_{jk}(s) - (\ell_0)_{jk} r_0^{ik}\|_{L^p} \\
&= \|(\ell^{ik}(s) - r_0^{ik})\ell_{jk}(s) - r_0^{ik}((\ell_0)_{jk} - \ell_{jk}(s))\|_{L^p} \\
&\le \|(\ell^{ik}(s) - r_0^{ik})\ell_{jk}(s)\|_{L^p} + \|r_0^{ik}((\ell_0)_{jk} - \ell_{jk}(s))\|_{L^p} \\
&\le \|\ell^{ik}(s) - r_0^{ik}\|_{L^{2p}} \|\ell_{jk}(s)\|_{L^{2p}} + \|r_0^{ik}\|_{L^{2p}} \|(\ell_0)_{jk} - \ell_{jk}(s)\|_{L^{2p}} \\
&\to 0
\end{aligned}$$

as $s \searrow 0$ in view of (7.4), (7.5), (7.6), and (7.7). Hence, $r_0 = (\ell_0)^{-1}$ almost everywhere. At points x in the set of measure zero, where $\ell(0)(x)$ is degenerate, we replace $\ell(0)(x)$ by $\ell(t)(x)$ for a fixed $t > 0$. The convergence result still holds, but now $\ell(0)$ is positive definite everywhere. ∎

Theorem 7.4. *For any $A > 0$, there exist $\alpha_1, \beta, S > 0$ such that the following holds. Let $(M^4, \ell(t))_{t\in(0,T]}$ be a smooth four-dimensional solution to the Ricci flow, with $B_{\ell(t)}(x_0, 10) \Subset M$, $T \le 1$, satisfying a uniform Sobolev inequality for all $t \in (0, T]$,*

$$\left(\int_{B_{\ell(t)}(x_0,2)} |f|^4 \, d\ell(t)\right)^{1/2} \le A\left(\int_{B_{\ell(t)}(x_0,2)} |\nabla f|^2 \, d\ell(t) + \int_{B_{\ell(t)}(x_0,2)} |f|^2 \, d\ell(t)\right)$$

for any f compactly contained in $B_{\ell(t)}(x_0, 2)$ for any $t \in (0, T]$, where ∇ refers to the covariant derivative with respect to $\ell(t)$. We further assume

$$\begin{aligned}
\int_{B_{\ell(t)}(x_0,2)} |\mathrm{Rm}(\ell(t))|^2 \, d\ell(t) + \int_{B_{\ell(t)}(x_0,2)} |\mathrm{Rm}(\ell(t))| \, d\ell(t) \le \alpha_1, \\
|\mathrm{Rc}(\ell(t))| + |\nabla \mathrm{Rc}(\ell(t))|^{2/3} \le \frac{\alpha_1}{t} \quad \text{on } B_{\ell(s)}(x_0, 2)
\end{aligned} \tag{7.10}$$

for all $t, s \in (0, T]$. Then, we have

$$\int_{B_{\ell(t)}(x_0,1/2)} |\nabla(\ell(t) - \ell(s))|^2_{\ell(t)}\, d\ell(t) \le |t-s|^\beta \tag{7.11}$$

for all $t, s \in (0, T] \cap (0, S]$, with $s < t$, where ∇ refers to the covariant derivative with respect to $\ell(t)$.

Proof. We first prove that the space-time integral of $|\nabla \mathrm{Rc}|^2$ can be locally, uniformly bounded in the setting we are considering. This estimate shall in turn be used to prove the L^2 gradient estimate (7.11). In the following c refers to a universal constant independent of the solution. Let $\eta\colon M \to \mathbb{R}^+$ be a Perelman cut-off function, with $\eta(\cdot,t) = e^{-t}$ on $B_{\ell(t)}(x_0, 1)$ and $\eta(\cdot, t) = 0$ on $M - B_{\ell(t)}(x_0, \frac{5}{4})$, $\frac{\partial}{\partial t}\eta \le \Delta_{\ell(t)}\eta$, $|\nabla\eta|^2 \le c\eta$ with (see [33, Section 7] for details of the construction). Then, using the Sobolev inequality, Hölder's inequality, and the fact that

$$\int_{B_{\ell(t)}(x_0,2)} \eta^2(t)|\mathrm{Rm}|^2(t)\, d\ell(t) \le \alpha_1,$$

we see that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_M \eta^2(\cdot,t)|\mathrm{Rm}|^2(\cdot,t)\, d\ell(t)\\
&= \int_M \Big(|\mathrm{Rm}|^2\Big(\frac{\partial}{\partial t}\eta^2\Big) + \eta^2\frac{\partial}{\partial t}(|\mathrm{Rm}|^2) - \eta^2 \mathrm{R}|\mathrm{Rm}|^2\Big)\, d\ell(t)\\
&\le \int_M \big(|\mathrm{Rm}|^2\Delta(\eta^2) + \eta^2\Delta(|\mathrm{Rm}|^2) - 2\eta^2|\nabla\mathrm{Rm}|^2 + c\eta^2|\mathrm{Rm}|^3\big)\, d\ell(t)\\
&= \int_M \big(\eta\langle \mathrm{Rm} * \nabla\mathrm{Rm}, \nabla\eta\rangle_\ell - 2\eta^2|\nabla\mathrm{Rm}|^2 + c\eta^2|\mathrm{Rm}|^3\big)\, d\ell(t)\\
&\le \int_{B_{\ell(t)}(x_0,2)} \big(-\eta^2|\nabla\mathrm{Rm}|^2 + c|\mathrm{Rm}|^2 + c\eta^2|\mathrm{Rm}|^3\big)\, d\ell(t)\\
&= \int_{B_{\ell(t)}(x_0,2)} \Big(-\frac{1}{2}\eta^2|\nabla\mathrm{Rm}|^2 - \frac{1}{2}\big[|\nabla(\eta\mathrm{Rm})|^2 - |\nabla\eta|^2|\mathrm{Rm}|^2 - 2\eta\langle \mathrm{Rm}\nabla\eta, \nabla\mathrm{Rm}\rangle\big]\\
&\qquad + c|\mathrm{Rm}|^2 + c\eta^2|\mathrm{Rm}|^3\Big)\, d\ell(t)\\
&\le \alpha_1 c + \int_{B_{\ell(t)}(x_0,2)} \Big(-\frac{1}{4}|\nabla(\eta\mathrm{Rm})|^2 + c(\eta|\mathrm{Rm}|)^2|\mathrm{Rm}|\Big)\, d\ell(t)\\
&\le \alpha_1 c - \int_{B_{\ell(t)}(x_0,2)} \frac{1}{4}|\nabla(\eta\mathrm{Rm})|^2\, d\ell(t)\\
&\qquad + c\Big(\int_{B_{\ell(t)}(x_0,2)} |\eta\mathrm{Rm}|^4\, d\ell(t)\Big)^{1/2}\Big(\int_{B_{\ell(t)}(x_0,2)} |\mathrm{Rm}|^2\, d\ell(t)\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_1 c + \Big(cA\sqrt{\alpha_1} - \frac{1}{4}\Big) \int_{B_{\ell(t)}(x_0,2)} |\nabla(\eta \mathrm{Rm})|^2 \, d\ell(t) \\
&\le \alpha_1 c - \frac{1}{8} \int_{B_{\ell(t)}(x_0,2)} |\nabla(\eta \mathrm{Rm})|^2 \, d\ell(t)
\end{aligned}
$$

if α_1 is small enough. Hence, integrating from s to t, we see that

$$
\int_s^t \int_{B_{\ell(r)}(x_0,1)} |\nabla \mathrm{Rm}|^2(r) \, dx \, dr \le c \int_M \eta^2 |\mathrm{Rm}|^2(\cdot, s) + \alpha_1 ct \le \alpha_1 c \tag{7.12}
$$

with $\alpha_1 c \le 1$, without loss of generality. We now turn to the proof of the integral gradient estimate, (7.11). This is similar to the L^p-estimate obtained for $\ell(t)$ in (7.1), but uses the space-time L^2 bound on the gradient of the Ricci curvature (7.12) that we just derived, instead of the bound on the Riemannian curvature. In the following, $|\cdot|$ refers to $|\cdot|_{\ell(t)}$, and Rc to $\mathrm{Rc}(\ell(t))$. Defining $\Omega := B_{\ell(t_0)}(x_0, 1/2)$, we see that $\Omega \subseteq B_{\ell(r)}(x_0, 1)$ for all $r \in (0, t_0)$ if $t_0 \le 1$ in view of [33, Corollary 3.3] and the fact that the condition (7.10) holds. Differentiating the function

$$
f(t) := \int_\Omega |\nabla(\ell(t) - \ell(s))|^2_{\ell(t)} d\ell(t),
$$

for $s < t \le t_0$, and using Young's inequality, we get

$$
\begin{aligned}
&\frac{\partial}{\partial t} f(t) = \frac{\partial}{\partial t} \int_\Omega |\nabla(\ell(t) - \ell(s))|^2_{\ell(t)} \, d\ell(t) \\
&\le \int_\Omega \big(c|\mathrm{Rc}||\nabla(\ell(t) - \ell(s))|^2 + 2\langle \nabla \mathrm{Rc}, \nabla(\ell(t) - \ell(s))\rangle_{\ell(t)}\big) \, d\ell(t) \\
&\qquad + c \int_\Omega |\ell(t) - \ell(s)||\nabla \mathrm{Rc}||\nabla(\ell(t) - \ell(s))| \, d\ell(t) \\
&\le \frac{c\alpha_1}{t} f(t) + c \int_\Omega |\nabla \mathrm{Rc}||\nabla(\ell(t) - \ell(s))| \, d\ell(t) \\
&\qquad + c \int_\Omega |\ell(t) - \ell(s)||\nabla \mathrm{Rc}||\nabla(\ell(t) - \ell(s))| \, d\ell(t) \\
&\le \frac{c\alpha_1}{t} f(t) + \frac{c\alpha_1}{t-s} f(t) + \frac{c}{\alpha_1}(t-s) \int_\Omega |\nabla \mathrm{Rc}|^2 \, d\ell(t) \\
&\qquad + c\Big(\int_\Omega |\ell(t) - \ell(s)|^2 |\nabla \mathrm{Rc}|^{4/3} \, d\ell(t)\Big)^{1/2} (\alpha_1 t^{-1} f(t))^{1/2} \\
&\le \frac{c\alpha_1}{t-s} f(t) + \frac{c}{\alpha_1}(t-s) \int_\Omega |\nabla \mathrm{Rc}|^2 \, d\ell(t) + c \int_\Omega |\ell(t) - \ell(s)|^2 |\nabla \mathrm{Rc}|^{4/3} \, d\ell(t) \\
&\le \frac{c\alpha_1}{t-s} f(t) + \frac{c}{\alpha_1}(t-s) \int_\Omega |\nabla \mathrm{Rc}|^2 \, d\ell(t) \\
&\qquad + c\Big(\int_\Omega |\ell(t) - \ell(s)|^6 \, d\ell(t)\Big)^{1/3} \Big(\int |\nabla \mathrm{Rc}|^2 d\ell(t)\Big)^{2/3}
\end{aligned}
$$

$$\leq \frac{c\alpha_1}{t-s} f(t) + \frac{c}{\alpha_1}(t-s)\int_\Omega |\nabla \mathrm{Rc}|^2\, d\ell(t) + c(t-s)^{1/3}\Big(\int |\nabla \mathrm{Rc}|^2\, d\ell(t)\Big)^{2/3}$$
$$\leq \frac{c\alpha_1}{t-s} f(t) + \frac{c}{\alpha_1}(t-s)^{1/3}\Big(\int_\Omega |\nabla \mathrm{Rc}|^2\, d\ell(t) + 1\Big),$$

since

$$\int_\Omega |\ell(t) - \ell(s)|^6 \leq c(t-s)$$

for sufficiently small α_1, in view of Theorem 7.1, and $(t-s) \leq (t-s)^{1/3}$, for $t < s \leq T \leq 1$. Hence,

$$\frac{\partial}{\partial t} f(t) \leq \frac{\alpha_2}{t-s} f(t) + Z(t)$$

for $\alpha_2 := c\alpha_1$, and

$$Z(t) := c\alpha_2^{-1}(t-s)^{1/3}\Big(\int_\Omega |\nabla \mathrm{Rc}(t)|^2\, d\ell(t) + 1\Big)$$

for $t \leq S(n, \alpha_1) \leq 1$. Hence, $F(t) := f(t+s)$ for $t \in (0, S-s)$ then satisfies

$$\frac{\partial}{\partial t} F(t) \leq \frac{\alpha_2}{t} F(t) + \tilde{Z}(t),$$

where

$$\tilde{Z}(r) = c\alpha_2^{-1} r^{1/3}\Big(\int_\Omega |\nabla \mathrm{Rc}(\ell(s+r))|^2\, d\ell(s+r) + 1\Big),$$

for $r \in (0, T-s)$. Thus, for $\alpha_2 \leq 1/6$, we obtain

$$\begin{aligned}
F(t) &\leq t^{\alpha_2}\int_0^t \frac{\tilde{Z}(r)}{r^{\alpha_2}}\, dr \\
&= c\alpha_2^{-1} t^{\alpha_2}\int_0^t r^{1/3-\alpha_2}\Big(\int_\Omega |\nabla \mathrm{Rc}(\ell(r+s))|^2\, d\ell(r+s) + 1\Big)\, dr \\
&\leq c\alpha_2^{-1} t^{\alpha_2}\int_0^t \Big(\int_\Omega |\nabla \mathrm{Rc}(\ell(r+s))|^2\, d\ell(r+s)\, dr + 1\Big) \\
&= c\alpha_2^{-1} t^{\alpha_2}\Big(\int_s^{s+t}\int_\Omega |\nabla \mathrm{Rc}(\ell(r))|^2\, d\ell(r)\, dr + 1\Big) \\
&\leq c t^{\alpha_2}\Big(\alpha_2^{-1}\int_s^{s+t}\int_{B_{\ell(r)}(x_0,1)} |\nabla \mathrm{Rc}(\ell(r))|^2\, d\ell(r)\, dr + 1\Big) \\
&\leq c t^{\alpha_2}
\end{aligned}$$

for $t \in (0, S-s)$ in view of Lemma C.1, and the fact that inequality (7.12) holds. That is,

$$f(t) \leq c(t-s)^{\alpha_2}$$

for $t \in (s, S)$. By choosing $\beta = \alpha_2/2$, we obtain

$$f(t) = \int_{\Omega} |\nabla(\ell(t) - \ell(s))|^2_{\ell(t)} \, d\ell(t) \leq (t-s)^{\beta}$$

for $t \in (s, S)$, as required. ∎

8. The Ricci flow related solution

In the sections before Section 7, we constructed a solution $g(t)_{t\in(0,T]}$ to the Ricci–DeTurck flow coming out of $W^{2,2}$ initial data g_0 on a four-dimensional manifold, and we proved estimates for such solutions. In this section we construct a Ricci flow related solution $\ell(t)_{t\in(0,T]}$ coming from the Ricci–DeTurck flow solution constructed in the sections preceding Section 7. We show in the setting we are considering, that the Ricci flow related solution $\ell(t)$ converges back to some starting value ℓ_0 locally in the $W^{1,2}$-norm, as $t \to 0$. We shall see that the tensor ℓ_0 is non-negative definite, up to a set of measure zero. Note that since g_0 and ℓ_0 are only defined up to a measure zero, we can arbitrarily change distance induced by g_0 respectively by ℓ_0 by changing g_0, respectively ℓ_0, on a set of measure zero, if we try and use the usual definition of distance with respect to a Riemannian metric, as the following example shows.

Example 8.1. Let g, h be smooth Riemannian metrics on $M = \mathbb{B}_1(0) \subseteq \mathbb{R}^n$, $r > 0$ small so that $B_h(0, r) \Subset \mathbb{B}_1(0)$ and $x \neq y, x, y \in B_h(0, r/4)$, and let

$$\gamma \colon [0,1] \to \mathbb{B}_1(0)$$

be a smooth length minimising geodesic with respect to h from x to y. We define a new metric $\tilde{g}$, which is the same as g except on the line γ. On γ we define $\tilde{g}(\gamma(s)) = b^2 h(\gamma(s))$ for all $s \in [0,1]$, for some $b \in \mathbb{R}, b > 0$.

The metric $\tilde{g}$ is still a well-defined Riemannian metric, with

$$\frac{1}{N}\delta \leq \tilde{g} \leq N\delta$$

for some $N > 0$, $N \in \mathbb{R}$, in view of the smoothness of g and the definition of $\tilde{g}$. This ensures then that $\tilde{g}_{ij}$ is a Borel-measurable function, since g_{ij} is smooth and $g = \tilde{g}$ almost everywhere. Using the fact that $\frac{1}{N}\delta \leq \tilde{g} \leq N\delta$ for some $N > 0$, we see then that $\tilde{g}$ is in L^p for any $p \in (0, \infty]$.

We also have, for any piecewise smooth $\sigma \colon [0,1] \to \mathbb{B}_1(0)$ with $\sigma(0) \neq \sigma(1)$ that $\tilde{g}_{ij} \circ \sigma \colon I \to \mathbb{R}$ is Borel measurable, since both $\tilde{g}_{ij}$ and σ are, and

$$\frac{1}{N^2}\delta_{ij} \leq \tilde{g}_{ij} \circ \sigma \leq N^2 \delta_{ij},$$

since this is true for $\widetilde{g}_{ij}$. This means ℓ: $[0, 1] \to \mathbb{R}$,

$$\ell(s) := \sqrt{\widetilde{g}_{ij}(\sigma(s))\partial_s \sigma^i(s)\partial_s \sigma^j(s)}$$

is a well-defined L^1-function, and

$$L_{\widetilde{g}}(\sigma) := \int_0^1 \sqrt{\widetilde{g}_{ij}(\sigma(s))\partial_s \sigma^i(s)\partial_s \sigma^j(s)}\, ds$$

satisfies

$$0 < \frac{1}{N} L_\delta(\sigma) \leq L_{\widetilde{g}}(\sigma) < N L_\delta(\sigma) < \infty$$

for all such σ.

If we define

$$d(\widetilde{g})(p, q) := \inf_{\sigma \in B_{p,q}} L_{\widetilde{g}}(\sigma)$$

for all $p, q \in M$, where $B_{p,q}$ refers to the space of continuous, piecewise smooth curves between p and q in $M = \mathbb{B}_1(0)$, then we see that $(\mathbb{B}_1(0), d(\widetilde{g}))$ is a well-defined metric space, that is, $d(\widetilde{g})$ is symmetric, satisfies the triangle inequality, and $d(\widetilde{g})(p, q) \geq 0$ for all $p, q \in M$ with equality if and only if $p = q$. Furthermore,

$$d(\widetilde{g})(x, y) \leq L_{\widetilde{g}}(\gamma)(x, y) = L_{bh}(x, y) = bd(h)(x, y) < d(g)(x, y)$$

if $b > 0$ is chosen small enough, and hence

$$d(\widetilde{g})(x, y) < d(g)(x, y)$$

if $b > 0$ is chosen small enough. That is, *if we use the usual definition for distance with respect to a Riemannian metric, distance can change if we change the Riemannian metric on a set of measure zero.*

In particular, this example shows that we cannot be sure that

$$d(g(t))(x, y) \to d(g_0)(x, y)$$

everywhere, as $t \searrow 0$, in the case that we have a family of smooth metrics $g(t)$ which convergences in the L^1-sense (or another weak sense) to a $g_0 \in L^1$, if we define $d(g_0)$ in the usual way,

$$d(g_0)(x, y) = \inf_{\sigma \in B_{x,y}} L_{g_0}(\gamma),$$

where $B_{x,y}$ is the set of smooth curves going from x to y. If g_0 is bounded from above and below by a smooth metric, we can change g_0 on a smooth curve between two given points x and y (as in the example above), so that $d(g(t))(x, y)$ does not converge to $d(g_0)(x, y)$, but we still have $g(t) \to g_0$ in L^1 as $t \searrow 0$.

Nevertheless, we will see for solutions $g(t)$ to the Ricci–DeTurck flow constructed in the previous sections, that $d(g(t))(x,y)$ does converge to some metric $d_0(x,y)$ as $t \searrow 0$, where d_0 is defined in a similar fashion to the usual definition of $d(g_0)$. However, it is necessary to restrict further the class of admissible curves $B_{x,y}$ between x and y to the class $C_{\varepsilon,x,y}$ of so-called *ε-approximative Lebesgue curves* between x and y, and then to take a limit inferior as $\varepsilon \to 0$ of the lengths.

Definition 8.2. We give two definitions.

(i) For $p \in [1,\infty)$, we say g is an L^p-metric, if the following holds. Let g be a Riemannian metric, such that $g(x): T_xM \times T_xM \to \mathbb{R}$ is defined, symmetric, positive-definite for all $x \in M$, and locally, writing

$$\tilde{g}_{ij}(\tilde{x}) := g(x)\Big(\frac{\partial}{\partial_i}(x), \frac{\partial}{\partial_j}(x)\Big)$$

for any smooth coordinates

$$\varphi: U \to \varphi(U) = \tilde{U} \subseteq \mathbb{R}^n, \quad \tilde{g}_{vv}: \tilde{U} \to \mathbb{R}$$

is in $L^p(\tilde{U})$ for all $v \in \mathbb{R}^n$, where v is any fixed length one (with respect to δ) vector in $\mathbb{R}^n$, and $\tilde{g}_{vv}(x) := \tilde{g}_{ij}(x)v^iv^j$.

(ii) For $x, y \in M$, we define the set $C_{\varepsilon,x,y}(g)$ of *ε-approximative Lebesgue curves* with respect to g from x to y in M to be the set of paths $\gamma: [a,b] \to M$, which can be written as the union of finitely many so-called *parametrised Lebesgue lines*

$$\gamma_i: [a_{i-1}, a_i] \to M,$$

where $i \in \{1,\dots,N\}$, $a_0 = a$, $a_N = b$, $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \dots \cup \gamma_N$. That is,

$$\gamma(s) := \gamma_i(s)$$

if $s \in [a_{i-1}, a_i]$, and

$$\begin{aligned} d_h(x,\gamma_1(a)) &+ d_h(\gamma_1(a_1),\gamma_2(a_1)) + d_h(\gamma_2(a_2),\gamma_3(a_2)) \\ &+ \dots + d_h(\gamma_{N-1}(a_{N-1}),\gamma_N(a_{N-1})) + d_h(\gamma_N(b),y) \le \varepsilon, \end{aligned}$$

and a *parametrised Lebesgue line* is defined as follows: A *parametrised Lebesgue line* for an L^1-metric g on M, is a smooth curve $\ell: [b,c] \to M$ with $|b-c| \le 1/4$, such that there exist smooth coordinates $\varphi: \Omega \to \varphi(\Omega) = \mathbb{B}_1(0)$ for some $p \in M$, such that $\ell([b,c]) \subseteq \Omega$ and the curve $\sigma := \varphi \circ \ell: [b,c] \to \mathbb{B}_1(0)$ in these coordinates is a line in the direction e_1, with speed one,

$$\sigma(s) = -\frac{(b+c)}{2}e_1 + se_1, \quad \sigma(b) = -k, \quad \sigma(c) = k,$$

where $k = (c-b)/2$, and for $f(s) := \sqrt{\tilde{g}_{11}(\sigma(s))}$, we have

$$f \in L^1\big([b,c]\big) \quad \text{and} \quad \int_b^c f(s)\,ds = \lim_{\alpha\to 0} \frac{\int_{T_\alpha(\sigma)} \sqrt{\tilde{g}_{11}(x)}\,dx}{\omega_{n-1}\alpha^{n-1}},$$

where here $T_\alpha(\sigma)$ refers to an α-tubular neighbourhood of σ with respect to δ,

$$T_\alpha(\sigma) = \left\{ se_1 + \alpha v \mid |v| = 1, \langle v, e_1\rangle = 0, s \in \left(-\frac{(c-b)}{2}, \frac{(c-b)}{2}\right)\right\}.$$

Note that if g is smooth, then

$$\int_b^c f(s)\,ds = \lim_{\alpha\to 0} \frac{\int_{T_\alpha(\sigma)} \sqrt{\tilde{g}_{11}(x)}\,dx}{\omega_{n-1}\alpha^{n-1}}$$

always holds. Also, an ε-*approximative Lebesgue curve* γ is the union of finitely smooth curves, but may itself be discontinuous, and hence non-smooth.

In the setting that a Riemannian metric is L^∞ (or weaker), there are various notions of distance and convergence of distance which may be defined – and there are many papers in this area investigating the properties thereof – their relation to one another and to the underlying measures. For one overview, as well as independent results and proofs thereof, we refer to the paper [5]. Further notions and convergence results may be found in the papers [1, 2, 6, 12, 18, 22], as well as the works cited in these papers. Earlier works can be found in [7]. In our setting it is sufficient to define distance by considering the class of ε-*approximative Lebesgue curves* instead of the class of piecewise smooth curves or continuous curves, and then to take the lim inf as $\varepsilon \to 0$ of the lengths; see Theorem 8.3 (iv) below.

Theorem 8.3. *Let (M,h) be a smooth four-dimensional Riemannian manifold satisfying (2.4), $1 < a < \infty$ and g_0 satisfy the assumptions of Theorem 6.3, and let $(M, g(t))_{t\in(0,T]}$, $T \le 1$ be the smooth solution to (1.1) appearing in the conclusions of Theorem 6.3. Then*

(i) *there exists a constant $c(a)$ and a smooth solution $\Phi\colon M \times (0,T] \to M$ to (1.3) with $\Phi(T/2) = \mathrm{Id}$ such that*

$$\Phi(t) := \Phi(\cdot, t)\colon M \to M$$

is a diffeomorphism, and

$$d_h(\Phi(t)(x), \Phi(s)(x)) \le c(a,n)\sqrt{|t-s|}$$

for all $x \in M$. The metrics $\ell(t) := (\Phi(t))^* g(t)$, $t \in (0, T]$ solve the Ricci flow equation. Furthermore, there are well-defined limit maps

$$\Phi(0)\colon M \to M, \quad \Phi(0) := \lim_{t \searrow 0} \Phi(t),$$
$$W(0)\colon M \to M, \quad W(0) := \lim_{t \searrow 0} W(t),$$

where $W(t)$ is the inverse of $\Phi(t)$ and these limits are obtained uniformly on compact subsets, and $\Phi(0)$, $W(0)$ are homeomorphisms inverse to another.

(ii) *For the Ricci flow solution $\ell(t)$ from* (i)*, there is a value $\ell_0(\cdot) = \lim_{t \searrow 0} \ell(\cdot, t)$ well defined up to a set of measure zero, where the limit exists in the L^p-sense, for any $p \in [1, \infty)$, such that, ℓ_0 is positive definite, and in $W^{1,2}_{\mathrm{loc}}$ and for any $y_0 \in M$ and $0 < s < t$, we have*

$$\begin{aligned}
&\int_{B_1(y_0)} |\ell(s) - \ell_0|^p_{\ell(t)}\, d\ell(t) \le c(g_0, h, p, y_0)s \\
&\int_{B_1(y_0)} |(\ell(0))^{-1} - (\ell(s))^{-1}|^p_{\ell(t)}\, d\ell(t) \le c(g_0, h, p, y_0)|s|^{1/4} \\
&\int_{B_1(y_0))} |\nabla \ell_0|^2_{\ell(t)}\, d\ell(t) \le c(g_0, h, p, y_0)t^{\sigma} \\
&\int_{B_1(y_0)} |\mathrm{Rm}(\ell)|^2(x,t)\, d\ell(x,t) + \int_0^t \int_{B_{\ell(s)}(y_0,1)} |\nabla \mathrm{Rm}(\ell)|^2(x,s)\, d\ell(x,s)\, ds \\
&\qquad \le c(g_0, h, p, y_0) \\
&\sup_{B_1(y_0)} |\nabla^j \mathrm{Rc}(\ell(t))|^2 t^{j+2} \to 0 \quad \text{as } t \searrow 0 \text{ for all } j \in \mathbb{N}_0,
\end{aligned}$$

where $\sigma > 0$ is a universal constant, $c(g_0, h, p, y_0)$ is a constant depending upon g_0, h, p, y_0, but not on t, and ∇ refers to the gradient with respect to $\ell(t)$.

(iii) *The limit maps*

$$\Phi(0)\colon M \to M, \quad \Phi(0) := \lim_{t \searrow 0} \Phi(t),$$
$$W(0)\colon M \to M, \quad W(0) := \lim_{t \searrow 0} W(t)$$

from (i) *are also obtained in the $W^{1,p}_{\mathrm{loc}}$-sense for $p \in [1, \infty)$. Furthermore, for any smooth coordinates $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$, and open sets $\tilde{U} \subset\subset U$ and $\tilde{V} \subset\subset V$ with $W(s)(\tilde{V}) \subset\subset U$ and $\Phi(s)(\tilde{U}) \subset\subset V$ for all $s \in [0, \hat{T}]$, for some $0 < \hat{T} < T$, the functions*

$$(\ell_0)_{ij} \circ W(0)\colon \tilde{V} \to \mathbb{R}$$

are in L^p_{loc} for all $p \in [1, \infty)$, and

$$(g_0)_{\alpha\beta}\colon \tilde{V} \to \mathbb{R} \quad \textit{and} \quad (\ell_0)_{ij}\colon W(0)(\tilde{V}) \to \mathbb{R}$$

are related by the identity

$$(g_0)_{\alpha\beta} = D_\alpha(W(0))^i D_\beta(W(0))^j ((\ell_0)_{ij} \circ W(0))$$

on $\widetilde{V}$. *In particular,* ℓ_0 *is isometric to* g_0 *almost everywhere through the map* $W(0)$ *which is in* $W^{1,p}_{\mathrm{loc}}$, *for all* $p \in [1, \infty)$.

(iv) *We define* $d_t = d(g(t))$ *and* $\widetilde{d}_t = d(\ell(t)) = \Phi(t)^* d_t$. *There are well-defined limit metrics*

$$d_0, \widetilde{d}_0 \colon M \times M \to \mathbb{R}_0^+, \quad d_0(x, y) = \lim_{t \searrow 0} d_t(x, y),$$
$$\widetilde{d}_0 := M \times M \to \mathbb{R}_0^+, \quad \widetilde{d}_0(p, q) = \lim_{t \searrow 0} \widetilde{d}_t(p, q),$$

and they satisfy

$$\widetilde{d}_0(x, y) = d_0(\Phi(0)(x), \Phi(0)(y)).$$

That is, $(M, \widetilde{d}_0)$ *and* (M, d_0) *are isometric to one another through the map* $\Phi(0)$.

The metric d_0 *satisfies*

$$d_0(x, y) := \liminf_{\varepsilon \searrow 0} \inf_{\gamma \in C_{\varepsilon,x,y}} L_{g_0}(\gamma),$$

where $C_{\varepsilon,x,y}$ *is the space of Lebesgue curves between* x *and* y *with respect to* g_0.

Proof. (i) For $r \in (0, T)$, we define $\psi_r \colon M \times (0, T] \to M$ to be the solution to

$$\frac{\partial}{\partial t}\psi_r(y, t) = V(\psi_r(y, t), t), \quad \psi_r(y, r) = y,$$

where

$$V(y, t) := g^{ij}\big(\Gamma^k_{ij}(g) - \Gamma(h)^k_{ij}\big)(y, t)\frac{\partial}{\partial x^k}(y) \quad \forall y \in M, t \in (0, T).$$

The fundamental theorem of time dependent flows (see [21, Theorem 9.4.8]) tells us that the $\psi_r(\cdot, s) \colon M \to M$ are smooth diffeomorphisms for all $r, s \in (0, T]$, and that $\psi_{t_1}(\psi_{t_0}(p, t_1), s) = \psi_{t_0}(p, s)$ for all $t_0, t_1, s \in (0, T]$, and in particular, that

$$\psi_{t_1}(\psi_{t_0}(\cdot, t_1), t_0) = \mathrm{Id}(\cdot)$$

for all $t_0, t_1 \in (0, T]$. We shall use these facts freely in the following.

The maps $\Phi(s), W(s) \colon M \to M$ are defined by

$$\Phi(s)(x) := \psi_{T/2}(x, s) \quad \text{and} \quad W(s)(x) := \psi_s(x, T/2)$$

for $s \in (0, T]$, and $\Phi, W \colon M \times (0, T] \to M$ are defined by

$$\Phi(x, s) = \Phi(s)(x) \quad \text{and} \quad W(x, s) = W(s)(x)$$

for $s \in (0, T]$. From the fact that $\psi_{t_1}(\psi_{t_0}(\cdot, t_1), t_0) = \mathrm{Id}(\cdot)$, we have

$$W(s) \circ \Phi(s)(\cdot) = \psi_s(\Phi(s)(\cdot), T/2) = \psi_s(\psi_{T/2}(\cdot, s), T/2) = \mathrm{Id}(\cdot)$$

for all $s \in (0, T]$. Defining

$$\ell(s) := (\Phi(s))^* g(s)$$

for $s \in (0, T]$, we obtain a smooth solution to Ricci flow with $\ell(T/2) = g(T/2)$.

We define $\Phi(0)$ by

$$\Phi(0)(x) := \lim_{s_i \searrow 0} \Phi(s_i)(x) = \lim_{s_i \searrow 0} \psi_{T/2}(x, s_i),$$

and $W(0)\colon M \to M$ by

$$W(0)(x) := \lim_{t_i \searrow 0} W(t_i)(x) = \lim_{t_i \searrow 0} \psi_{t_i}(x, T/2).$$

In the following we show that these limits exist, and are independent of the sequences $t_i \searrow 0$ and $s_i \searrow 0$ chosen. We have

$$\Big| \frac{\partial}{\partial t} \psi_{T/2}(x, t) \Big|_h = |V(\psi_{T/2}(x, t), t)|_h \leq \frac{\varepsilon}{\sqrt{t}},$$

due to the fact that $|{}^h\nabla g|^2 \leq \varepsilon/t$. Hence,

$$\begin{aligned} d_h(\Phi(s)(x), \Phi(t)(x)) &= d_h(\psi_{T/2}(x, s), \psi_{T/2}(x, t)) \\ &\leq 2\varepsilon|\sqrt{t} - \sqrt{s}| \leq 2\varepsilon\sqrt{|t - s|} \end{aligned}$$

for all $t, s \in (0, T]$, which shows that $\Phi(0)\colon M \to M$ is obtained uniformly and is well defined:

$$d_h(\Phi(0)(x), \psi_{T/2}(x, t)) = d_h(\Phi(0)(x), \Phi(t)(x)) \leq c\sqrt{t}$$

for all $x \in M$.

We now turn to the construction and properties of W. We can estimate

$$d_h(\psi_{t_i}(x, s), x) \leq \varepsilon\sqrt{s}$$

for all $s \in (t_i, T/2]$ and for all $x \in M$, in view of the fact that $|\frac{\partial}{\partial s}(\psi_{t_i}(x, s))|_h \leq \varepsilon/\sqrt{s}$. In particular, writing everything with respect to fixed coordinates φ, where

$$\frac{99}{100}\delta \leq h \leq \frac{101}{100}\delta \quad \text{and} \quad |Dh|^2 + |D^2h|^2 \leq \varepsilon$$

on a large ball of radius 1000 and centre point x, we have $\psi_{t_i}(x,s), \psi_{t_j}(x,s)$ stays in this ball. We write $x, h, \dots$ for $\varphi(x), \varphi_*(h), \dots$ For $s \geq s_0 := \max(t_j, t_i)$, we have

$$\begin{aligned}|\psi_{t_i}(x,s) - \psi_{t_j}(x,s)| &\leq |\psi_{t_i}(x,s) - x| + |x - \psi_{t_j}(x,s)| \\ &\leq 4\varepsilon\sqrt{s}.\end{aligned}$$

We also have for such s,

$$\begin{aligned}&\frac{\partial}{\partial s}|\psi_{t_i}(x,s) - \psi_{t_j}(x,s)|^2 \\ &\quad = |\psi_{t_i}(x,s) - \psi_{t_j}(x,s)||V(\psi_{t_i}(x,s),s) - V(\psi_{t_j}(x,s),s)| \\ &\quad \leq 2|DV(m,s)||\psi_{t_i}(x,s) - \psi_{t_j}(x,s)|^2 \\ &\quad \leq c(a) \sup_{y \in \mathbb{B}_{1000}(x)} \left(|^h\nabla^2 g| + |^h\nabla g|^2 + \varepsilon\right)|\psi_{t_i}(x,s) - \psi_{t_j}(x,s)|^2 \\ &\quad \leq \frac{\varepsilon}{s}|\psi_{t_i}(x,s) - \psi_{t_j}(x,s)|^2,\end{aligned}$$

where m is some value lying on the line between $\psi_{t_i}(x,s)$ and $\psi_{t_j}(x,s)$ in the euclidean ball $\mathbb{B}_{1000}(x)$. Here we used $|^h\nabla g^j|^2 t^j \leq \varepsilon$ when t is sufficiently small. Hence, writing $f(s) := |\psi_{t_i}(x,s) - \psi_{t_j}(x,s)|^2$ we have

$$\frac{\partial}{\partial s} f(s) \leq \frac{\varepsilon}{s} f(s)$$

for $s \geq \max(t_i, t_j) = s_0$, which implies $\frac{\partial}{\partial s}(s^{-\varepsilon} f(s)) \leq 0$, and hence

$$f(s) \leq s^{\varepsilon}((s_0)^{-\varepsilon} f(s_0))$$

for all $s \in [s_0 = \max(t_i, t_j), T/2]$. But $f^2(s_0) = |\psi_{t_i}(x,s_0) - \psi_{t_j}(x,s_0)|^2 \leq 16\varepsilon^2 s_0$ from the above estimate, and so we get

$$(s_0)^{-\varepsilon} f(s_0) \leq (s_0)^{-\varepsilon} 2\varepsilon s_0 \leq 2\varepsilon(s_0)^{1-\varepsilon} = 2\varepsilon(\max(t_i, t_j))^{1-\varepsilon} \to 0$$

as $\max(t_j, t_i) \to 0$, and hence

$$\begin{aligned}f(s) &= |\psi_{t_i}(x,s) - \psi_{t_j}(x,s)|^2 \\ &\leq 2s^{\varepsilon}\varepsilon(\max(t_i, t_j))^{1-\varepsilon} \leq 2T^{\varepsilon}\varepsilon(\max(t_i, t_j))^{1-\varepsilon} \to 0 \qquad (8.1)\end{aligned}$$

as $t_i, t_j \to 0$, for all $s \in (\max(t_i, t_j), T/2]$, for all $x \in M$. This shows, $(\psi_{t_i}(x,s))_{i \in \mathbb{N}}$ with $t_i \searrow 0$ is Cauchy, and hence $\lim_{t_i \searrow 0} \psi_{t_i}(x,s)$ exists for all $s \in (0, T]$. In particular,

$$W(0)(x) := \lim_{t \searrow 0} \psi_t(x, T/2) = \lim_{t \searrow 0} W(t)(x)$$

is well defined, and achieved uniformly,

$$\begin{aligned} d_h(W(0)(x), W(t)(x)) &= \lim_{s\to 0} d_h(\psi_s(x, T/2), \psi_t(x, T/2)) \\ &\le \sqrt{2}T^{\varepsilon/2}\varepsilon(t)^{(1-\varepsilon)/2} \to 0 \end{aligned}$$

for $t \searrow 0$, in view of (8.1).

We show now that $\Phi(0)$ is the inverse of $W(0)$. $\Phi(0)$ and $W(0)$ are continuous, by construction, and are the uniform limits of continuous functions,

$$\sup_{z\in M} d_h(\Phi(0)(z), \psi_{T/2}(z, t_i)) \to 0 \quad \text{as } i \to \infty$$

and

$$\sup_{x\in M} d_h(W(0)(x), \psi_{t_i}(x, T/2)) \to 0 \quad \text{as } i \to \infty.$$

For $x \in M$, for any $\sigma > 0$ if i is large enough, we have

$$d_h\big(\Phi(0)(W(0)(x)), \Phi(0)(\psi_{t_i}(x, T/2))\big) \le \sigma,$$

and

$$d_h(\Phi(0)(z), \psi_{T/2}(z, t_i)) \le \sigma$$

for all $z \in M$. This implies

$$\begin{aligned} &d_h\big(\Phi(0)(W(0)(x)), x\big) \\ &\quad\le d_h\big(\Phi(0)(W(0)(x)), \Phi(0)(\psi_{t_i}(x, T/2))\big) + d_h\big(\Phi(0)(\psi_{t_i}(x, T/2)), x\big) \\ &\quad= d_h\big(\Phi(0)(W(0)(x)), \Phi(0)(\psi_{t_i}(x, T/2))\big) \\ &\qquad\quad + d_h\big(\Phi(0)(\psi_{t_i}(x, T/2)), \psi_{T/2}(\psi_{t_i}(x, T/2), t_i)\big) \\ &\quad\le 2\sigma. \end{aligned}$$

Hence, $\Phi(0)(W(0)(x)) = x$, as $x \in M$ and $\sigma > 0$ were arbitrary. Similarly, for $z \in M$, for any $\sigma > 0$ if i is large enough, we have

$$d_h\big(W(0)(\Phi(0)(z)), W(0)(\psi_{T/2}(z, t_i))\big) \le \sigma$$

and

$$d_h\big(W(0)(x), \psi_{t_i}(x, T/2)\big) \le \sigma$$

for all $x \in M$, and hence

$$\begin{aligned} &d_h\big(W(0)(\Phi(0)(z)), z\big) \\ &\quad\le d_h\big(W(0)(\Phi(0)(z)), W(0)(\psi_{T/2}(z, t_i))\big) + d_h\big(W(0)(\psi_{T/2}(z, t_i)), z\big) \\ &\quad= d_h\big(W(0)(\Phi(0)(z)), W(0)(\psi_{T/2}(z, t_i))\big) \\ &\qquad\quad + d_h\big(W(0)(\psi_{T/2}(z, t_i)), \psi_{t_i}(\psi_{T/2}(z, t_i), T/2)\big) \\ &\quad\le 2\sigma. \end{aligned}$$

Hence, $W(0)(\Phi(0)(z)) = z$ for all $z \in M$, as $z \in M$ and $\sigma > 0$ were arbitrary. That is, $W(0)$ is the inverse of $\Phi(0)$.

We further have that

$$\Phi(s)(B_{1-\varepsilon}(x_0)) \subseteq \Phi(0)(B_{1-\varepsilon/4}(\tilde{x}_0))$$

for all $s = s(\varepsilon) > 0$ small enough, as we now explain, where

$$\tilde{x}_0 = W(0) \circ \Phi(s)(x_0) \colon W(0) \circ \Phi(s) \to \mathrm{Id}$$

uniformly as $s \searrow 0$, as was shown above. This means

$$W(0) \circ \Phi(s)(B_{1-\varepsilon}(x_0)) \subseteq B_{1-3\varepsilon/4}(x_0) \subseteq B_{1-\varepsilon/4}(\tilde{x}_0)$$

for s small enough, and hence taking $\Phi(0)$ of both sides, the claim follows. This is (i).

(ii) Let $y_0 \in M$ and $p \in [0, \infty)$, and $\varepsilon \le \alpha_0(p, n = 4)$ be the constant from Theorem 7.1, Corollary 7.3, and assume also that $\varepsilon \le \alpha_1(4, C_S(4))$, where $\alpha_1(n, A)$ is the constant from Theorem 7.4 with $n = 4$ and $A = C_S(4)$, and $C_S(n)$ is the Sobolev constant from Theorem B.1. The construction of our solution, see Theorem 6.1, Theorem 6.3 and the Tensor Sobolev inequality, Theorem B.1 (v) guarantee that, without loss of generality,

$$\int_{B_2(x)} \left(|{}^h\nabla g(t)|^4 + |{}^h\nabla^2 g(t)|^2 \right) dh \le \delta^4(a)$$

for all $x \in M$ and $\delta(a)$ is the constant from Corollary 5.4. We also have, without loss of generality,

$$|{}^h\nabla^3 g|^2 t^3 + |{}^h\nabla^2 g|^2 t^2 + |{}^h\nabla g|^2 t \le \varepsilon^2$$

on $\overline{B_{200}(y_0)}$ for $t \in (0, T)$ in view of (e_t), and hence

$$|\mathrm{Rm}(g(t))| + |{}^{g(t)}\nabla \mathrm{Rm}|^{2/3} \le \frac{\varepsilon}{t} \tag{8.2}$$

on $\overline{B_{200}(y_0)}$ for $t \in (0, T)$, after reducing the time interval if necessary.

By choosing $R_1 = R_1(y_0, g_0) > 0$ small enough, we can guarantee that

$$\int_{B_{R_1}(x_0)} \left(|{}^h\nabla g_0|^4 + |{}^h\nabla^2 g_0|^2 \right) dh \le \frac{\varepsilon}{2}$$

for all x_0 in the compact set $\overline{B_{100}(y_0)}$ in view of Lemma B.3. By scaling once, we have for all such x_0 that

$$\int_{B_8(x_0)} \left(|{}^h\nabla g_0|^2 + |{}^h\nabla^2 g_0|^2 \right) dh \le \frac{\varepsilon}{2},$$

and

$$\int_{B_{20}(x)} \left(|^h\nabla g(t)|^4 + |^h\nabla^2 g(t)|^2\right) dh \le \delta^4(a)$$

for all $x \in M$ and all x_0 in $\overline{B_{100}(y_0)}$, which implies

$$\int_{B_{20}(x)} \left(|^h\nabla g(t)|^2 + |^h\nabla^2 g(t)|^2\right) dh \le \delta(a)$$

for all $x \in M$, in view of Hölder's inequality and the fact that $\delta(a)$ is, without loss of generality, less than $\mathrm{Vol}_h(B_{20}(x))$ for all $x \in M$. Using Corollary 5.4, we see

$$\int_{B_4(x_0)} \left(|^h\nabla^2 g(t)|^2 + |^h\nabla g(t)|^2\right) dh \le \varepsilon$$

for all x_0 in $\overline{B_{100}(y_0)}$, and hence using the fact that, without loss of generality, $|\mathrm{Rm}(h)| \le \varepsilon$, we have

$$\int_{B_4(x_0)} |\mathrm{Rm}(g(t))|^2_{g(t)}\, dg(t) \le 2\varepsilon,$$

for all $t \le T$, after reducing the time interval if necessary. This estimate with (8.2) show that the Ricci flow related solution ℓ restricted to $\Omega = B_4(x_0)$ for any such x_0 satisfies all the conditions of Theorem 7.1, Corollary 7.3 and Theorem 7.4 (after scaling once more by a factor 5), and hence the estimates obtained there hold. These estimates change at most by a factor when we scale the solution back to the original solution, the constant depending on the scaling factor, h and x_0, g_0 and p. These scaled estimates are (ii) for the given p. As $p \in [0, \infty)$ was arbitrary, (ii) holds.

(iii) From the definition of ℓ, in local coordinates, we have

$$\begin{aligned}
\ell_{ij}(s)(x) &= D_i\Phi^\alpha(s)(x) D_j\Phi^\beta(s)(x) g_{\alpha\beta}(s)(\Phi(s)(x)),\\
g_{\alpha\beta}(s)(\Phi(s)(x)) &= D_\alpha W^i(s)(\Phi(s)(x)) D_\beta W^j(s)(\Phi(s)(x))\ell_{ij}(x,s),\\
g_{\alpha\beta}(s)(y) &= D_\alpha W^i(s)(y) D_\beta W^j(s)(y)\ell_{ij}(W(s)(y),s),
\end{aligned}$$

where we have chosen smooth coordinates as in the statement of the claim of (iii) of this theorem, $y \in \widetilde{V}$, $x \in \widetilde{U}$. Notice that $g(t) \to g(0)$ and $g(t)^{-1} \to g(0)^{-1}$ in the L^p_{loc}-sense for all $p \in [1, \infty)$, in view of Corollary 5.7. Hence, we may apply Theorem 9.1, and we see that (iii) holds.

(iv) For $x, y \in M$, we define

$$d_0(x,y) := \liminf_{\varepsilon \to 0} \inf_{\gamma \in C_{\varepsilon,x,y}} L_{g_0}(\gamma),$$

where $C_{\varepsilon,x,y}$ is the space of ε-*approximative Lebesgue curves* with respect to g_0 joining x and y, defined in Definition 8.2.

Let $x, y \in B_{R/c(a)}(x_0)$, where $B_{R/c(a)}(x_0)$ is the ball with respect to h, for some fixed $x_0 \in M$, where $c(a)$ is a large constant to be determined in the proof. Since $g_0 \in W^{2,2}(B_{2R}(x_0))$, we know from Lemma B.3 that for any $\sigma > 0$, there exists $r(\sigma, R, a) > 0$ such that

$$\int_{B_{2r}(z)} \left(|{}^h\nabla^2 g_0|^2 + |{}^h\nabla g_0|^4\right) dh \le \sigma^4$$

for all $z \in B_R(x_0)$.

Scaling g_0 and h once, by the same large constant K, and still calling the new scaled metrics g_0, h, and the new radius $\sqrt{K}R$ will still be denoted by R, we have

$$\int_{B_2(z)} \left(|{}^h\nabla^2 g_0|^2 + |{}^h\nabla g_0|^4\right) dh \le \sigma^4$$

for all $z \in B_R(x_0)$ and, without loss of generality,

$$\sup_M \sum_{i=1}^{4} |{}^h\nabla^i \mathrm{Rm}(h)| \le \sigma^4.$$

Hence, using Corollary 5.4, Hölder's inequality and Lemma B.1 (v),

$$\begin{gathered}\int_{B_2(z)} \left(|{}^h\nabla^2 g(t)|^2 + |{}^h\nabla g(t)|^4\right) dh \le \sigma, \quad \sum_{i=1}^{4} |{}^h\nabla^i \mathrm{Rm}(h)| \le \sigma, \\ \frac{1}{400a} h \le g(t) \le 400ah, \quad |{}^h\nabla^j g(t)| \le \frac{c(j,a)}{t^j},\end{gathered} \tag{8.3}$$

for all $j \in \mathbb{N}$, for all $z \in B_R(x_0)$, and for all $0 < t \le S_2(400a, \sigma)$, and after scaling once more by $1/2S_2$ for all $t \in (0, 2]$.

We first show that

$$d_0(x, y) \le \liminf_{t \to 0} d_t(x, y). \tag{8.4}$$

Let $\varepsilon > 0$ be given. Taking any $0 < t \le \varepsilon^4$ and scaling by $\hat{g}(s) = \frac{1}{t} g(st)$, and denoting the new radius by $\hat{R}$, that is

$$\hat{R} = \frac{1}{\sqrt{t}} R,$$

and

$$\hat{h} = \frac{1}{t} h, \quad \hat{g}_0 = \frac{1}{t} g_0,$$

we see, in view of (8.3), that we obtain a new solution $\hat{g}(s)$, $s \in [0, 2]$, such that

$$\begin{gathered}\int_{\hat{B}_2(z)} \left(|{}^h\nabla^2 \hat{g}(s)|^2 + |{}^h\nabla \hat{g}(s)|^4\right) dh \le 2\sigma, \quad \frac{1}{400a} \hat{h} \le \hat{g}(s) \le 400a\hat{h}, \\ |{}^h\nabla^j \hat{g}(s)| \le \frac{c(j,a)}{s^j}, \quad \sum_{i=1}^{4} |{}^h\nabla^i \mathrm{Rm}(\hat{h})| \le \sigma, \quad |{}^h\nabla^j \hat{g}(1)| \le c(j, a, \sigma)\end{gathered}$$

for all $j \in \mathbb{N}$, for all $z \in \widehat{B}_{\widehat{R}}(x_0)$, and for all $s \leq 1$, where $c(j,a,\sigma) \to 0$ for $\sigma \to 0$, where $\widehat{B}_s(m)$ refers to a ball of radius m with respect to $\widehat{h}$. For later use, note that $\widehat{R} \geq 1/\varepsilon^2$. Let γ be a length minimising geodesic between x and y with respect to $\widehat{g}(1)$. Writing $\widehat{g}(1)$ in geodesic coordinates at any $z \in \widehat{B}_{\widehat{R}}(x_0)$ on a ball of radius one, we have

$$\begin{gathered}\big(1 - |\alpha(\sigma,a)|\big)\delta \leq \widehat{g}(1) \leq \big(1 + |\alpha(\sigma,a)|\big)\delta \quad \text{on } \mathbb{B}_{10}(0),\\ \frac{1}{400a}\widehat{h} \leq \widehat{g}(s) \leq 400a\widehat{h} \quad \text{for } s \in [0,2],\end{gathered}$$

where $\alpha(\sigma,a) \to 0$ as $\sigma \to 0$.

In the following, any constant $c(\sigma,a)$ with $c(\sigma,a) \to 0$ as $\sigma \to 0$ shall be denoted by $\alpha(\sigma,a)$, although it may differ from the one just defined, and $\alpha(\sigma,a)$ is not necessarily larger than zero.

We can break γ up into N pieces

$$\gamma_1 = \gamma|_{[0,1]}, \quad \gamma_2 = \gamma|_{[1,2]}, \quad \gamma_{N-1} = \gamma|_{[N-2,N-1]}, \quad \gamma_N = \gamma|_{[N-1,B]}$$

each with length one with respect to $\widehat{g}(1)$, except for the last piece which has length less than or equal to one. Due to

$$\frac{1}{400a}\widehat{h} \leq \widehat{g}(1) \leq 400a\widehat{h},$$

we have $N \leq c(a)\widehat{R}$. After rotating once, we may assume that any length one piece of γ, going from $\gamma(i)$ to $\gamma(i+1)$, $i \in \{0,2,\ldots,N-2\}$, in geodesic coordinates, with respect to $\widehat{g}(1)$ centred at $z = \gamma(i+1/2)$ lies in $\overline{\mathbb{B}_2(0)}$, and is (in these coordinates) the line segment

$$v\colon \Big[-\frac{1}{2},\frac{1}{2}\Big] \to \overline{\mathbb{B}_2(0)}, \quad v(s) = se_1.$$

We ignore the last piece of γ for the moment.

Using Corollary 5.7 and $|\mathrm{Rm}(\widehat{h})| \leq \sigma$, we have

$$\int_{\mathbb{B}_1(0)} |\widehat{g}(t) - \widehat{g}_0|^2\, dh \leq 2\sigma t$$

for all $t \leq 1$, and hence

$$\int_{\mathbb{B}_1(0)} |\delta - \widehat{g}_0|^2 \leq \alpha(\sigma,a).$$

Let $\varepsilon > 0$ be given. Using Lemma 10.2 we see, by choosing $\sigma = \sigma(\varepsilon) > 0$ small enough, that there exists an $x \in \mathbb{B}^{n-1}_{\varepsilon}(0)$ such that $\sqrt{\widehat{g}_{11}(0)}(\cdot,x)\colon [-1/2,1/2] \to \mathbb{R}^n$ is measurable, $\ell\colon[-1/2,1/2]\to\mathbb{R}^n$, $\ell(t)=(t,x)$ is a Lebesgue line between $(-1/2,x)$ and $(1/2,x)$, and

$$\int_{-1/2}^{1/2}\sqrt{\widehat{g}_{11}(0)(s,x)}\,ds\le 1+\varepsilon=(1+\varepsilon)d_\delta((-1/2,x),(1/2,x))$$
$$\le(1+2\varepsilon)d_{\widehat{g}(1)}((-1/2,x),(1/2,x)),$$

which tells us for the original curve $\gamma|_{[0,N-1]}$, that there exist Lebesgue curves

$$v_i\colon[i-1,i]\to\widehat{B}_{\widehat{R}}(x_0)$$

for all $i\in\{1,\dots,N-1\}$ with respect to $\widehat{g}(0)$ such that

$$d_{\widehat{h}}(v_i(i-1),\gamma(i-1))\le c(a)\varepsilon\quad\text{and}\quad d_{\widehat{h}}(v_i(i),\gamma(i))\le c(a)\varepsilon,$$

and

$$L_{\widehat{g}_0}(v_i)\le(1+\varepsilon)d_{\widehat{g}(1)}(\gamma(i-1),\gamma(i))$$

for all $i\in\{1,\dots,N-1\}$. The curves v_i are the curves ℓ constructed above. Adding up all the curve segments, we have

$$\sum_{i=1}^{N-2}d_{\widehat{h}}(v_i(i),v_{i+1}(i))\le N\varepsilon c(a)\le\widehat{R}\varepsilon c(a),$$

and hence

$$\sum_{i=1}^{N-2}d_{\widehat{h}}(v_i(i),v_{i+1}(i))+d_h(x,v_1(0))+d_h(y,v_{N-1}(N-1))\le\widehat{R}\varepsilon c(a)+2,$$

and also

$$\sum_{i=1}^{N-1}L_{\widehat{g}_0}(v_i)\le(1+\varepsilon)d_{\widehat{g}(1)}(x,\gamma(N-1))\le(1+\varepsilon)d_{\widehat{g}(1)}(x,y)$$

as γ was a length minimising, with respect to $\widehat{g}(1)$, geodesic between x and y. Scaling back the solution we had at the beginning of this proof of this claim, (8.4), that is defining $g(s)=tg(s\frac{1}{t})$, for the t we chose there, we see at time t that

$$\sum_{i=1}^{N-1}L_{g_0}(v_i)\le(1+\varepsilon)d_{g(t)}(x,y)$$

and

$$\sum_{i=1}^{N-2}d_h(v_i(i),v_{i+1}(i))+d_h(x,v_1(0))+d_h(y,v_{N-1}(N-1))$$
$$\le R\varepsilon c(a)+2\sqrt{t}\le R\varepsilon c(a)+2\varepsilon$$

in view of the choice of $t \leq \varepsilon^4$. That is, $v = \bigcup_{i=1}^{N-1} v_i$ is an $Rc(a)\varepsilon$-approximative Lebesgue curve and $L_{g_0}(v) \leq (1+\varepsilon)d_t(x,y)$. Hence,

$$\inf_{\gamma \in C_{Rc(a)\varepsilon,x,y}} L_{g_0}(\gamma) \leq (1+\varepsilon)d_t(x,y)$$

for all $t \leq T(\varepsilon, a, g_0, x, y, R, h)$, and this shows

$$d_0(x,y) = \liminf_{\varepsilon \to 0} \inf_{\gamma \in C_{\varepsilon,x,y}} L_{g_0}(\gamma) \leq \liminf_{t \to 0} d_t(x,y)$$

for all $x, y \in M$.

Now we show that

$$d_0(x,y) := \liminf_{\varepsilon \to 0} \inf_{\gamma \in C_{\varepsilon,x,y}} L_{g_0}(\gamma) \geq \limsup_{t \searrow 0} d_t(x,y)$$

for all $x, y \in M$. From the definition of $C_{\varepsilon,x,y}$, $\gamma \in C_{\varepsilon,x,y}$, may be written as $\gamma = \bigcup_{i=1}^{N} \gamma_i$, where each $\gamma_i : [a_i, b_i] \to M$ is a parametrised Lebesgue line. Let $\sigma : [c_i, d_i] \to \mathbb{B}_2(x)$ be one of the segments γ_i written in smooth coordinates, so that $\sigma(t) = te_1$. Since the coordinates are smooth, and $\frac{1}{400a} h \leq g(t) \leq 400ah$, we see that there exists a constant C depending possibly on the coordinates and a, such that $\frac{1}{C}\delta \leq h \leq C\delta$ and $\frac{1}{C}\delta \leq g(t) \leq C\delta$ in these coordinates.

Using Corollary 5.7 and $|\mathrm{Rm}(h)| \leq \sigma \leq 1$, we see that we have

$$\int_{\mathbb{B}_2(0)} |g(x,t) - g_0(x)|_\delta^2 \, dx \leq ct$$

with respect to these coordinates for all $t \leq 1$ for some constant $c = c(C)$.

Using Lemma 10.1, we see that for all $\varepsilon > 0$, there exists a $t_0 > 0$ such that

$$L_{g_0}(\sigma) := \int_{c_i}^{d_i} \sqrt{g(0)_{11}(s,0)} \geq (1-\varepsilon)d(g(t))(\sigma(c_i), \sigma(d_i))$$

for all $t \in (0, t_0)$ in these coordinates. That is,

$$L_{g_0}(\gamma_i) \geq (1-\varepsilon)d(g(t))\big(\gamma_i(a_i), \gamma_i(b_i)\big).$$

Hence, estimating on each Lebesgue line γ_i in this way, we see (setting $\gamma_{N+1}(a_{N+1}) := y$) that, there exists s_0 such that

$$\begin{aligned} L_{g_0}(\gamma) &\geq \sum_{i=1}^{N} (1-\varepsilon) d_t(\gamma_i(a_i), \gamma_i(b_i)) \\ &\geq \sum_{i=1}^{N} (1-\varepsilon) d_t(\gamma_i(a_i), \gamma_{i+1}(a_{i+1})) - \sum_{i=1}^{N} d_t(\gamma_i(b_i), \gamma_{i+1}(a_{i+1})) \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{N}(1-\varepsilon)d_t(\gamma_i(a_i),\gamma_{i+1}(a_{i+1})) - c\sum_{i=1}^{N} d_h(\gamma_i(b_i),\gamma_{i+1}(a_{i+1}))\\
&\geq (1-\varepsilon)d_t(\gamma_1(a_1),\gamma_{N+1}(a_{N+1})=y) - c\varepsilon\\
&\geq (1-\varepsilon)d_t(x,y) - 2c\varepsilon
\end{aligned}
$$

for $t \in (0, s_0)$. That is, for fixed $x, y \in M$, we have

$$
d_0(x,y) \geq \limsup_{t\searrow 0} d_t(x,y),
$$

in view of the definition of d_0 and the fact that $\varepsilon > 0$ was arbitrarily chosen in the argument above. Combining the lower and upper bound proved for $d_0(x, y)$, we have

$$
d_0(x,y) := \liminf_{\varepsilon\to 0} \inf_{\gamma\in C_{\varepsilon,x,y}} L_{g_0}(\gamma) = \lim_{t\searrow 0} d_t(x,y),
$$

as claimed.

The property $\tilde{d}_0 = \lim_{t\searrow 0} \tilde{d}_t$ now follows easily from the definitions and the fact that $\Phi(t)$ converges uniformly to $\Phi(0)$ as $t \searrow 0$:

$$
\begin{aligned}
\tilde{d}_0(x,y) &:= d_0(\Phi(0)(x),\Phi(0)(y))\\
&= d_t(\Phi(0)(x),\Phi(0)(y)) + \varepsilon(x,y,t)\\
&\leq (\geq)\ d_t(\Phi(0)(x),\Phi(t)(y))\\
&\qquad + (-)\ d_t(\Phi(t)(y),\Phi(0)(y)) + \varepsilon(x,y,t)\\
&\leq (\geq)\ d_t(\Phi(t)(x),\Phi(t)(y)) + (-)\ d_t(\Phi(t)(x),\Phi_0(x))\\
&\qquad + (-)\ d_t(\Phi(t)(y),\Phi(0)(y)) + \varepsilon(x,y,t)\\
&\leq (\geq)\ d_t(\Phi(t)(x),\Phi(t)(y)) + (-)\ c(n,a)d_h(\Phi(t)(x),\Phi(0)(x))\\
&\qquad + (-)\ c(n,a)d_h(\Phi(t)(y),\Phi(0)(y)) + \varepsilon(x,y,t)\\
&= \tilde{d}_t(x,y) + (-)\ c(n,a)d_h(\Phi(t)(x),\Phi(0)(x))\\
&\qquad + (-)\ c(n,a)d_h(\Phi(t)(y),\Phi(0)(y)) + \varepsilon(x,y,t),
\end{aligned}
$$

where $\varepsilon(x, y, t) \to 0$ for $t \searrow 0$. d.h. $\tilde{d}_t(x, y) \to \tilde{d}_0(x, y)$ for $t \searrow 0$. ∎

9. Metric convergence in Sobolev spaces

Theorem 9.1. *Let (M, h) be a four-dimensional Riemannian manifold satisfying (2.4). We assume that there are continuous maps $W(0)$, $\Phi(0)$: $M \to M$ inverse to one* *another such that for all compact sets $K \subseteq M$,*

$$\sup_{x\in K} d_h(\Phi(r)(x), \Phi(0)(x)) \to 0 \quad \text{as } r \to 0,$$
$$\sup_{y\in K} d_h(W(r)(y), W(0)(y)) \to 0 \quad \text{as } r \to 0,$$

where $\Phi, W\colon M \times (0,T) \to M$ are smooth maps such that $W(s)\colon M \to M$, $\Phi(s)$ are smooth diffeomorphisms inverse to one another for all $s \in (0,T)$.

Let $\ell(s)_{s\in(0,T)}$, $g(s)_{s\in(0,T)}$ be smooth families of Riemannian metrics isometric to one another through the smooth maps $\Phi(s)$ and $W(s)$:

$$\ell(s) = \Phi(s)^*(g(s)), \quad g(s) = W(s)^*\ell(s).$$

Assume that we have chosen smooth coordinates $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$, and open sets $\widetilde{U} \subset\subset U$ and $\widetilde{V} \subset\subset V$ with $W(s)(\widetilde{V}) \subset\subset U$ and $\Phi(s)(\widetilde{U}) \subset\subset V$ for all $s \in [0,S]$, $W(s)(V) \subset\subset U$ for all $s \in [0,S]$ for some $0 < S < T$. That is, in these coordinates, we have

$$\begin{aligned} \ell_{ij}(s)(x) &= D_i\Phi^\alpha(s)(x) D_j\Phi^\beta(s)(x) g_{\alpha\beta}(s)(\Phi(s)(x)),\\ g_{\alpha\beta}(s)(\Phi(s)(x)) &= D_\alpha W^i(s)(\Phi(s)(x)) D_\beta W^j(s)(\Phi(s)(x))\ell_{ij}(x,s),\\ g_{\alpha\beta}(s)(y) &= D_\alpha W^i(s)(y) D_\beta W^j(s)(y)\ell_{ij}(W(s)(y),s) \end{aligned} \tag{9.1}$$

for $x \in \widetilde{U}$, $y \in \widetilde{V}$. Assume further, that there exist Riemannian metrics $\ell(0)$ and $g(0)$ whose inverse exists almost everywhere, so that $g(0), \ell(0), g^{-1}(0), \ell^{-1}(0) \in L^p_{\mathrm{loc}}$ for all $p \in [1,\infty)$ such that

- (i) *$\ell(s) \to \ell(0)$, $\ell^{-1}(s) \to \ell^{-1}(0)$, $g(s) \to g(0)$, $g^{-1}(s) \to g^{-1}(0)$ as $s \to 0$ locally in L^p for all $p \in [1,\infty)$,*
- (ii) *for any compact set $K \subseteq M$, for all $s \in (0,T)$, $p \in [1,\infty)$, there exists a constant $c(K,h,p)$ such that $\|\ell(s)\|_{W^{1,2}(K)} + \|g(s)\|_{W^{1,4}(K)} \le c(K,h,p)$,*
- (iii) *there is a constant $a \ge 1$ such that $\frac{1}{a}h \le g(s) \le ah$ for all $s \in (0,T)$.*

Then, $D\Phi(s)$ is bounded in $L^p(\widetilde{U})$ and $DW(s)$ is bounded in $L^p(\widetilde{V})$ uniformly independent of $s \in (0,T)$. Furthermore, $\Phi(s) \to \Phi(0)$ and $W(s) \to W(0)$ locally in $W^{1,p}$ for any $p \in [1,\infty)$, $\ell(0)\circ W(0)$ is in L^p_{loc} for all $p \in [1,\infty)$, and $\ell(0)$ and g_0 are isometric to one another through the map $W(0)$ which is in $W^{1,p}_{\mathrm{loc}}$, for all $p \in [1,\infty)$,

$$(g_0)_{\alpha\beta} = D_\alpha(W(0))^i D_\beta(W(0))^j((\ell_0)_{ij}\circ W(0))$$

on $\widetilde{V}$ in the L^p-sense for all $p \in [1,\infty)$.

Proof. We consider in the following only $s \in (0, S)$. The first identity of (9.1) implies:

$$\begin{aligned} h^{ij}(x)\ell_{ij}(s)(x) &= h^{ij}(x) D_i \Phi^\alpha(s)(x) D_j \Phi^\beta(s)(x) g_{\alpha\beta}(s)(\Phi(s)(x)) \\ &\geq \frac{1}{a} h^{ij}(x) D_i \Phi^\alpha(s)(x) D_j \Phi^\beta(s)(x) h_{\alpha\beta}(\Phi(s)(x)), \end{aligned}$$

and hence using the fact that $h^{ij}\ell_{ij}(s)$ is locally uniformly bounded in L^p independent of s, we see that $D\Phi(s)$ is locally uniformly bounded in L^p, that is independent of s, for any $p \in [1, \infty)$:

$$\int_{\widetilde{U}} |h^{ij}(x) D_i(\Phi(s))^\alpha(x) D_j(\Phi(s))^\beta(x) h_{\alpha\beta}(\Phi(s)(x))|^p \, d_h(x) \leq c(p, \ldots) < \infty,$$

where $c(p, \ldots)$ is a constant depending on $p, \widetilde{U}, a, \ell, h$. Constants which only depend on $p, \widetilde{U}, \widetilde{V}, \varphi, \psi, \Phi, W, a, \ell, h, g, \ell^{-1}, h^{-1}, g^{-1}$, and importantly *do not depend on s* shall be denoted by $c(p, \ldots)$, although the value may differ from line to line. In view of the uniform convergence of $\Phi(s)$ and $W(s)$ we may assume, by choosing geodesic coordinates with respect to h around m and $\Phi(0)(m)$ without loss of generality, that

$$\widetilde{U} = B_r(m) =: B \quad \text{and} \quad \widetilde{V} = B_v(\Phi(0)(m)) =: \widehat{B},$$

so that $\frac{1}{2}\delta_{\alpha\beta} \leq h_{\alpha\beta} \leq 2\delta_{\alpha\beta}$ on $\widetilde{V}$ and $\frac{1}{2}\delta_{ij} \leq h_{ij} \leq 2\delta_{ij}$ on $\widetilde{U}$, and

$$\widetilde{B} = B_{\tilde{r}}(m) \subseteq W(s)(\widehat{B}) \subseteq B = B_r(m)$$

for s sufficiently small. With respect to these coordinates, we have

$$\int_B \sum_{i,\alpha=1}^n |D_i \Phi^\alpha(s)(x)|^{2p} \, dx \leq c(p, \ldots)$$

for s sufficiently small. Using the third identity of (9.1) and these coordinates, we see that

$$\begin{aligned} &\int_{\widehat{B}} \Big(\sum_{\alpha,i=1}^n (D_\alpha W^i(s)(y))^2 \Big)^p \, dy \\ &= \int_{\widehat{B}} |\delta^{\alpha\beta} D_\alpha W^i(s)(y) D_\beta W^j(s)(y) \delta_{ij}|^p \, dy \\ &\leq c(p, \ldots) \int_{\widehat{B}} |h^{\alpha\beta}(y) D_\alpha W^i(s)(y) D_\beta W^j(s)(y) h_{ij}(W(s)(y), s)|^p \, dy \\ &= c(p, \ldots) \int_{\widehat{B}} |DW(s)|^{2p}_{h,h(W(s))}(y) \, dy \\ &\leq c(p, \ldots) \int_{\widehat{B}} |DW(s)|^{2p}_{h,\ell(W(s),s)}(y)(1 + |h \circ W(s)|^{2p}_{\ell(W(s),s)}(y)) \, dy \\ &\qquad \text{[in view of (D.1) of Theorem D.1]} \end{aligned}$$

$$
\begin{aligned}
&= c(p,\dots)\int_{\widehat{B}} |h^{\alpha\beta} g_{\alpha\beta}(s)|^p(y)\big(1+|h|^{2p}_{\ell(s)}\circ W(s)\big)(y)\,dy\\
&\le c(p,\dots)\bigg(\int_{\widehat{B}} |h^{\alpha\beta} g_{\alpha\beta}(s)(y)|^{2p}\,dy\bigg)^{1/2}\bigg(\int_{\widehat{B}}\big(1+|h|^{4p}_{\ell(s)}\circ W(s)(y)\big)\,dy\bigg)^{1/2}\\
&\le c(p,\dots)\bigg(1+\bigg(\int_{\widehat{B}} |h|^{4p}_{\ell(s)}\circ W(s)(y)\,dy\bigg)^{1/2}\bigg)\\
&= c(p,\dots)\bigg(1+\bigg(\int_{\widehat{B}} |h|^{4p}_{\ell(s)}\circ W(s)(y)\|DW(s)\|(y)\|D\Phi(s)\circ W(s)\|(y)\,dy\bigg)^{1/2}\bigg)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{since } DW(s)\cdot D\Phi(s)\circ W(s) = ID]\\
&\le c(p,\dots)\bigg(1+\bigg(\int_{W(s)(\widehat{B})\subseteq B_r(m)} |h|^{4p}_{\ell(s)}(x)\|D\Phi(s)(x)\|\,dx\bigg)^{1/2}\bigg)\\
&\le c(p,\dots)
\end{aligned}
$$

since $(\ell)^{-1}$, g and $D\Phi(s)$ are bounded locally in L^p independently of s, for s sufficiently small, and we have used the transformation formula, and the notation $\|A\|$ to represent $\det(A)$.

We also have

$$
\begin{aligned}
&\int_{\widetilde{B}} |D(g_s\circ\Phi(s))|^{4-\varepsilon}(x)\,dx\\
&\le \int_{W(s)(\widehat{B})} |D(g_s\circ\Phi(s))|^{4-\varepsilon}(x)\,dx\\
&= \int_{W(s)(\widehat{B})} |((Dg_s)\circ\Phi(s))(x)\cdot D\Phi(s)(x)|^{4-\varepsilon}\,dx\\
&\le \bigg(\int_{W(s)(\widehat{B})} (|Dg_s|\circ\Phi(s))^{4-\varepsilon/2}(x)\,dx\bigg)^{1/q}\bigg(\int_{W(s)(\widehat{B})} |D\Phi(s)|^{r(4-\varepsilon)}(x)\,dx\bigg)^{1/r}\\
&\qquad\qquad\qquad\qquad\qquad [\text{where } q = (4-(\varepsilon/2))/(4-\varepsilon),\ r = 1/(1-(1/q))]\\
&\le c(p(\varepsilon),\dots)\bigg(\int_{W(s)(\widehat{B})} |Dg_s|^{4-\varepsilon/2}\circ\Phi(s)(x)\,dx\bigg)^{1/q}\\
&= c(p(\varepsilon),\dots)\\
&\qquad\times\bigg(\int_{W(s)(\widehat{B})} |Dg_s|^{4-\varepsilon/2}\circ\Phi(s)(x)\cdot\|DW(s)\circ\Phi(s)(x)\|\|D\Phi(s)(x)\|\,dx\bigg)^{1/q}\\
&\qquad\qquad\qquad\qquad\qquad [\text{since } I = D(W(s)\circ\Phi(s)) = (DW(s)\circ\Phi(s))\cdot D\Phi(s)]\\
&= c(p(\varepsilon),\dots)\bigg(\int_{\widehat{B}} |Dg_s|^{4-\varepsilon/2}(y)\|DW(s)\|(y)\,dy\bigg)^{1/q}
\end{aligned}
$$

$$\le c(p(\varepsilon),\dots)\Big(\int_{\widehat{B}} |Dg_s|^4(y)\,dy\Big)^{1/qv}\Big(\int_{\widehat{B}} \|DW(s)\|^u\,dy\Big)^{1/qu}$$
$$[\text{where } v = 4/(4-\varepsilon/2),\ u = 1/(1-1/v)]$$
$$\le c(p(\varepsilon),\dots)$$

for sufficiently small $s > 0$, where $c(p(\varepsilon),\dots)$ is independent of s, and $2 > \varepsilon > 0$ is arbitrary, since Dg is bounded in L^4 due to the assumptions, and $DW(s)$ is bounded uniformly in every L^p for every fixed p, and we used

$$\Big(\int_{W(s)(\widehat{B})} |D\Phi(s)|^{r(4-\varepsilon)}(x)\,dx\Big)^{1/r} \le c(p,\varepsilon,\dots).$$

Hence, for any sequence $t_i > 0$ with $t_i \to 0$ as $i \to \infty$, we can find a subsequence s_i with $s_i \searrow 0$ such that $(g_{s_i})_{\alpha\beta}\circ\Phi(s_i)$ converges strongly in L^p to some $Z_{\alpha\beta}$ on $\widetilde{B}$ for all p as s_i goes to zero, in view of the Sobolev embedding theorems (see, for example, [14, Theorem 7.26]). Also, Z satisfies

$$\frac{1}{C(a)}\delta_{\alpha\beta} \le Z_{\alpha\beta} \le C(a)\delta_{\alpha\beta}$$

on $\widetilde{B}$ since $\frac{1}{2}\delta_{\alpha\beta} \le h_{\alpha\beta}(\cdot) \le 2\delta_{\alpha\beta}$ on $\varphi(s)(\widetilde{B}) \subseteq \widetilde{V}$ and $\frac{1}{a}h \le g(s) \le ah$.

For a sequence $0 < s_i \searrow 0$, we write $g(i) = g(s_i)$, $\Phi(i) = \Phi(s_i)$, and $\ell(i) := \ell(s_i)$. Using (9.1), we see

$$\delta_s^k = \ell(i)^{rk}(x)D_s\Phi(i)^\alpha(x)D_r\Phi(i)^\beta(x)g(i)_{\alpha\beta}\circ\Phi(i)(x),$$

and hence

$$\begin{aligned}
0 &= \ell(i)^{rs}(x)D_s\Phi(i)^\alpha(x)D_r\Phi(i)^\beta(x)g(i)_{\alpha\beta}\circ\Phi(i)(x)\\
&\qquad - \ell(j)^{rs}(x)D_s\Phi(j)^\alpha(x)D_r\Phi(j)^\beta(x)g(j)_{\alpha\beta}\circ\Phi(j)(x)\\
&= \ell_0^{rs}(x)Z_{\alpha\beta}(x)(D_s\Phi(i)^\alpha(x)D_r\Phi(i)^\beta(x)\\
&\qquad - D_s\Phi(j)^\alpha(x)D_r\Phi(j)^\beta(x)) + \mathrm{err}(i,j)(x)\\
&= |D\Phi(i) - D\Phi(j)|^2_{\ell_0,Z}(x) + \mathrm{err}(i,j)(x),
\end{aligned}$$

where $\mathrm{err}(i,j)$ is an error term which goes to 0 in the L^p-sense, on $\widetilde{B}$ for any $p \in [2,\infty)$ as $i,j \to \infty$, and hence, using (D.2) of Theorem D.1, we see

$$\begin{aligned}
&\int_{\widetilde{B}} |D\Phi(i) - D\Phi(j)|^p_{\delta,\delta}(x)\,dx\\
&\quad\le c(p,\dots)\int_{\widetilde{B}} |D\Phi(i) - D\Phi(j)|^p_{h,Z}(x)\,dx\\
&\quad\le c(p,\dots)\int_{\widetilde{B}} (1 + |\ell_0|_h^2)^{p/2}(x)|D\Phi(i) - D\Phi(j)|^p_{\ell_0,Z}(x)\,dx
\end{aligned}$$

$$\leq c(p,\ldots)\Big(\int_{\widetilde{B}} \big(1+|\ell_0|_h^2\big)^p(x)\,dx\Big)^{1/2}\Big(\int_{\widetilde{B}} |D\Phi(i)-D\Phi(j)|^{2p}_{\ell_0,Z}(x)\,dx\Big)^{1/2}$$
$$= c(p,\ldots)\Big(\int_{\widetilde{B}} (\mathrm{err}(i,j))^{2p}(x)\,dx\Big)^{1/2},$$

where

$$\int_{\widetilde{B}} |\mathrm{err}(i,j)|^{2p} \to 0$$

as $i, j \to \infty$. That is, $D\Phi(i)|_{\widetilde{B}}$ is Cauchy in $L^p(\widetilde{B})$. Since $\Phi(i) \to \Phi(0)$ in locally, uniformly, and hence locally in L^p for any $p < \infty$, we see $\Phi(i) \to \Phi(0)$ in $W^{1,p}(\widetilde{B})$ as $i \to \infty$. In fact, $\Phi(s) \to \Phi(0)$ in $W^{1,p}(\widetilde{B})$ as $s \searrow 0$. If this were not the case, then we could find a sequence of times $t_i \to 0$ such that

$$|\Phi(t_i) - \Phi(0)|_{W^{1,p}(\widetilde{B})} \geq \sigma > 0.$$

Repeating the above argument, we see that a subsequence $\Phi(s_i)$ of $\Phi(t_i)$ converges to $\Phi(0)$ in $W^{1,p}(\widetilde{B})$, which contradicts

$$|\Phi(t_i) - \Phi(0)|_{W^{1,p}(\widetilde{B})} \geq \sigma > 0.$$

We now show that a subsequence of $\ell_{t_i} \circ W(t_i)$ converges in L^p locally for any sequence $t_i \searrow 0$. For $0 < 4\delta < 3$, we have (where here the norm $|\cdot|$ refers to the euclidean norm)

$$\begin{aligned}
&\int_{\widehat{B}} |D(\ell_s \circ W(s))|^{1+\delta}\,dy \\
&\qquad= \int |(D\ell \circ W(s))DW(s)|^{1+\delta}\,dy \\
&\qquad\leq \int_{\widehat{B}} |D\ell|^{1+2\delta} \circ W(s)\,dy + \int_{\widehat{B}} |DW(s)|^v\,dy \\
&\qquad\qquad\qquad\qquad\qquad [\text{with } v = 1/(1-(1/q)),\ q = (1+2\delta)/(1+\delta)] \\
&\qquad\leq \int_{\widehat{B}} |D\ell|^{1+2\delta} \circ W(s)\,dy + c(v(\delta),\ldots) \\
&\qquad= \int_{\widehat{B}} \big(|D\ell|^{1+2\delta} \circ W(s)\big)\|D\Phi(s) \circ W(s)\|\|DW(s)\|\,dy + c(v(\delta),\ldots) \\
&\qquad= \int_{W(s)(\widehat{B})} |D\ell|^{1+2\delta}\,dx\|D\Phi(s)\| + c(v(\delta),\ldots)
\end{aligned}$$

$$\leq \int_{W(s)(\widehat{B})} |D\ell|^{1+4\delta}\,dx + \int_{W(s)(\widehat{B})} \|D\Phi(s)\|^{\widehat{p}}\,dx + c(v(\delta),\dots)$$

$$[\text{with } \widehat{p} = 1/(1-(1/\widehat{q})),\ \widehat{q} = (1+4\delta)/(1+2\delta)]$$

$$\leq c(v(\delta),\dots),$$

where $c(v(\delta),\dots)$ is independent of s. Hence, a subsequence $(s_i)_{i\in\mathbb{N}}$ of $(t_i)_{i\in\mathbb{N}}$ satisfies $\ell_{jk}(s_i)\circ W(s_i) \to R_{jk}$ in $L^{\alpha(\delta)}(\widehat{B})$ for some $R \in L^{\alpha}(\widehat{B})$ from the Sobolev embedding theorems (see, for example, [14, Theorem 7.26]). In particular, $\ell_{jk}(s_i)\circ W(s_i) \to R_{jk}$ almost everywhere on $\widehat{B}$.

On the other hand, the transformation formula for smooth diffeomorphisms shows us that

$$\begin{aligned}
\int_{\widehat{B}} |\ell_s\circ W(s)|^p\,dy &= \int_{\widehat{B}} |\ell_s\circ W(s)|^p \|D\Phi(s)\|\circ W(s)\|DW(s)\|\,dy \\
&= \int_{W(s)(\widehat{B})} |\ell_s|^p \|D\Phi(s)\|\,dx \\
&\leq c(p,\dots)
\end{aligned}$$

in view of Hölder's theorem, since ℓ_s and $D\Phi(s)$ are uniformly bounded in $L^p(B)$ for all $p \in [1,\infty)$. This shows us that $\ell(s_i)\circ W(s_i) \to R$ in $L^p(\widehat{B})$ for all $p \in [1,\infty)$ and that $R \in L^p(\widehat{B})$. Similarly, $\ell(s_i)^{-1}\circ W(s_i) \to R^{-1}$ with R^{-1} in L^p for all $p \in [1,\infty)$ after taking a subsequence

$$|D(\ell^{ij}(s)\circ W(s))|^{1+\delta} = |\ell^{ik}(s)\ell^{jl}(s)D\ell_{kl}(s)\circ W(s)|^{1+\delta},$$

and hence a subsequence of $\ell^{ij}(s_k)\circ W(s_k)$ converges in $L^{\alpha(\delta)}$ to some H^{ij} in $L^{\alpha(\delta)}$. We also have

$$\begin{aligned}
\int_{\widehat{B}} |\ell_s^{-1}\circ W(s)|^p\,dy &= \int_{\widehat{B}} |\ell_s^{-1}\circ W(s)|^p \|D\Phi(s)\|\circ W(s)\|DW(s)\|\,dy \\
&= \int_{W(s)(\widehat{B})} |\ell_s^{-1}|^p \|D\Phi(s)\|\,dx \\
&\leq c(p,\dots),
\end{aligned}$$

and so $\ell^{ij}(s_{k_r})\circ W(s_{k_r}) \to H^{ij}$ in $L^p(\widehat{B})$ for all $p \in [1,\infty)$, where $(s_{k_r})_{r\in\mathbb{N}}$ is a subsequence of $(s_k)_{k\in\mathbb{N}}$. We also have

$$\delta^j_m = \ell^{jk}(s_{i_r})\circ W(s_{i_r})\ell_{km}\,(s_{i_r})\circ W(s_{i_r}) \to H^{jk}R_{km}$$

in $L^p(\widehat{B})$, and hence almost everywhere, and hence H is the inverse of R almost everywhere. After changing the function H on a set of measure zero, H is the inverse of R everywhere.

Using (9.1), we see

$$\delta^\gamma_\beta = g^{\gamma\alpha}(y)(s) D_\alpha W^i(s)(y) D_\beta W^j(s)(y) \ell_{ij}(W(s)(y), s),$$

and writing $W(k) := W(s_{i_k}), g(k) := g(s_{i_k}), \ldots$, we have

$$\begin{aligned}
0 &= g(k)^{\beta\alpha}(y) D_\alpha W^i(k)(y) D_\beta W^j(k)(y) \ell(k)_{ij}(W(k)(y)) \\
&\quad - g(l)^{\beta\alpha}(y) D_\alpha W^i(l)(y) D_\beta W^j(l)(y) \ell(l)_{ij}(W(l)(y)) \\
&= g(0)^{\beta\alpha}(y)(D_\alpha W^i(k)(y) D_\beta W^j(k)(y) - D_\alpha W^i(l)(y) D_\beta W^j(l)(y)) R_{ij}(y) \\
&\quad + \operatorname{err}(k,l)(y) \\
&= |DW(k) - DW(l)|^2_{g(0),R} + \operatorname{err}(k,l),
\end{aligned}$$

where $\operatorname{err}(k,l) \to 0$ in $L^p(\widehat{B})$ for all $p \in [1,\infty)$. Hence, using (D.1) of Theorem D.1, we have

$$\begin{aligned}
\int_{\widehat{B}} |DW(k) - DW(l)|^{2p}_{\delta,\delta}\, dy &\le c(p,\ldots) \int_{\widehat{B}} |DW(k) - DW(l)|^{2p}_{g(0),h}\, dy \\
&\le c(p,\ldots) \int_{\widehat{B}} \big(|DW(k) - DW(l)|^{2p}_{g(0),R}\big)\big(1 + |h|^{2p}_R\big)\, dy \\
&\le c(p,\ldots) \left(\int_{\widehat{B}} |DW(k) - DW(l)|^{4p}_{g(0),R}\, dy \right)^{1/2} \\
&= c(p,\ldots) \left(\int_{\widehat{B}} |\operatorname{err}(k,l)|^{4p}\, dy \right)^{1/2},
\end{aligned}$$

and hence $W(k)$ is Cauchy in $W^{1,p}(\widehat{B})$, and hence converges. Here we used that

$$\int_{\widehat{B}} |h|^{4p}_R\, dy$$

is bounded, which follows from the fact that $R^{-1} = H \in L^q$ for all $q \in [1,\infty)$. Using a similar argument to the one we used for Φ, we see that $W(s) \to W(0)$ in $W^{1,p}(\widehat{B})$ as $s \searrow 0$, i.e., that the convergence $W(t_i) \to W(0)$ in $W^{1,p}(\widehat{B})$ holds for all sequences $0 < t_i \to 0$.

We saw that $\ell(s) \circ W(s)$ converges in $L^p(\widehat{B})$ for all $p \in [1,\infty)$ as $s \searrow 0$. We would like to further show that the limit function is $\ell(0) \circ W(0)$.

Using the change of variable formula for smooth diffeomorphisms, we see for the same coordinates from above, for any $B_r(y_0) \subset\subset \widehat{B}$ and any cut-off smooth non-negative function η with support in $B_{r-2\varepsilon}(y_0)$ that

$$\begin{aligned}
&\int_{B_r(y_0)} \ell_s \circ W(s) \cdot \eta\, dy \\
&\qquad = \int_{B_{r-2\varepsilon}(y_0)} \ell_s \circ W(s) \cdot \eta \circ \Phi(s) \circ W(s) \cdot \|D\Phi(s)\| \circ W(s) \|DW(s)\|\, dy
\end{aligned}$$

$$= \int_{W(s)(B_{r-2\varepsilon}(y_0))} \ell_s \cdot \eta\circ\Phi(s) \cdot \|D\Phi(s)\|\, dx$$
$$= \int_{W(\delta)(B_{r-\varepsilon/4}(y_0))} \ell_0 \cdot \eta\circ\Phi(0) \cdot \|D\Phi(0)\|\, dx + \mathrm{err}(s)$$

for any $\delta \leq s$, where $\mathrm{err}(s) \to 0$ in L^p_{loc} as $s \searrow 0$ since $D\Phi(s) \to D\Phi(0)$ in L^p_{loc} and $\Phi(s) \to \Phi(0)$ uniformly as $s \searrow 0$ and $\ell_s \to \ell_0$ in L^p_{loc} as $s \searrow 0$, and $\eta\circ\Phi(s)$ has compact support in $W(s)(B_{r-2\varepsilon}(y_0)) \subseteq W(\delta)(B_{r-\varepsilon/4}(y_0))$ for s, δ sufficiently small. Observe that $\Phi(0)$, $W(0)$ are homeomorphisms which are continuous representatives of $W^{1,p}_{\mathrm{loc}}$-functions with $p > n$, and so they both satisfy the *Lusin N-property* (see [25, Corollary B]), and hence the change of area formula is valid for $\Phi(0)$ and $W(0)$ (see [24, Proposition 1.1]):

$$\int_{W(\delta)(B_{r-\varepsilon/4}(y_0))} \ell_0 \cdot \eta\circ\Phi(0) \cdot \|D\Phi(0)\|\, dx + \mathrm{err}(s)$$
$$= \int_{\Phi(0)(W(\delta)(B_{r-\varepsilon/4}(y_0)))} \ell_0\circ W(0) \cdot \eta\, dy + \mathrm{err}(s) \to \int_{B_r} \ell_0\circ W(0) \cdot \eta\, dy$$

as $s \searrow 0$. As this is true for any continuous η and ball $B_r(y_0)$ of this type, we see that

$$\ell(s)\circ W(s) \to R = \ell_0\circ W(0)$$

almost everywhere and in L^p_{loc}, since $\ell(s)\circ W(s)$ converges in L^p_{loc} for $s \searrow 0$, for all $p \in [1, \infty)$,

$$\int \eta\ell(s)\circ W(s)\, dy \to \int \eta R\, dy = \int \eta\ell(0)\circ W(0)\, dy,$$

and hence

$$\int \eta(R - \ell(0)\circ W(0))\, dy = 0$$

for all non-negative cut-off functions of this type. Hence, using the *fundamental lemma of the calculus of variations*, we have,

$$R - \ell(0)\circ W(0) = 0$$

in $L^1(\hat{B})$, and hence $R = \ell(0)\circ W(0)$ almost everywhere in $B_r(y_0)$. Returning to the last identity in (9.1), we see that

$$g(0)_{\alpha\beta} = D_\alpha W^i(0) D_\beta W^j(0) \ell_{ij}(0)\circ W(0)$$

almost everywhere and in the L^p-sense.

An almost identical argument shows that $g_{\alpha\beta}(s)\circ\Phi(s)$ converges to $g_{\alpha\beta}(0)\circ\Phi(0)$ in L^p, as we now explain. Let $C = W(0)(B_r(y_0))$ and η a cut-off function with compact support in $W(0)(B_{r-4\varepsilon}(y_0))$. Then $\eta = 0$ outside of $W(s)(B_{r-2\varepsilon}(y_0))$ and $W(0)(B_{r-4\varepsilon}(y_0)) \subseteq W(s)(B_{r-2\varepsilon}(y_0))$ for sufficiently small $s > 0$. Hence,

$$\begin{aligned}
\int_C g(s)\circ\Phi(s)\eta\, dx &= \int_{W(0)(B_{r-4\varepsilon}(y_0))} g(s)\circ\Phi(s)\eta\, dx \\
&= \int_{W(s)(B_{r-2\varepsilon}(y_0))} g(s)\circ\Phi(s)\eta\, dx \\
&= \int_{W(s)(B_{r-2\varepsilon}(y_0))} g(s)\circ\Phi(s)\cdot \eta\circ W(s)\circ\Phi(s)\cdot \|DW(s)\|\circ\Phi(s)\|D\Phi(s)\|\, dx \\
&= \int_{B_{r-2\varepsilon}(y_0)} g(s)\cdot\eta\circ W(s)\cdot\|DW(s)\|\, dy \\
&= \int_{B_{r-2\varepsilon}(y_0)} g(0)\cdot\eta\circ W(0)\cdot\|DW(0)\|\, dy + \mathrm{err}(s) \\
&= \int_{W(0)(B_{r-2\varepsilon}(y_0))} g(0)\circ\Phi(0)\cdot\eta\, dx + \mathrm{err}(s) \to \int_C g(0)\circ\Phi(0)\eta\, dx
\end{aligned}$$

as $s \searrow 0$, where we have once again used that the change of variables formula is valid for $\Phi(0)$ and $W(0)$. Hence, since $g(s)\circ\Phi(s) \to Z$ in L^p_{loc}, we have

$$g(s)\circ\Phi(s) \to Z = g(0)\circ\Phi(0)$$

in $L^p(C)$ as $s \searrow 0$ for all $p \in [1,\infty)$, in view of the *fundamental lemma of the calculus of variations*. Hence,

$$g(s)\circ\Phi(s) \to g(0)\circ\Phi(0)$$

in L^p_{loc} as $s \searrow 0$ for all $p \in [1,\infty)$. Returning to the first identity in (9.1), we see that this implies

$$\ell_{ij}(0)(x) = D_i\Phi^\alpha(0)(x)D_j\Phi^\beta(0)(x)g_{\alpha\beta}(0)(\Phi(0)(x))$$

almost everywhere and in the L^p-sense. ■

10. Distance convergence in Sobolev spaces

Lemma 10.1. *In the following $\mathbb{B}^k_r(0)$ is a k-dimensional ball of radius $r > 0$ in $\mathbb{R}^k$ and middle point 0. Let $c > 1$ and $g\colon \mathbb{B}^n_2(0) \times [0,1] \to \mathbb{R}^{n\times n}$ be a family of non-negative two-tensors, such that*

$$\int_{\mathbb{B}^n_2(0)} |g(t)(z) - g(0)(z)|^2_\delta\, dz \le ct, \quad \frac{1}{c}\delta \le g(t) \le c\delta$$

for all $t \in [0,1)$, where $g(t)$ are smooth for all $t > 0$ and $g(0)$ is in L^2.

Let $\sigma\colon[-1,1]\to\mathbb{B}^n_2(0)$, $\sigma(s)=se_1$ *be a* Lebesgue line *with respect to* $g(0)$, *that is, the function*

$$\sqrt{g(0)_{11}(\cdot,\cdot)}\colon[-1,1]\times\mathbb{B}^{n-1}_1(0)\subseteq\mathbb{B}^n_2(0)\to\mathbb{R}^+_0$$

is measurable,

$$\sqrt{g(0)_{11}(s,0)}\colon\mathbb{B}^{n-1}_1(0)\to\mathbb{R}^+_0$$

is measurable,

$$\sqrt{g(0)_{11}(s,\cdot)}\colon\mathbb{B}^{n-1}_1(0)\to\mathbb{R}^+_0$$

is measurable for almost all $s\in[-1,1]$,

$$\sqrt{g(0)_{11}(\cdot,x)}\colon[-1,1]\to\mathbb{R}^+_0$$

is measurable for almost all $x\in\mathbb{B}^{n-1}_1(0)$, *and*

$$\begin{aligned}\int_{-1}^{1}\sqrt{g(0)_{11}(s,0)}\,ds &= \lim_{\alpha\to0}\frac{1}{\omega_{n-1}\alpha^{n-1}}\int_{-1}^{1}\int_{\mathbb{B}^{n-1}_\alpha(0)}\sqrt{g(0)_{11}(s,x)}\,dx\,ds\\ &= \lim_{\alpha\to0}\frac{1}{\omega_{n-1}\alpha^{n-1}}\int_{T_\alpha(\sigma)}\sqrt{g(0)_{11}(z)}\,dz,\end{aligned}$$

where dx *is an* $(n-1)$*-dimensional Lebesgue measure, and* dz *is an* n*-dimensional Lebesgue measure,*

$$T_\alpha(\sigma):=\left\{\sigma(s)+\beta(0,v)\mid s\in[-1,1],v\in\mathbb{R}^{n-1},|v|=1,\beta\in\mathbb{R},|\beta|\le\alpha\right\}$$

is an α *tubular neighbourhood of* σ, $\omega_{n-1}\alpha^{n-1}$ *is the (Lebesgue)* $(n-1)$*-dimensional volume of*

$$T_\alpha(\sigma(s)):=\left\{\sigma(s)+\beta(0,v)\mid v\in\mathbb{R}^{n-1},|v|=1,\beta\in\mathbb{R},|\beta|\le\alpha\right\}.$$

Then, for all $\varepsilon>0$, *there exists a* $t_0>0$ *such that*

$$L_{g_0}(\sigma):=\int_{-1}^{1}\sqrt{g(0)_{11}(s,0)}\ge(1-\varepsilon)\,d(g(t))(\sigma(-1),\sigma(1))$$

for all $t\in(0,t_0)$.

Proof. We calculate

$$\begin{aligned}L_{g_0}(\sigma)=\int_{-1}^{1}\sqrt{g(0)_{11}(s,0)}\,ds&\ge\frac{\int_{y\in T_\alpha(\sigma)}\sqrt{g(0)_{11}(y)}\,dy}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))}-R(\alpha)\\ =\frac{\int_{y\in T_\alpha(\sigma)}\left(\sqrt{g(0)_{11}(y)}-\sqrt{g(t)_{11}(y)}\right)+\sqrt{g(t)_{11}(y)}\,dy}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))}&-R(\alpha)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\int_{y\in T_\alpha(\sigma)} -|\sqrt{g(0)_{11}(y)} - \sqrt{g(t)_{11}(y)}| + \sqrt{g(t)_{11}(y)}\, dy}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} - R(\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad [\text{using } |\sqrt{a}-\sqrt{b}| \leq \sqrt{|a-b|} \text{ for } a,b\in\mathbb{R}^+] \\
&\geq \frac{\int_{y\in T_\alpha(\sigma)} -\sqrt{|g(0)_{11}(y) - g(t)_{11}(y)|} + \sqrt{g(t)_{11}(y)}\, dy}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} - R(\alpha) \\
&\geq -\frac{\left(\int_{y\in T_\alpha(\sigma)} |g(0)_{11}(y) - g(t)_{11}(y)|^2\, dy\right)^{1/4}}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} (2\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0)))^{3/4} \\
&\quad + \frac{\int_{y\in T_\alpha(\sigma)} \sqrt{g(t)_{11}(y)}\, dy}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} - R(\alpha) \\
&\geq \frac{-2ct^{1/4}(\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0)))^{3/4}}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} + \frac{\int_{x\in\mathbb{B}^{n-1}_\alpha(0)} \int_{-1}^{1} \sqrt{g(t)_{11}(s,x)}\, ds\, dx}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} - R(\alpha) \\
&\geq \frac{-ct^{1/4}}{(\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0)))^{1/4}} + \frac{\int_{x\in\mathbb{B}^{n-1}_\alpha(0)} L_{g(t)}(\sigma_x)\, dx}{\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0))} - R(\alpha) \\
&\geq \frac{-ct^{1/4}}{(\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0)))^{1/4}} + \inf_{x\in\mathbb{B}^{n-1}_\alpha(0)} L_{g(t)}(\sigma_x) - R(\alpha) \\
&\geq \frac{-ct^{1/4}}{(\mathrm{Vol}(\mathbb{B}^{n-1}_\alpha(0)))^{1/4}} + \inf\left\{d_t((-1,x),(1,x)) \mid x \in \overline{\mathbb{B}^{n-1}_\alpha(0)}\right\} - R(\alpha), \quad (10.1)
\end{aligned}
$$

where $R(\alpha)$ is independent of t and $R(\alpha) \to 0$ as $\alpha \searrow 0$, and $\sigma_x\colon [-a,a] \to \mathbb{R}^n$ is $\sigma_x(s) = (s,x)$. Now using the equivalence of $g(t)$ to δ with the constant c, we see that

$$
\begin{aligned}
&\inf\{d_t((-1,x),(1,x)) \mid x \in \overline{\mathbb{B}^{n-1}_\alpha(0)}\} \\
&\quad \geq d_t((-1,0),(1,0)) - \sup_{x\in\overline{\mathbb{B}^{n-1}_\alpha(0)}} d_t((-1,0),(-1,x)) - \sup_{x\in\overline{\mathbb{B}^{n-1}_\alpha(0)}} d_t((1,0),(1,x)) \\
&\quad \geq d_t((-1,0),(1,0)) - c|\alpha|
\end{aligned}
$$

independently of t, and hence choosing $t = \alpha^{200n}$ and $\alpha = \alpha(\varepsilon)$ small enough, we see that

$$L_{g_0}(\sigma) \geq (1-\varepsilon) d_t((-1,0),(1,0))$$

in view of (10.1), and the fact that

$$d_t((-1,0),(1,0)) \geq \frac{1}{\sqrt{c}} > 0$$

(since $g(t) \geq \frac{1}{c}\delta$). ∎

Lemma 10.2. *For all $\varepsilon > 0$, there exists $\alpha > 0$ such that if g is an L^2-metric on $\mathbb{B}_1(0) \subseteq \mathbb{R}^n$, the standard ball of radius one and middle point 0 in $\mathbb{R}^n$, with*

$$\int_{\mathbb{B}_1(0)} |g - \delta|^2 \leq \alpha,$$

then there exists an $x \in \mathbb{B}_\varepsilon^{n-1}(0)$ such that

$$\sqrt{g_{11}}(\cdot, x) \colon \Big[-\frac{1}{2}, \frac{1}{2}\Big] \to \mathbb{R}^n$$

is measurable, $\sigma\colon [-1/2, 1/2] \to \mathbb{R}^n$, $\sigma(t) = x + e_1 t$ is a Lebesgue line between $(-1/2, x)$ and $(1/2, x)$ and

$$\int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)}\, ds \leq 1 + \varepsilon = d_\delta\Big(\Big(-\frac{1}{2}, x\Big), \Big(\frac{1}{2}, x\Big)\Big) + \varepsilon.$$

Proof. Fubini's theorem tells us that the function

$$f\colon \mathbb{B}_{1/4}^{n-1}(0) \to \mathbb{R}_0^+, \quad f(x) := \int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)}\, ds$$

is well defined for almost all $x \in \mathbb{B}_{1/4}^{n-1}(0)$ and defines an L^1-function, also denoted by f, and the function

$$\hat{f}\colon [-1,1] \to \mathbb{R}_0^+, \quad \hat{f}(s) := \int_{\mathbb{B}_{1/4}^{n-1}(0)} \sqrt{g_{11}(s,x)}\, dx$$

is well defined for almost all $s \in [-1/2, 1/2]$ and defines an L^1-function, and

$$\begin{aligned} \int_{\mathbb{B}_{1/4}^{n-1}(0)} f(x)\, dx &= \int_{-1/2}^{1/2} \int_{\mathbb{B}_{1/4}^{n-1}(0)} \sqrt{g_{11}(s,x)}\, dx\, ds \\ &= \int_{\mathbb{B}_{1/4}^{n-1}(0)} \int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)}\, ds\, dx = \int_{\mathbb{B}_{1/4}^{n-1}(0) \times [-1/2,1/2]} \sqrt{g_{11}(z)}\, dz. \end{aligned}$$

This also implies that almost every $x \in \mathbb{B}_{1/4}^{n-1}(0)$ is a Lebesgue point of f, that is

$$\frac{\int_{\mathbb{B}_r^{n-1}(x)} |f(y) - f(x)|\, dy}{\omega_{n-1} r^{n-1}} \to 0$$

as $r \searrow 0$ for almost every $x \in \mathbb{B}_{1/4}^{n-1}(0)$ (see [13, Corollary 1, Section 1.7]), and as a consequence

$$\frac{\int_{\mathbb{B}_r^{n-1}(x)} f(y)\, dy}{\omega_{n-1} r^{n-1}} \to f(x)$$

as $r \searrow 0$ for almost every $x \in \mathbb{B}_{1/4}^{n-1}(0)$. That is, almost every curve

$$v_x\colon [-1/2, 1/2] \to \mathbb{B}_1(0), \quad v_x(s) := (s, x)$$

for $x \in \mathbb{B}_{1/4}^{n-1}(0)$ is a parametrised Lebesgue line.

We wish to estimate the measure m of the set $Z \subseteq \mathbb{B}^{n-1}_{1/4}(0)$ of $x \in \mathbb{B}^{n-1}_{1/4}(0)$ such that

$$\int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)}\, ds$$

is well defined and for which

$$\int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)}\, ds \geq 1 + \widehat{\alpha},$$

where $\widehat{\alpha} := (\alpha)^{1/8}$ ($\to 0$ as $\alpha \to 0$). We will see that $m \leq a(n)\alpha^{1/8}$. Using $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$ for $a, b \in \mathbb{R}^+$, we see that

$$\begin{aligned}
\alpha &\geq \int_{\mathbb{B}^{n-1}_{1/4}(0)} \int_{-1/2}^{1/2} |g_{11}(s,x) - 1|^2\, ds\, dx \\
&= \int_{\mathbb{B}^{n-1}_{1/4}(0)} \int_{-1/2}^{1/2} \big(\sqrt{|g_{11}(s,x) - 1|}\big)^4\, ds\, dx \\
&\geq \int_{\mathbb{B}^{n-1}_{1/4}(0)} \int_{-1/2}^{1/2} |\sqrt{g_{11}(s,x)} - 1|^4\, ds\, dx \\
&\geq \frac{\big(\int_{\mathbb{B}^{n-1}_{1/4}(0)} \int_{-1/2}^{1/2} |\sqrt{g_{11}(s,x)} - 1|\, ds\, dx\big)^4}{\big(\mathrm{Vol}(\mathbb{B}^{n-1}_{1/4}(0) \times [-1/2, 1/2])\big)^3} \\
&= \frac{1}{c(n)} \left(\int_{\mathbb{B}^{n-1}_{1/4}(0)} \int_{-1/2}^{1/2} |\sqrt{g_{11}(s,x)} - 1|\, ds\, dx \right)^4 \\
&\geq \frac{1}{c(n)} \left(\int_{\mathbb{B}^{n-1}_{1/4}(0)} \left| \int_{-1/2}^{1/2} (\sqrt{g_{11}(s,x)} - 1)\, ds \right| dx \right)^4 \\
&\geq \frac{1}{c(n)} \left(\int_{Z} \left| \int_{-1/2}^{1/2} (\sqrt{g_{11}(s,x)} - 1)\, ds \right| dx \right)^4 \\
&= \frac{1}{c(n)} \left(\int_{Z} \left| \int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)} ds - 1 \right| dx \right)^4 \\
&= \frac{1}{c(n)} \left(\int_{Z} \int_{-1/2}^{1/2} \sqrt{g_{11}(s,x)} ds - 1\, dx \right)^4 \\
&\geq \frac{1}{c(n)} \left(\int_{Z} \widehat{\alpha}\, dx \right)^4 \geq \frac{1}{c(n)} m^4 (\widehat{\alpha})^4,
\end{aligned}$$

which implies

$$m^4 \leq \frac{c(n)\alpha}{\widehat{\alpha}^4} = c(n)\alpha^{1/2},$$

that is,

$$m = \mathcal{L}^{n-1}(Z) \leq (c(n))^{1/4}\alpha^{1/8} \leq \alpha^{1/20}$$

for $\alpha \leq 1/(c(n))^4$. For $\varepsilon > 0$ given, we now choose $\alpha = \varepsilon^{100n}$, so that $m \leq \varepsilon^{5n}$. But then

$$\mathcal{L}^{n-1}(Z^c \cap \mathbb{B}_\varepsilon^{n-1}(0)) > 0.$$

Otherwise, $\mathcal{L}^{n-1}(Z^c \cap \mathbb{B}_\varepsilon^{n-1}(0)) = 0$, that is, $\mathcal{L}^{n-1}(Z \cap \mathbb{B}_\varepsilon^{n-1}(0)) = \omega_{n-1}\varepsilon^n$, and as a consequence

$$\varepsilon^{5n} \geq m = \mathcal{L}^{n-1}(Z) \geq \mathcal{L}^{n-1}(Z \cap \mathbb{B}_\varepsilon^{n-1}(0)) = \omega_{n-1}\varepsilon^n,$$

which is a contradiction.

Using this, with the fact that for almost every $x \in \mathbb{B}_\varepsilon^{n-1}(0)$ the curve

$$v_x : [-1/2, 1/2] \to \mathbb{B}_1(0), \quad v_x(s) := (s, x)$$

is a parametrised Lebesgue line, we see that it is possible to choose an $x \in \mathbb{B}_\varepsilon^{n-1}(0)$ such that

$$\int_{-1/2}^{1/2} \sqrt{g(0)_{11}(s,x)}\, ds \leq 1 + \hat{\alpha} = 1 + \varepsilon^{100n/8} \leq 1 + \varepsilon,$$

and so that v_x is a parametrised Lebesgue line, as claimed. ∎

11. Uniqueness

Lemma 11.1 (L^2-lemma, cf. proof of [10, Lemma 6.1]). *Let M be n-dimensional and g_1, g_2 be two smooth solutions on $M \times [0,T]$ to the h-Ricci–DeTurck flow, and let*

$$\ell := g_1 - g_2, \quad \tilde{\ell}^{ab} := \frac{1}{2}(g_1^{ab} + g_2^{ab}), \quad \hat{\ell}^{ab} := \frac{1}{2}(g_1^{ab} - g_2^{ab}).$$

Then the quantity $|\ell|_h^2$ satisfies the evolution equation:

$$\begin{aligned} \frac{\partial}{\partial t}|\ell|^2 = \tilde{\ell}^{abh}\nabla_a {}^h\nabla_b |\ell|^2 - 2|{}^h\nabla\ell|^2_{\tilde{\ell},h} + \ell * \hat{\ell} * {}^h\nabla^2(g_1 + g_2) \\ + *\ell * \hat{\ell} * g_1^{-1} * {}^h\nabla g_1 * {}^h\nabla g_1 + \ell * g_2^{-1} * \hat{\ell} * {}^h\nabla g_1 * {}^h\nabla g_1 \\ + \ell * g_2^{-1} * g_2^{-1} * {}^h\nabla\ell * {}^h\nabla g_1 + \ell * g_2^{-1} * g_2^{-1} * {}^h\nabla g_2 * {}^h\nabla\ell \\ + \ell * \hat{\ell} * g_1 * \mathrm{Rm}(h) + \ell * \ell * (g_2)^{-1} * \mathrm{Rm}(h), \end{aligned} \tag{11.1}$$

where $T * S$ refers to contractions of the tensors T and S involving h^{-1} and

$$|Z|^2_{g,h} = g^{ij} h^{ks} h^{vr} Z_{ikv} Z_{jsr}$$

for a zero-three tensor Z.

Proof. The formula was proved in [10] for the case that $h = \delta$ is the standard metric on a euclidean ball $B_1(0)$, and hence the curvature $\mathrm{Rm}(h) = 0$. We carry out a similar argument to the one given there, explaining why the term arising from the curvature $\mathrm{Rm}(h)$ in the evolution equation of $|\ell|^2$ in this setting can be written as

$$\ell * \widehat{\ell} * g_1 * \mathrm{Rm}(h) + \ell * \ell * (g_2)^{-1} * \mathrm{Rm}(h).$$

We then have

$$\begin{aligned}
\frac{\partial}{\partial t}\ell &= g_1^{ab\,h}\nabla_a{}^h\nabla_b g_1 + g_1^{-1} * g_1^{-1} * {}^h\nabla g_1 * {}^h\nabla g_1 \\
&\quad - (g_1)^{kl}(g_1)_{ip}h^{pq}R_{jkql}(h) - (g_1)^{kl}(g_1)_{jp}h^{pq}R_{ikql}(h) \\
&\quad - g_2^{ab\,h}\nabla_a{}^h\nabla_b g_2 - g_2^{-1} * g_2^{-1} * {}^h\nabla g_2 * {}^h\nabla g_2 \\
&\quad + (g_2)^{kl}(g_2)_{ip}h^{pq}R_{jkql}(h) + (g_2)^{kl}(g_2)_{jp}h^{pq}R_{ikql}(h) \\
&= \frac{1}{2}(g_1^{ab} + g_2^{ab})^h\nabla_a{}^h\nabla_b\ell + \frac{1}{2}(g_1^{ab} - g_2^{ab})^h\nabla_a{}^h\nabla_b(g_1 + g_2) \\
&\quad + (g_1^{-1} - g_2^{-1}) * g_1^{-1} * {}^h\nabla g_1 * {}^h\nabla g_1 + g_2^{-1} * (g_1^{-1} - g_2^{-1}) * {}^h\nabla g_1 * {}^h\nabla g_1 \\
&\quad + g_2^{-1} * g_2^{-1} * {}^h\nabla\ell * {}^h\nabla g_1 + g_2^{-1} * g_2^{-1} * {}^h\nabla g_2 * {}^h\nabla\ell, \\
&\quad - \widehat{\ell}^{kl}h^{pq}\,(g_1)_{ip}R_{ikql}(h) - \widehat{\ell}^{kl}h^{pq}\,(g_1)_{jp}R_{jkql}(h) \\
&\quad - (g_2)^{kl}\ell_{ip}h^{pq}R_{jkql}(h) + (g_2)^{kl}\ell_{jp}h^{pq}R_{ikql}(h),
\end{aligned}$$

which we can write as

$$\begin{aligned}
\frac{\partial}{\partial t}\ell &= \widetilde{\ell}^{ab\,h}\nabla_a{}^h\nabla_b\ell + \widehat{\ell}^{ab\,h}\nabla_a{}^h\nabla_b(g_1 + g_2) \\
&\quad + \widehat{\ell} * g_1^{-1} * {}^h\nabla g_1 * {}^h\nabla g_1 + g_2^{-1} * \widehat{\ell} * {}^h\nabla g_1 * {}^h\nabla g_1 \\
&\quad + g_2^{-1} * g_2^{-1} * {}^h\nabla\ell * {}^h\nabla g_1 + g_2^{-1} * g_2^{-1} * {}^h\nabla g_2 * {}^h\nabla\ell \\
&\quad + \widehat{\ell} * g_1 * \mathrm{Rm}(h) + \ell * (g_2)^{-1} * \mathrm{Rm}(h).
\end{aligned}$$

The formula now follows from this equality, combined with the facts that

$$\frac{\partial}{\partial t}|\ell|_h^2 = 2\Big(\ell, \frac{\partial}{\partial t}\ell\Big)_h, \quad 2(\ell, \widetilde{\ell}^{ab\,h}\nabla_a{}^h\nabla_b\ell)_h = \widetilde{\ell}^{ab\,h}\nabla_a{}^h\nabla_b|\ell|_h^2 - 2|{}^h\nabla\ell|_{\widetilde{\ell},h}^2. \quad \blacksquare$$

Using the previous evolution equation for the difference between two solutions of the h-Ricci–DeTurck flow, we are now able to show that the solution constructed in Theorem 6.1 is unique among all solutions satisfying (a_t), (b_t), and (c_t) with ε sufficiently small. The proof below slightly resembles the argument used by Struwe to prove a uniqueness result for the harmonic map flow in two dimensions, see the argument given in the proof of uniqueness in the proof of [34, Theorem 6.6, Chapter III].

Theorem 11.2. *Let M be four-dimensional and $g(t)$, $0 \le t \le S$, be a solution of the h-Ricci–DeTurck flow with initial condition $g_0 \in W^{2,2}_{\mathrm{loc}} \cap W^{1,\infty}_{\mathrm{loc}}$ satisfying* (a) *and* (b)*. Assume additionally that $g(t)$ satisfies the estimates* (a_t), (b_t), (c_t), *and* (d_t) *from Theorem* 6.3 *for all $0 \le t \le S$. Then there exists a time $T_{\max} = T_{\max}(n,a) \in (0,S)$, so that the solution is unique for all $0 \le t \le T_{\max}$.*

Proof. We let g_1, g_2 be two solutions and as above we let $l = g_1 - g_2$. Next we multiply (11.1) with η^4, where η is a cut-off function which is equal to one on $B_{1/2}(x)$ and zero outside of $B_{3/4}(x)$, and integrating by parts, with respect to ${}^h\nabla$, and using Young's and Hölder's inequalities, we obtain the estimate

$$
\begin{aligned}
&\partial_t \int_M \eta^4 |l|^2 + 2\int_M \eta^4 |{}^h\nabla l|^2_{\tilde{\ell},h} \\
&\quad \le C \int_M \eta^3 \big(\eta |l| |{}^h\nabla \tilde{l}| |{}^h\nabla l| + |{}^h\nabla \eta| |\tilde{l}| |l| |{}^h\nabla l|\big) + \int_M \eta^4 |{}^h\nabla l|^2_{\tilde{\ell},h} \\
&\qquad + C\Big(\int_M \eta^4 \big(|l|^4 + |\hat{l}|^4\big)\Big)^{1/2} \Big(\int_{B_1(x)} |{}^h\nabla^2 (g_1+g_2)|^2\Big)^{1/2} \\
&\qquad + C\Big(\int_M \eta^4 \big(|l|^4 + |\hat{l}|^4\big)\Big)^{1/2} \Big(\int_M \eta^4 |{}^h\nabla (g_1+g_2)|^4\Big)^{1/2} \\
&\qquad + C\int_M \eta^4 \big(|l|^2 + |\hat{l}|^2\big),
\end{aligned}
$$

where the term $2\int_M \eta^4 |{}^h\nabla l|^2_{\tilde{\ell},h}$ is the second term, up to a change of sign, appearing on the right-hand side of equation (11.1). Using the Sobolev embedding theorem (see Theorem B.1) and the assumption (b_t), it follows that

$$
\int_{B_1(x)} |{}^h\nabla^2 (g_1+g_2)|^2 + \Big(\int_M \eta^4 |{}^h\nabla (g_1+g_2)|^4\Big)^{1/2} \le C\varepsilon.
$$

Using the estimate $|\hat{l}| \le C|l|$ and again the Sobolev inequality, we conclude

$$
\Big(\int_M \eta^4 \big(|\hat{l}|^4 + |l|^4\big)\Big)^{1/2} \le C\int_{B_1(x)} \big(|{}^h\nabla l|^2 + |l|^2\big),
$$

and hence

$$
\begin{aligned}
\partial_t \int_M \eta^4 |l|^2 + \int_M \eta^4 |{}^h\nabla l|^2_{\tilde{\ell},h} &\le C\int_M \big(\eta^4 |l| |{}^h\nabla \tilde{l}| |{}^h\nabla l| + |{}^h\nabla \eta| \eta^3 |\tilde{l}| |l| |{}^h\nabla l|\big) \\
&\quad + C\sqrt{\varepsilon} \int_{B_1(x)} \big(|{}^h\nabla l|^2 + |l|^2\big) + C\int_M \eta^4 \big(|l|^2 + |\hat{l}|^2\big), \qquad (11.2)
\end{aligned}
$$

We will estimate the first two terms appearing on the right-hand side of (11.2). In preparation thereof, we first note that from (b_t), we also have

$$\Big(\int_M \eta^4\big(|{}^h\nabla l|^4 + |{}^h\nabla \tilde{l}|^4\big)\Big)^{1/2} \le C\varepsilon.$$

The first term $\int_M \eta^4(\eta|l||{}^h\nabla\tilde{l}||{}^h\nabla l|)$ on the right-hand side of (11.2) we estimate as follows:

$$\begin{aligned}\int_M \eta^4|l||{}^h\nabla\tilde{l}||{}^h\nabla l| &\le \frac{1}{4}\int_M \eta^4|{}^h\nabla l|^2_{\tilde{\ell},h} + C\int_M \eta^4|l|^2|{}^h\nabla\tilde{l}|^2\\ &\le \frac{1}{4}\int_M \eta^4|{}^h\nabla l|^2_{\tilde{\ell},h} + C\Big(\int_M \eta^4|l|^4\Big)^{1/2}\Big(\int_M \eta^4|{}^h\nabla\tilde{l}|^4\Big)^{1/2}\\ &\le \frac{1}{4}\int_M \eta^4|{}^h\nabla l|^2_{\tilde{\ell},h} + C\Big(\int_{B_1(x)} |{}^h\nabla l|^2 + |l|^2\Big)C\varepsilon. \qquad (11.3)\end{aligned}$$

The second term $\int_M |{}^h\nabla\eta|\eta^3|\tilde{l}||l||{}^h\nabla l|$ of (11.2) is estimated as follows:

$$\begin{aligned}\int_M |{}^h\nabla\eta|\eta^3|\tilde{l}||l||{}^h\nabla l| &\le \Big(\int_M \eta^4|\tilde{l}|^2|l|^2\Big)^{1/2}\Big(\int_M |{}^h\nabla\eta|\eta^2|{}^h\nabla l|^2\Big)^{1/2}\\ &\le C\Big(\int_M \eta^4\big(|\tilde{l}|^4 + |l|^4\big)\Big)^{1/2}\Big(\int_M \eta^4|{}^h\nabla l|^4\Big)^{1/4}\\ &\le C\Big(\int_{B_1(x)} |l|^2 + |{}^h\nabla l|^2\Big)(\varepsilon)^{1/4}. \qquad (11.4)\end{aligned}$$

Using (11.3) and (11.4), we can estimate the left-hand side of (11.2) by

$$\begin{aligned}&\partial_t\int_M \eta^4|l|^2 + \int_M \eta^4|{}^h\nabla l|^2_{\tilde{\ell},h}\\ &\quad\le C\Big(\int_M |l|^2 + |{}^h\nabla l|^2\Big)(\varepsilon)^{1/4} + C\int_M \eta^4|l|^2,\end{aligned}$$

and hence

$$\partial_t\int_M \eta^4|l|^2 + \frac{1}{ca}\int_M \eta^4|{}^h\nabla l|^2 \le C\int_{B_1(x)} |l|^2 + C\sqrt{\varepsilon}\int_{B_1(x)} |{}^h\nabla l|^2.$$

After integrating in time, we obtain for every $x \in M$,

$$\begin{aligned}&\int_{B_{1/2}(x)} |l|^2(t) + \frac{1}{ca}\int_0^t\int_{B_{1/2}(x)} |{}^h\nabla l|^2\,ds\\ &\quad\le C\int_0^t\int_{B_1(x)} |l|^2(s)\,ds + C(\varepsilon)^{1/4}\int_0^t\int_{B_1(x)} |{}^h\nabla l|^2\,ds. \qquad (11.5)\end{aligned}$$

Next we let $1 > \sigma > 0$ be arbitrary, and we conclude from Corollary 5.7 that

$$\sup_{x\in M}\int_{B_1(x)}|l|^2(t) < \sigma$$

for every $0 \le t \le C\sigma$, where C is a constant only depending on n and a. In the following we let $T_{\max}$ be the smallest time, so that

$$\sup_{x\in M}\int_{B_1(x)}|l|^2(T_{\max}) = \sigma.$$

For any $x \in M$, we can cover the ball $B_1(x)$ by finitely many balls $B_{1/2}(x_i)$, $1 \le i \le N = N(h)$ (see Section B). We conclude from (11.5) that for $t \le T_{\max}$,

$$\begin{aligned}
\int_{B_1(x)}|l|^2(t) &+ \frac{1}{ca}\int_0^t\int_{B_1(x)}|{}^h\nabla l|^2 \\
&\le N\sup_i\Big(C\int_0^t\int_{B_{1/2}(x_i)}|l|^2(t) + C\sqrt{\varepsilon}\int_0^t\int_{B_{1/2}(x_i)}|{}^h\nabla l|^2\Big) \\
&\le CNt\sigma + CN\varepsilon^{1/4}\sup_i\int_0^t\int_{B_1(x_i)}|{}^h\nabla l|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
\sup_{x\in M}&\Big(\int_{B_1(x)}|l|^2(t) + \frac{1}{ca}\int_0^t\int_{B_1(x)}|{}^h\nabla l|^2\Big) \\
&\le CNt\sigma + CN\varepsilon^{1/4}\sup_{x\in M}\int_0^t\int_{B_1(x)}|{}^h\nabla l|^2 \\
&\le CNt\sigma + \frac{1}{2}\sup_{x\in M}\Big(\int_{B_1(x)}|l|^2(t) + \frac{1}{ca}\int_0^t\int_{B_1(x)}|{}^h\nabla l|^2\Big)
\end{aligned}$$

if $\varepsilon > 0$ is sufficiently small. Hence,

$$\sup_{x\in M}\Big(\int_{B_1(x)}|l|^2(t) + \frac{1}{ca}\int_0^t\int_{B_1(x)}|{}^h\nabla l|^2\Big) \le 2CNt\sigma \le \frac{\sigma}{2}$$

for all $0 \le t \le 1/4CN$, which implies that $T_{\max} \ge 1/4CN$, and since $\sigma > 0$ was arbitrary this finishes the proof of the theorem. ∎

12. An application

Here we present an application for $W^{2,2} \cap L^\infty$-metrics on four-dimensional manifolds in the setting that scalar curvature is weakly bounded from below. For the case that the metric is C^0 we refer to the paper of [4] for related results.

Definition 12.1. Let M be a four-dimensional smooth closed manifold and g be a $W^{2,2} \cap L^\infty$-Riemannian metric (positive definite everywhere), such that $g, g^{-1} \in L^\infty$ and let $k \in \mathbb{R}$. Locally the scalar curvature may be written

$$\mathrm{R}(g) = g^{jk}\left(\partial_i \Gamma(g)^i_{jk} - \partial_j \Gamma^i_{ik} \Gamma^i_{ip} \Gamma^p_{jk} - \Gamma^i_{jp} \Gamma^p_{ik}\right),$$

where

$$\Gamma(g)^m_{ij} = \frac{1}{2} g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}),$$

and hence $\mathrm{R}(g)$ is well defined in the L^2-sense for a $W^{2,2}$ Riemannian metric. Let $k \in \mathbb{R}$. We say the scalar curvature $\mathrm{R}(g)$ is weakly bounded from below by k, $\mathrm{R}(g) \geq k$, if this is true almost everywhere, for all local smooth coordinates.

Theorem 12.2. *Let (M, h) be four-dimensional closed and satisfy (2.4). Assume that (M, g_0) is a $W^{2,2}$-metric such that $\frac{1}{a} h \leq g_0 \leq ah$ for some $\infty > a > 1$ and $\mathrm{R}(g_0) \geq k$ in the weak sense of Definition 12.1. Then the solution $g(t)_{t \in (0,T)}$ to the Ricci–DeTurck flow, respectively $\ell(t)_{t \in (0,T)}$ to the Ricci flow constructed in Theorem 8.3, with initial value $g(0) = g_0$, has $\mathrm{R}(g(t)) \geq k$ and $\mathrm{R}(\ell(t)) \geq k$ for all $t \in (0, T)$.*

Proof. The solution $g(t)$ to Ricci–DeTurck flow constructed in the main theorem is smooth for all $t > 0$ and satisfies $g(t) \to g_0$ in the $W^{2,2}$-sense and $\frac{1}{400a} h \leq g(t) \leq 400ah$ for all $t \in (0, T)$. Hence, $\mathrm{R}(g(t)) \to \mathrm{R}(g_0)$ as $t \searrow 0$ in the L^2_{loc}-sense, and in the pointwise sense almost everywhere, where $\mathrm{R}(g_0)$ is the L^2 quantity defined above (convergence of a sequence of functions in the L^2_{loc}-sense to an L^2_{loc}-function implies convergence of the sequence almost everywhere). This means $(\mathrm{R}(g(t)) + k)_- \to 0$ in the L^2-sense as $t \searrow 0$, and hence

$$\varphi(t) := \int_M (\mathrm{R}(g(t)) + k)_-^2 \, dg(t) = \int_M (\mathrm{R}(\ell(t)) + k)_-^2 \, d\ell(t) \to 0$$

as $t \searrow 0$.

The integrand $V(t) := (\mathrm{R}(\ell(t)) + k)_-^2$ is differentiable in space and time for all $t > 0$ and this yields that φ is differentiable in time for all $t > 0$. The derivative of V is zero for all $(x, t) \in M \times (0, T)$ with $\mathrm{R}(\ell)(x, t) + k \geq 0$.

By Sard's theorem (see [26, Section 2]), we know, for almost all k, that the sets

$$\{x \in M \mid \mathrm{R}(x, g(t)) + k < 0\}$$

have smooth boundary for almost every $t > 0$. Sard's theorem applied once to R yields that

$$W_k := \{(x, t) \in M \times (0, T) \mid \mathrm{R}(x, t) = -k\}$$

is smooth for almost all $k \in \mathbb{R}$, and then Sard's theorem applied to $\Psi_k\colon W_k \to \mathbb{R}$, $\Psi_k(x,t) = t$ (for such k) yields that

$$\{x \in M \mid \mathrm{R}(x,t) = -k\}$$

is smooth for almost all $t \in (0,T)$. Let $Z \subseteq \mathbb{R}$ denote the set of such $k \in \mathbb{R}$. For such $k \in Z$, we define

$$U_k(t) := \{x \in M \mid \mathrm{R}(x, g(t)) + k < 0\}$$

if $t \in (0,T)$ is a time such that $\{x \in M \mid \mathrm{R}(x,g(t)) + k < 0\}$ has smooth boundary, and we define

$$U_k(t) := \emptyset$$

for all other $t \in (0,T)$. Using the fundamental theorem of calculus for $0 < t_1 < t_2$, we compute

$$\begin{aligned}
&\psi(t_2) - \psi(t_1) = e^{-kt_2}\varphi(t_2) - e^{-kt_1}\varphi(t_1)\\
&= \int_{t_1}^{t_2} \frac{d}{d\tau}\int_M e^{-k\tau} V(\tau)\, d\ell(\tau)\, d\tau\\
&= \int_{t_1}^{t_2} e^{-k\tau}\int_M \Big(\frac{d}{d\tau}V(\tau) - \mathrm{R}(\tau)V(\tau) - kV(\tau)\Big)\, d\ell(\tau)\, d\tau\\
&= \int_{t_1}^{t_2}\int_{U_k(\tau)} e^{-k\tau}\Big(2\frac{\partial}{\partial\tau}(-\mathrm{R}(\tau)-k)(-\mathrm{R}(\tau)-k) - (\mathrm{R}(\tau)+k)^2\mathrm{R}(\tau)\Big)\, d\ell(\tau)\, d\tau\\
&\qquad - \int_{t_1}^{t_2}\int_{U_k(\tau)} ke^{-k\tau}V(\tau)\, d\ell(\tau)\, d\tau\\
&= \int_{t_1}^{t_2}\int_{U_k(\tau)} e^{-k\tau}\big(2\Delta_{\ell(\tau)}(\mathrm{R}(\tau)+k)\big)(\mathrm{R}(\tau)+k) + 4(\mathrm{R}(\tau)+k)|\mathrm{Rc}(\tau)|^2\\
&\qquad - \int_{t_1}^{t_2}\int_{U_k} e^{-k\tau}(\mathrm{R}(\tau)+k)^3\, d\ell(\tau)\, d\tau\\
&\qquad + k\int_{t_1}^{t_2}\int_{U_k(\tau)} e^{-k\tau}(\mathrm{R}(\tau)+k)^2 - k\int_{t_1}^{t_2}\int_{U_k(\tau)} e^{-k\tau}V(\tau)\, d\ell(\tau)\, d\tau\\
&\le \int_{t_1}^{t_2}\int_{U_k(\tau)} e^{-k\tau}\big(2\Delta_{\ell(\tau)}(\mathrm{R}(\tau)+k)\big)(\mathrm{R}(\tau)+k)\\
&\qquad - \int_{t_1}^{t_2}\int_{U_k(\tau)} e^{-k\tau}(\mathrm{R}(\tau)+k)^3\, d\ell(\tau)\, d\tau\\
&\qquad\qquad [\text{since } (\mathrm{R}(\tau)+k)(\tau) < 0 \text{ on } U_k(\tau) \text{ and } V = (\mathrm{R}(\tau)+k)^2]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} e^{-k\tau} \int_{U_k(\tau)} -2|\nabla(\mathrm{R}(\ell(\tau)) + k)|^2 \, d\ell(\tau)\, d\tau \\
&\quad + \int_{t_1}^{t_2} e^{-k\tau} \int_{\partial U_k(\tau)} (\mathrm{R}(\tau) + k)\ell(\tau)\big(\nu(\ell(\tau)), \nabla(\mathrm{R}(\tau) + k)\big)\, d\sigma(\tau)\, d\tau \\
&\quad + \int_{t_1}^{t_2} \int_{U_k(\tau)} e^{-k\tau}\big(|(\mathrm{R}(\tau) + k)_-|^3\big)\, d\ell(\tau)\, d\tau \\
&= \int_{t_1}^{t_2} \int_{U_k(\tau)} -2e^{-k\tau} |\nabla(\mathrm{R}(\tau) + k)_-|^2 \, d\ell(\tau)\, d\tau \\
&\quad + \int_{t_1}^{t_2} \int_{U_k(\tau)} e^{-k\tau}\big(|(\mathrm{R}(\tau) + k)_-|^3\big)\, d\ell(\tau)\, d\tau \\
&= \int_{t_1}^{t_2} -2e^{-k\tau} \int_M |\nabla(\mathrm{R}(\tau) + k)_-|^2 \, d\ell(\tau)\, d\tau + \int_{t_1}^{t_2} e^{-k\tau} \int_M |(\mathrm{R}(\tau) + k)_-|^3 \\
&\le -A(M,h) \int_{t_1}^{t_2} e^{-k\tau} \Big(\int_M |(\mathrm{R}(\tau) + k)_-|^4 \, d\ell(\tau)\Big)^{1/2} d\tau \\
&\quad + \int_{t_1}^{t_2} e^{-k\tau} \Big(\int_M |(\mathrm{R}(\tau) + k)_-|^4 \, d\ell(\tau)\Big)^{1/2} \Big(\int_M |(\mathrm{R}(\tau) + k)_-|^2 d\ell(\tau)\Big)^{1/2} d\tau \\
&\le \int_{t_1}^{t_2} e^{-k\tau} (-A(M) + A(M)/2) \Big(\int_M |(\mathrm{R}(\tau) + k)_-|^4 \, d\ell(\tau)\Big)^{1/2} d\tau \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{for sufficiently small } t_2] \\
&\le 0,
\end{aligned}
$$

where we have used the Sobolev inequality and $A(M)$ is the Sobolev constant, and we used that

$$\int_M |(\mathrm{R}(\tau) + k)_-|^2 \, d\ell(\tau) \le A(M)/2$$

for $\tau \le t_2$ and t_2 sufficiently small, since

$$\int_M |(\mathrm{R}(t) + k)_-|^2 \, d\ell(t) \to 0 = \int_M |(\mathrm{R}(t) + k)_-|^2 \, dg(t) \to 0$$

as $t \searrow 0$. Hence, since $\psi(0) = 0$, $\psi(t) = 0$ for all $t \in [0, T)$. That is, $\mathrm{R}(\ell(t)) \ge k$ for all $t \in (0, T)$ in the smooth sense. $\mathrm{R}(g(t)) \ge k$ for all $t \in (0, T)$ follows from the fact that $(M, \ell(t))$ and $(M, g(t))$ are isometric to one another. For general $k \in \mathbb{R}$, we can take a sequence $(k_i)_{i\in\mathbb{N}}$ with $k_i \to k$ and $k_i \in Z$. ∎

Remark 12.3. From this theorem we see that for a metric $g_0 \in L^\infty \cap W^{2,2}(M^4)$ with $\frac{1}{a}h \le g_0 \le ah$ for some positive constant $a > 0$: g_0 has scalar curvature $\ge k$ in the weak sense of Definition 12.1 if and only if there exists a sequence of smooth Riemannian metrics $g_{i,0}$ with $\frac{1}{b}h \le g_{i,0} \le bh$ for some $1 < b < \infty$ and $\mathrm{R}(g_{i,0}) \ge k$, and $g_{i,0} \to g_0 \in W^{2,2}(M^4)$ if and only if the Ricci–DeTurck flow of g_0 constructed in Theorem 6.5 has $\mathrm{R}(g(t)) \geq k$ for all $t \in (0,T)$. In particular, we do not need to change the constant form k to $k - 1/i$ after the first implication ($\Longrightarrow$).

A. Short-time existence of smooth bounded data

We present here a standard existence result for Ricci–DeTurck flows with smooth bounded initial data, based on the method of Shi [29].

Theorem A.1. *Let (M,h) be n-dimensional and satisfy (2.4). We assume there are constants $1 < a < \infty$ and $0 < c_j < \infty$ for all $j \in \mathbb{N}$, and g_0 is a smooth metric on M satisfying*

$$\frac{1}{a}h \leq g_0 \leq ah, \quad \sup_M |{}^h\nabla^j g_0| \leq c_j < \infty.$$

Then there exists a smooth solution $(M, g(t))_{t\in[0,\hat{T}]}$ to (1.1) for some $\hat{T} > 0$, and constants $b_j(g_0,h,S) < \infty$ for all $S \leq \hat{T}$ such that

$$\sup_M |{}^h\nabla^j g(\cdot,t)| \leq b_j(g_0,h,S) < \infty$$

for all $t \in [0,S]$.

Proof. We will construct a short time solution to (1.1), that is

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= g^{ab}({}^h\nabla_a {}^h\nabla_b g_{ij}) - g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) \\
&\quad + \frac{1}{2} g^{ab} g^{pq} \big({}^h\nabla_i g_{pa} {}^h\nabla_j g_{qb} + 2 {}^h\nabla_a g_{jp} {}^h\nabla_q g_{ib} - 2 {}^h\nabla_a g_{jp} {}^h\nabla_b g_{iq} \\
&\quad - 2 {}^h\nabla_j g_{pa} {}^h\nabla_b g_{iq} - 2 {}^h\nabla_i g_{pa} {}^h\nabla_b g_{jq}\big), \\
&= g^{ab}({}^h\nabla_a {}^h\nabla_b g_{ij}) + (g^{-1} * g * \mathrm{Rm}(h) * h)_{ij} \\
&\quad + (g^{-1} * g^{-1} * {}^h\nabla g * {}^h\nabla g)_{ij}
\end{aligned} \tag{A.1}$$

with $g(0) = g_0$. The method is essentially the one given in [29], with some minor modifications.

We choose radii $R(i) \to \infty$ such that $B_i = B_{R(i)}(p)$ have smooth boundary, and $M = \bigcup_{i=1}^\infty B_i$. For fixed $R = R(i) \geq 1$, we modify g_0 to $g_{0,R} = \eta g_0 + (1-\eta)h$, where η is a smooth cut-off function with $\eta = 0$ outside of $B_{R/2}(p)$ and $\eta = 1$ on $B_{R/4}(p)$, $|{}^h\nabla^k \eta|^2 \leq c(k,h)$ (see Theorem B.1 (iv) for the existence of η). We still have

$$\sup_M |{}^h\nabla^j g_{0,R}| \leq \hat{c}_j(c_1,\ldots,c_j,h,n,a) < \infty, \quad \frac{1}{a}h \leq g_{0,R} \leq ah \tag{A.2}$$

for some constants $0 < \hat{c}_j(c_1, \ldots, c_j, h, n, a) < \infty$, which do not depend on R. Equation (A.1) is strictly parabolic and h and $g_{0,R}$ are smooth, and so we obtain a smooth solution $g_R(t)_{t\in[0,T)}$ to the Dirichlet problem associated to (A.1) with $g_R(0) = g_{0,R}$ and

$$g_R(t)|_{\partial B_R(p)} = (g_{0,R})|_{\partial B_R(p)} = h|_{\partial B_R(p)}$$

for a $T = T(B_R, g_{0,R}, h) > 0$ using the methods of [29, Sections 3 and 4] (which in turn uses [19, Theorem 7.1, Section VII]). Using the argument of [29, Lemma 3.1], we see that as long as a smooth solution exists and $|g_R(t) - g_{0,R}|_h^2 \leq \varepsilon(g_0, a, h)$, then

$$\frac{1}{2a}h < g_R(t) < 2ah, \qquad \sup_{B_R(p)} |{}^h\nabla^j g_R(t)| \leq r(R, g_{0,R}, h, j, S) < \infty$$

for all $t \leq S$ for constants $r(R, g_{0,R}, h, j, S) < \infty$. On the other hand, as long as

$$|g_R(t) - g_{0,R}|_h^2 \leq \varepsilon(g_0, a, h) \leq 1,$$

we have (write $g(t)$ for $g_R(t)$ and g_0 for $g_{0,R}$ for ease of reading):

$$\begin{aligned}
\frac{\partial}{\partial t}|g(t) - g_0|_h^2 &= g^{ab}({}^h\nabla_a {}^h\nabla_b)|g(t) - g_0|_h^2 - 2|{}^h\nabla g|_{g,h}^2 \\
&\quad + 2h^{ij}h^{kl}g^{ab}({}^h\nabla_a {}^h\nabla_b g_0)_{ik}(g(t) - g_0)_{jl} \\
&\quad + 2h^{ij}h^{kl}(g(t) - g_0)_{ik}(g^{-1} * g * \mathrm{Rm}(h) * h)_{ij}h^{ij} \\
&\quad + h^{ij}h^{kl}(g(t) - g_0)_{ik}({}^h\nabla g * {}^h\nabla g * g^{-1} * g^{-1})_{jl} \\
&\leq g^{ab}({}^h\nabla_a {}^h\nabla_b)|g(t) - g_0|^2 - |{}^h\nabla g|_{g,h}^2 + c(\hat{c}_2, , a, n),
\end{aligned}$$

where $\hat{c}_2$ is the constant defined above in (A.2), and is independent of R. Hence,

$$|g_R(t) - g_{0,R}|^2 \leq c(\hat{c}_2, a, n)t \leq \varepsilon(g_0, a, h)$$

remains true for $t \leq \hat{T} := \varepsilon(g_0, a, h)/c(\hat{c}_2, a, n)$ in view of the maximum principle. Hence, we may extend the solution smoothly to time $\hat{T} := \varepsilon(g_0, h)/c(\hat{c}_2, a, n) \leq 1$. As long as

$$|g_R(t) - g_{0,R}|_h^2 \leq \varepsilon(g_0, a, h),$$

we also have, using the arguments of [29, Lemmata 4.1 and 4.2] and the fact that $\sup_M |{}^h\nabla^j g_{0,R}| \leq \hat{c}_j < \infty$ for constants $\hat{c}_j$ which do not depend on R, interior estimates

$$\sup_{B_1(x_0)\times[0,S]} |{}^h\nabla^m g_R|^2 \leq b_m = c(m, \hat{c}_1, \ldots, \hat{c}_m, a, S, h)$$

for all $x_0 \in B_{R/10}(p)$ and for all $S \leq \hat{T}$. Building the limit of the solutions $g_{R(i)}$ as $i \to \infty$, after taking a subsequence if necessary, we obtain a smooth solution $g(t)_{t\in[0,\hat{T}]}$

to (A.1) with $g(0) = g_0$, in view of the theorem of Arzelà–Ascoli and the fact that $R(i) \to \infty$ as $i \to \infty$, satisfying

$$\sup_{M\times[0,S]} |{}^h\nabla^m g|^2 \leq b_m$$

for all $S \leq \hat{T}$, as required. ■

B. Geometry lemmata

Lemma B.1. *Let (M^n, h) satisfy (2.4), (M, h) is a smooth, connected, complete Riemannian manifold, without boundary, satisfying*

$$\begin{gathered}\sup_M {}^h|{}^h\nabla^i \mathrm{Rm}(h)| < \infty \quad \textit{for all } i \in \mathbb{N}_0,\\ \sum_{i=0}^{4} \sup_M {}^h|{}^h\nabla^i \mathrm{Rm}(h)| \leq \delta_0(n), \quad \mathrm{inj}(M, h) \geq 100,\end{gathered} \tag{B.1}$$

where $\delta_0(n)$ is a sufficiently small constant. Then there exist constants $C_S(n) > 0$ and a constant $c_0(n)$ such that:

(i) *for any f which is smooth and whose support has diameter less than* 4*,*

$$\left(\int_M f^{2n/(n-2)}\, dh\right)^{(n-2)/n} \leq C_S(n) \int_M |{}^h\nabla f|^2\, dh$$

and

$$\left(\int_M f^n\, dh\right)^{1/2} \leq C_S(n) \int_M |{}^h\nabla f|^{n/2}\, dh,$$

(ii) *there exists a $c_0(n)$ such that any ball $B_2(x)$ of radius* 2 *can be covered by $c_0(n)$ balls,*

$$(B_{1/2}(y_i))_{i=1}^{c_0(n)}.$$

(iii) *there exists a covering of M, $(B_1(x_i))_{i=1}^{\infty}$, by balls of M such that for any $i \in \mathbb{N}$,*

$$\sharp\{j \in \mathbb{N} \mid x_j \in B_4(x_i)\} \leq c_0(n),$$

where $\sharp C$ denotes the number of elements in the set C, and is defined to be infinity if C has infinitely many elements,

(iv) *for every $R > 1$, $x_0 \in M$, there exists a cut-off function $\eta\colon M \to [0, 1] \subseteq \mathbb{R}$ such that $\eta = 1$ on $B_R(x_0)$, $\eta = 0$ on $M \setminus (B_{C(n)R}(x_0))$, $|{}^h\nabla^2(\eta)| + |{}^h\nabla\eta|^2/\eta \leq C(n)/R^2$ on M, and $|{}^h\nabla^k\eta| \leq c(k, h)$ on M for all $k \in \mathbb{N}$,*

(v) *letting $\varepsilon > 0$ be given, and T a smooth zero two tensor satisfying*

$$\int_{B_1(x)} |^h\nabla T|^{n/2} + |^h\nabla^2 T|^{n/2} \le \varepsilon$$

for all $x \in M$. Then

$$\left(\int_{B_1(x)} |^h\nabla T|^n\right)^{1/2} \le c(n)C_S(n)\varepsilon.$$

Remark B.2. If the conditions (B.1) are replaced by

$$\sup_M {}^h|^h\nabla^i \mathrm{Rm}(h)| < \infty \text{ for all } i \in \mathbb{N}_0, \quad \mathrm{inj}(M,h) > 0,$$

then scaled versions of the statements (i)–(v) hold, as we now explain. If we scale h by a large constant $c(h)$, we obtain a new metric which satisfies B.1, and hence (i)–(v) hold for this new metric. Scaling back, we obtained scaled versions of the statements (i)–(v). For example, part one of (i) would be replaced by: there exists an $r_0 > 0$, such that

$$\left(\int_M f^{2n/(n-2)}\, dh\right)^{(n-2)/n} \le C_S(n) \int_M |^h\nabla f|^2\, dh$$

for any f, which is smooth and whose support has diameter less than r_0.

Proof. We can always find local geodesic coordinates for any $p_0 \in M$ on the ball $B_{50}(p_0)$ such that in these coordinates $\frac{99}{100}\delta \le h \le \frac{101}{100}\delta$ if $\delta_0(n)$ is sufficiently small. This implies that the first two statements hold in these coordinates, and hence on the manifold.

The third statement is proved as follows. First we construct a *maximal* set of disjoint balls $(B_{1/2}(x_i))_{i=1}^\infty$ for M, maximal in the sense that any ball $B_{1/2}(p)$ for an arbitrary $p \in M$ must intersect one of these balls. This construction is carried out as follows: first choose disjoint balls

$$B_{1/2}(x_1), \ldots, B_{1/2}(x_{n(R)})$$

with centres in $B_R(p_0)$, such that any newly chosen ball $B_{1/2}(y)$ with $y \in B_R(p_0)$ intersects one of the balls $B_{1/2}(x_1), \ldots, B_{1/2}(x_{n(R)})$. In the next step, choose balls

$$B_{1/2}(x_{n(R)}), \ldots, B_{1/2}(x_{n(2R)}),$$

with centres in $B_{2R}(p_0)$ such that the collection $B_{1/2}(x_1), \ldots, B_{1/2}(x_{n(2R)})$, is disjoint, and any newly chosen ball $B_{1/2}(y)$ with $y \in B_{2R}(p_0)$ intersects one of the balls $B_{1/2}(x_1), \ldots, B_{1/2}(x_{n(2R)})$. Continuing in this way, we obtain a collection of disjoint balls $(B_{1/2}(x_i))_{i=1}^\infty$, which are maximal.

This then implies that $(B_1(x_i))_{i=1}^\infty$ covers M: if $y \in M$ satisfies $y \notin \bigcup_{i=1}^\infty B_1(x_i)$, then

$$B_{1/2}(y) \cap B_{1/2}(x_i) = \emptyset$$

for all $i \in \mathbb{N}$, which contradicts the maximality, and hence

$$y \in \bigcup_{i=1}^\infty B_1(x_i).$$

In geodesic coordinates $\varphi\colon B_{50}(x_i) \to \mathbb{B}_{50}(0)$, there can be at most $c_1(n)$ euclidean balls $\mathbb{B}_{1/4}(\widetilde{x}_{k(j)})_{j=1}^{c_1(n)}$, $\varphi(x_{k(j)}) = \widetilde{x}_{k(j)}$, which are disjoint, and contained in $\mathbb{B}_{40}(0)$. Hence, there are at most $c_1(n)$ points, $(x_{k(j)})_{j=1}^{c_1(n)}$, which are contained in $B_{30}(x_i)$, and this implies (iii).

Statement (iv) is proved with the help of an *exhaustion function*. For $R > 1$ given, let $\eta(x) := \widetilde{\eta}(f(x)/R)$, for a smooth cut-off function $\widetilde{\eta}\colon \mathbb{R} \to [0,1] \subseteq \mathbb{R}$ with $\widetilde{\eta}(x) = 1$ for $|x| \le 1$ and $\widetilde{\eta}(x) = 0$ for $|x| \ge 2$, where $f\colon M \to \mathbb{R}^+$ is a smooth so-called *exhaustion function*, satisfying

$$\frac{1}{C(n)} d(x,x_0) \le f(x) \le \frac{1}{2}(d(x,x_0)+1), \quad |{}^h\nabla f| \le C(n), \quad |{}^h\nabla^2 f| \le C(n),$$

the existence of which is, for example, guaranteed by Shi [28, Theorem 3.6]. By slightly modifying f on geodesic balls of radius 1, we can also achieve

$$|{}^h\nabla^k f| \le C(k,h).$$

Differentiating η we see that

$$\begin{aligned}\frac{|{}^h\nabla\eta|^2}{\eta} &+ |{}^h\nabla^2\eta| \\ &\le \frac{1}{R^2}\left(\frac{|{}^h\nabla\widetilde{\eta}|^2\circ f}{\widetilde{\eta}\circ f}|{}^h\nabla f|^2 + |{}^h\nabla\widetilde{\eta}|\circ f\,|{}^h\nabla^2 f| + |{}^h\nabla^2\widetilde{\eta}|\circ f\,|{}^h\nabla f|^2\right) \le \frac{C(n)}{R^2},\end{aligned}$$

as, without loss of generality,

$$\frac{|{}^h\nabla\widetilde{\eta}|^2}{\widetilde{\eta}} \le c$$

for some universal constant c. Similarly, $|{}^h\nabla^k\eta|^2 \le c(k,h)$. This finishes (iv).

We now prove (v). Let $\eta\colon M \to \mathbb{R}$ be a smooth cut-off function with $\eta = 1$ on $B_1(x)$ and $\eta = 0$ outside of $B_{4/3}(x)$, and $B_1(x_1),\ldots,B_1(x_{c_0(n)})$ a covering of $B_2(x)$, which exists in view of (ii). Then using Kato's and Young's inequalities, we see

$$\begin{aligned}\left(\int_{B_1(x)} |{}^h\nabla T|^n\right)^{1/2} &\le \left(\int_{B_2(x)} \left(|{}^h\nabla T|\eta\right)^n\right)^{1/2} \le C_S\int_{B_2(x)} \left|{}^h\nabla\left(\eta|{}^h\nabla T|\right)\right|^{n/2} \\ &\le c(n)C_S\int_{B_2(x)} |{}^h\nabla\eta|^{n/2}|{}^h\nabla T|^{n/2} + \eta^{n/2}|{}^h\nabla^2 T|^{n/2}\end{aligned}$$

$$\leq \sum_{i=1}^{c_0(n)} 2c(n)C_S \int_{B_1(x_i)} |{}^h\nabla T|^{n/2} + |{}^h\nabla^2 T|^{n/2}$$
$$\leq C_S c(n) c_0(n)\varepsilon,$$

as required. ∎

Lemma B.3. *Let (M^n, h) be a smooth, connected, complete Riemannian manifold, without boundary, satisfying*

$$\nu(3) := \sum_{i=1}^{3} \sup_M {}^h|{}^h\nabla^i \mathrm{Rm}(h)| < \infty \quad \textit{for all } \mathrm{inj}(M,h) \geq i_0 > 0,$$

and let g_0 be in $W^{2,n/2}_{\mathrm{loc}}$ and satisfy

$$\frac{1}{a} h \leq g_0 \leq ah.$$

Then for any $\varepsilon > 0$, $R > 1$, $x_0 \in M$, there exists an r_1 such that

$$\sup_{x \in B_R(x_0)} \int_{B_{r_1}(x)} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \varepsilon.$$

In the case that

$$\int_M \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \infty,$$

then for any $\varepsilon > 0$, there exists an r_1 such that

$$\sup_{x \in M} \int_{B_{r_1}(x)} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \varepsilon.$$

Proof. As the conclusion is a scale invariant conclusion, it suffices to prove it after scaling g_0 and h by the same constant. We scale g_0 and h once so that h satisfies (2.4), hence the statements Lemma B.1 (i)–(v) hold for the new metrics, which we also denote by g_0 and h.

Using the covering from (iii), we consider only those x_i with $x_i \in B_{2R}(x_0)$, $i = 1, \dots, C(n, R)$, and cut-off functions $\eta_i \colon M \to [0, 1] \subseteq \mathbb{R}$ with $\mathrm{supp}(\eta_i) \subseteq B_{\frac{3}{2}}(x_i)$, $\eta_i = 1$ on $B_1(x_i)$, $|{}^h\nabla \eta_i|^2 \leq c(n)\eta_i$, we see using the Sobolev inequality

$$\left(\int_{B_{2R}(x_0)} |{}^h\nabla g_0|^n\right)^{1/2} \leq \left(\sum_{i=1}^{\infty} \int_{B_1(x_i)} |{}^h\nabla g_0|^n\right)^{1/2} \leq \sum_{i=1}^{C(n,R)} \left(\int_{B_1(x_i)} |{}^h\nabla g_0|^n\right)^{1/2}$$
$$\leq \sum_{i=1}^{C(n,R)} \left(\int_M \left(\eta_i |{}^h\nabla g_0|\right)^n\right)^{1/2} \leq \sum_{i=1}^{C(n,R)} \int_M |{}^h\nabla\left(\eta_i |{}^h\nabla g_0|\right)|^{n/2}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{C(n,R)} \int_M c(a,n)|{}^h\nabla \eta_i|^{n/2}|{}^h\nabla g_0|^{n/2} + c(n,a)|\eta_i|^{n/2}|{}^h\nabla^2 g_0|^{n/2} \\
&\le c(n,a) \sum_{i=1}^{C(n,R)} \int_{B_2(x_i)} |{}^h\nabla g_0|^{n/2} + |{}^h\nabla^2 g_0|^{n/2} \\
&= c(n,a) \sum_{i=1}^{C(n)} \int_M \chi_{B_2(x_i)}\big(|{}^h\nabla g_0|^{n/2} + |{}^h\nabla^2 g_0|^{n/2}\big) \\
&= c(n,a) \int_M \Bigg(\sum_{i=1}^{C(n,R)} \chi_{B_2(x_i)}\Bigg)\big(|{}^h\nabla g_0|^{n/2} + |{}^h\nabla^2 g_0|^{n/2}\big) \\
&\le c(n,a)c_0(n)\Bigg(\int_{B_{2R}(x_0)} |{}^h\nabla g_0|^{n/2} + |{}^h\nabla^2 g_0|^{n/2}\Bigg) = K(n,a,R,x_0) < \infty,
\end{aligned}
$$

where we used

$$\sum_{i=1}^{\infty} \chi_{B_2(x_i)}(\cdot) \le c_0(n)$$

in the last inequality, which follows from (iii).

We claim: For any $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that

$$\int_{B_r(x)} |{}^h\nabla g_0|^n + \int_{B_r(x)} |{}^h\nabla^2 g_0|^{n/2} < \varepsilon$$

for all $x \in B_R(x_0)$, as we now show.

Assume there are points $x_i \in \overline{B_R(x_0)}$, $i \in \mathbb{N}$ and radii $r(i) > 0$, $r(i) \to 0$ as $i \to \infty$, such that

$$\int_{B_{r(i)}(x_i)} |{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2} \ge \varepsilon.$$

Taking a subsequence, we see that $x_i \to x$ as $i \to \infty$, and hence

$$\int_{B_\sigma(x)} |{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2} \le \frac{\varepsilon}{2}$$

for $\sigma > 0$ small enough. In view of the fact that

$$\int_{B_{2R(x_0)}} |{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2} < \infty$$

we have

$$f_j := \chi_{B_{\frac{1}{j}}(x)}|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2} \le g := |{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2},$$

is in L_1, $f_j \to 0$ almost everywhere as $j \to \infty$, and

$$\int_{B_{2R}(x_0)} g < \infty$$

implies

$$\int_{\chi_{B_{1/j}(x)}} |{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2} = \int_{B_{2R}(x_0)} f_j \to 0$$

in view of the dominated convergence theorem. But for i large enough,

$$B_{r(i)}(x_i) \subset B_\sigma(x),$$

which leads to a contradiction. Hence, there exists an $r > 0$ such that

$$\int_{B_r(x)} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \varepsilon$$

for all $x \in B_R(x_0)$.

In the case that

$$\int_M \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \infty,$$

choose $R > 0$ so that

$$\int_{(B_{R/10}(x_0))^c} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \frac{\varepsilon}{2}.$$

This implies

$$\int_{B_\sigma(x)} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \frac{\varepsilon}{2}$$

for all $x \in (B_{R/2}(x_0))^c$ and for any $0 < \sigma < 1$. Repeating the argument above, we find a $\sigma > 0$ such that

$$\int_{B_\sigma(x)} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \frac{\varepsilon}{2}$$

for all $x \in (B_R(x_0))$. Hence,

$$\int_{B_\sigma(x)} \left(|{}^h\nabla g_0|^n + |{}^h\nabla^2 g_0|^{n/2}\right) < \frac{\varepsilon}{2}$$

for all $x \in M$, as required. ∎

C. Estimates for ordinary differential equations

Lemma C.1. *Let $\varepsilon < 1$ and $f\colon [0,T] \to \mathbb{R}_0^+$, $Z\colon (0,1] \to \mathbb{R}_0^+$ be smooth, and satisfy*

$$f(0) = 0, \quad \frac{\partial}{\partial t} f(t) \leq \frac{\varepsilon}{t} f(t) + Z(t).$$

Then

$$f(t) \leq t^{\varepsilon} \lim_{t_0 \to 0} \int_{t_0}^{t} \frac{Z(s)}{s^{\varepsilon}}\, ds.$$

Proof. Note that $F(t) := t^{-\varepsilon} f(t)$ satisfies

$$\frac{\partial}{\partial t} F(t) \leq -\varepsilon t^{-1-\varepsilon} f(t) + t^{-\varepsilon} \frac{\varepsilon}{t} f(t) + t^{-\varepsilon} Z(t) \leq t^{-\varepsilon} Z(t). \tag{C.1}$$

Using that f is smooth, and hence $f(t) \leq Ct$ for small $t > 0$ and some constant C, we see

$$F(t) \leq C t^{-\varepsilon+1} \to 0$$

as $t \searrow 0$. Integrating (C.1) from $t_0 > 0$ to t, we see

$$F(t) \leq F(t_0) + \int_{t_0}^{t} \frac{Z(s)}{s^{\varepsilon}}\, ds \to \lim_{t_0 \to 0} \int_{t_0}^{t} \frac{Z(s)}{s^{\varepsilon}}\, ds$$

as $t_0 \searrow 0$, and hence, from the definition of $F(t)$,

$$f(t) \leq t^{\varepsilon} \lim_{t_0 \to 0} \int_{t_0}^{t} \frac{Z(s)}{s^{\varepsilon}}\, ds. \qquad \blacksquare$$

Lemma C.2. *Let $\varepsilon < 1$ and $f\colon [0,T] \to \mathbb{R}_0^+$, be smooth, and satisfy*

$$f(0) = 0, \quad \frac{\partial}{\partial t} f(t) \leq \frac{\varepsilon}{t} f(t) + c.$$

Then

$$f(t) \leq \frac{c}{1-\varepsilon} t.$$

Proof. For $Z(s) = c$, we have

$$\lim_{t_0 \to 0} \int_{t_0}^{t} \frac{Z(s)}{s^{\varepsilon}}\, ds = \lim_{t_0 \to 0} \int_{t_0}^{t} \frac{c}{s^{\varepsilon}}\, ds = c \frac{1}{1-\varepsilon} t^{1-\varepsilon},$$

and so

$$t^{\varepsilon} \lim_{t_0 \to 0} \int_{t_0}^{t} \frac{Z(s)}{s^{\varepsilon}}\, ds = t^{\varepsilon} \lim_{t_0 \to 0} \int_{t_0}^{t} \frac{c}{s^{\varepsilon}}\, ds = \frac{c}{1-\varepsilon} t. \qquad \blacksquare$$

D. Metric norm comparisons

We compare the norms of tensor with respect to different metrics.

Theorem D.1. *Let*

$$\ell = (\ell_{ij})_{i,j\in\{1,\dots,n\}}, \quad g = (g_{ij})_{i,j\in\{1,\dots,n\}}, \quad h = (h_{\alpha\beta})_{\alpha,\beta\in\{1,\dots,n\}}, (u_{\alpha\beta})_{\alpha,\beta\in\{1,\dots,n\}}$$

be positive definite symmetric matrices and

$$\begin{aligned}(\ell)^{-1} &= (\ell^{ij})_{i,j\in\{1,\dots,n\}}, & (g)^{-1} &= (g^{ij})_{i,j\in\{1,\dots,n\}},\\ (h)^{-1} &= (h^{\alpha\beta})_{\alpha,\beta\in\{1,\dots,n\}}, & (u^{-1}) &= (u^{\alpha\beta})_{\alpha,\beta\in\{1,\dots,n\}}\end{aligned}$$

the inverses thereof. Let $S=(S^i_\alpha)_{i,\alpha\in\{1,\dots,n\}}$, $T=(T_{ij})_{i,j\in\{1,\dots,n\}}$, $N=(N^{ij})_{i,j\in\{1,\dots,n\}}$ be matrices in $\mathbb{R}^{n\times n}$. Then the following estimates hold:

$$|S|^2_{h,\ell} := h^{\alpha\beta}(y)S^i_\alpha S^j_\beta \ell_{ij} \le c(n)|S|^2_{h,g}\big(1+|\ell|^2_g\big), \tag{D.1}$$

$$|S|^2_{h,\ell} \le c(n)|S|^2_{u,\ell}\big(1+|u|^2_h\big), \tag{D.2}$$

where $|\ell|^2_g = g^{ij}g^{kl}\ell_{ik}\ell_{jl} = |g^{-1}|^2_\ell$ and $|u|^2_h = h^{\alpha\gamma}h^{\beta\gamma}u_{\alpha\beta}u_{\gamma\sigma} = |h^{-1}|^2_u$, and

$$\begin{aligned}|T|^2_g &:= g^{ik}g^{jl}T_{ij}T_{kl} \le c(n)|T|^2_\ell|\ell|^2_g,\\ |N|^2_g &:= g_{ik}g_{jl}N^{ij}N^{kl} \le c(n)|N|^2_\ell|g|^2_\ell,\\ \frac{\det(g)}{\det(\ell)} &\le |g|^n_\ell,\end{aligned}$$

where $|g|^2_\ell = (\ell^{ij}\ell^{kl}g_{ik}g_{jl})$.

Proof. We regard g,ℓ as positive definite symmetric linear maps from $V\otimes V$ to $\mathbb{R}$, where $V=\mathbb{R}^n$ and h,u as positive definite symmetric linear maps from $Y\otimes Y$ to $\mathbb{R}$ for another copy of $Y := \mathbb{R}^n$. Then

$$\begin{gathered} g,\ell\colon V\otimes V\to\mathbb{R}, \quad h,u\colon Y\otimes Y\to\mathbb{R},\\ g(v^ie_i, v^je_j) = v^iv^jg_{ij}, \quad \ell(w^ie_i,w^je_j) = w^iw^jg_{ij},\\ h(z^\alpha e_\alpha, z^\beta e_\beta) = z^\alpha z^\beta h_{\alpha\beta}, \quad u(z^\alpha e_\alpha, z^\beta e_\beta) = z^\alpha z^\beta u_{\alpha\beta},\end{gathered}$$

and we regard S, T, and N as linear maps

$$S\colon Y^*\times V\to\mathbb{R},\ \ T\colon V\times V\to\mathbb{R},\ \ N\colon V^*\times V^*\to\mathbb{R},\ \ S(w_\alpha e^\alpha, v^ie_i) = S^\alpha_i w_\alpha v^i.$$

From the theory of tensors, $|S|^2_{h,\ell}$, $|S|^2_{u,\ell}$, $|\ell^{-1}|^2_g$, $|T|^2_g$, $\det(g)/\det(\ell)$, $|N|^2_\ell$, $|g|^2_\ell$, etc. are all quantities which are independent of coordinates. If $(\widetilde{e}_i)_{i\in\{1,\dots,n\}}$, $(\widehat{e}_\alpha)_{\alpha\in\{1,\dots,n\}}$ are bases, and

$$\widetilde{\ell}_{ij} = \ell(\widetilde{e}_i,\widetilde{e}_j), \quad \widetilde{g}_{ij} = g(\widetilde{e}_i,\widetilde{e}_j), \quad \widehat{h}_{\alpha\beta} = h(\widehat{e}_\alpha,\widehat{e}_\beta), \quad \widehat{u}_{\alpha\beta} = u(\widehat{e}_\alpha,\widehat{e}_\beta)$$

with inverses given by $\widetilde{\ell}^{ij}$, $\widetilde{g}^{ij}$, $\widehat{h}^{\alpha\beta}$, $\widehat{u}^{\alpha\beta}$, then the quantities defined above as $|S|^2_{\ell,h}$, $|S|^2_{u,\ell}$, $|T|^2_g$, $\det(g)/\det(\ell)$, $|\ell^{-1}|^2_g$ calculated using

$$\widetilde{g}_{ij},\ \ \widetilde{g}^{ij},\ \ \widehat{h}_{\alpha\beta},\ \ \widehat{u}_{\alpha\beta},\ \ \widehat{h}^{\alpha\beta},\ \ \widehat{u}^{\alpha\beta}\widetilde{\ell}_{ij},\ \ \widetilde{\ell}^{ij},\ \ \widetilde{T}_{ij},\ \ \widetilde{\widetilde{S}}{}^{i}_{\alpha}$$

in place of

$$g_{ij},\ \ g^{ij},\ \ h_{\alpha\beta},\ \ h^{\alpha\beta},\ \ \ell_{ij},\ \ \ell^{ij},\ \ T_{ij},\ \ S^i_\alpha,$$

then the result is the same; see, for example, [15].

We can always choose a basis for Y such that $\widehat{h}_{\alpha\beta} = \delta_{\alpha\beta}$, $\widehat{u}_{\alpha\beta} = r_\alpha\delta_{\alpha\beta}$ and a basis for V such that $\widetilde{g}_{ij} = \lambda_i\delta_{ij}$, $\widetilde{\ell}_{ij} = \sigma_i\delta_{ij}$. That is, without loss of generality, we have $h_{\alpha\beta} = \delta_{\alpha\beta}$, $u_{\alpha\beta} = r_\alpha\delta_{\alpha\beta}$ and $g_{ij} = \lambda_i\delta_{ij}$, $\ell_{ij} = \sigma_i\delta_{ij}$. Then

$$\begin{aligned}
|T|^2_g &= \Big(\sum_{i,j=1}^n \frac{1}{\lambda_i}\frac{1}{\lambda_j}T_{ij}T_{ij}\Big) = \Big(\sum_{i,j=1}^n \frac{1}{\sigma_i}\frac{1}{\sigma_j}\frac{\sigma_i}{\lambda_i}\frac{\sigma_j}{\lambda_j}T_{ij}T_{ij}\Big)\\
&\le c(n)\Big(\sup_{i,j\in\{1,\dots,n\}}\frac{1}{\sigma_i}\frac{1}{\sigma_j}T_{ij}T_{ij}\Big)\Big(\sup_{i\in\{1,\dots,n\}}\frac{\sigma_i^2}{\lambda_i^2}\Big)\\
&\le c(n)\Big(\sum_{i,j=1}^n \frac{1}{\sigma_i}\frac{1}{\sigma_j}T_{ij}T_{ij}\Big)\Big(\sum_{i=1}^n\frac{\sigma_i^2}{\lambda_i^2}\Big)\\
&= c(n)|T|^2_\ell(g^{ij}g^{kl}\ell_{ik}\ell_{jl}) = c(n)|T|^2_\ell|\ell|^2_g
\end{aligned}$$

and

$$\begin{aligned}
|N|^2_g &= \Big(\sum_{i,j=1}^n \lambda_i\lambda_j N^{ij}N^{ij}\Big) = \Big(\sum_{i,j=1}^n \sigma_i\sigma_j\frac{\lambda_i}{\sigma_i}\frac{\lambda_j}{\sigma_j}N^{ij}N^{ij}\Big)\\
&\le c(n)\Big(\sup_{i,j\in\{1,\dots,n\}}\sigma_i\sigma_j N^{ij}N^{ij}\Big)\Big(\sup_{i\in\{1,\dots,n\}}\frac{\lambda_i^2}{\sigma_i^2}\Big)\\
&\le c(n)\Big(\sum_{i,j=1}^n \sigma_i\sigma_j N^{ij}N^{ij}\Big)\Big(\sum_{i=1}^n\frac{\lambda_i^2}{\sigma_i^2}\Big)\\
&= c(n)|N|^2_\ell(g_{ij}g_{kl}\ell^{ik}\ell^{jl}) = c(n)|N|^2_\ell|g|^2_\ell.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\det(g)}{\det(\ell)} &= \frac{\lambda_1\lambda_2\dots\lambda_n}{\sigma_1\sigma_2\dots\sigma_n} = \frac{\lambda_1}{\sigma_1}\cdot\frac{\lambda_2}{\sigma_2}\cdots\frac{\lambda_n}{\sigma_n}\\
&\le\Big(\sup_{i\in\{1\dots n\}}\frac{\lambda_i^2}{\sigma_i^2}\Big)^{n/2} \le \Big(\sum_{i=1}^n\frac{\lambda_i^2}{\sigma_i^2}\Big)^{n/2} = |g|^n_\ell
\end{aligned}$$

and

$$
\begin{aligned}
|S|^2_{h,\ell} &:= h^{\alpha\beta}(y) S^i_\alpha S^j_\beta \ell_{ij} = \sum_{\alpha,i=1}^{n} S^i_\alpha S^i_\alpha \sigma_i = \sum_{\alpha,i=1}^{n} S^i_\alpha S^i_\alpha \lambda_i \frac{\sigma_i}{\lambda_i} \\
&\le c(n)\Big(\sup_{\alpha,i\in\{1,\dots,n\}} S^i_\alpha S^i_\alpha \lambda_i\Big) \sup_{i\in\{1,\dots,n\}} \frac{\sigma_i}{\lambda_i} \\
&\le c(n)\Big(\sup_{\alpha,i\in\{1,\dots,n\}} S^i_\alpha S^i_\alpha \lambda_i\Big)\Big(1+\sup_{i\in\{1,\dots,n\}} \frac{\sigma_i^2}{\lambda_i^2}\Big) \\
&\le c(n)\Big(\sum_{\alpha,i=1}^{n} S^i_\alpha S^i_\alpha \lambda_i\Big)\Big(1+\sum_{i=1}^{n} \frac{\sigma_i^2}{\lambda_i^2}\Big) \\
&= c(n)|S|^2_{h,g}\Big(1+\sum_{i=1}^{n} \frac{\sigma_i^2}{\lambda_i^2}\Big) = c(n)|S|^2_{h,g}\big(1+|\ell|^2_g\big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|S|^2_{h,\ell} &:= h^{\alpha\beta}(y) S^i_\alpha S^j_\beta \ell_{ij} = \sum_{\alpha,i=1}^{n} S^i_\alpha S^i_\alpha \sigma_i = \sum_{\alpha,i=1}^{n} r_\alpha\Big(\frac{1}{r_\alpha} S^i_\alpha S^i_\alpha \sigma_i\Big) \\
&\le c(n)\Big(\sup_{\alpha\in\{1,\dots,n\}} r_\alpha\Big)\Big(\sup_{\alpha,i\in\{1,\dots,n\}} \frac{1}{r_\alpha} S^i_\alpha S^i_\alpha \sigma_i\Big) \\
&\le c(n)\Big(\sum_{\alpha=1}^{n} r_\alpha\Big)\Big(\sum_{\alpha,i=1}^{n} \frac{1}{r_\alpha} S^i_\alpha S^i_\alpha \sigma_i\Big) = c(n)\Big(\sum_{\alpha=1}^{n} r_\alpha\Big)|S|^2_{u,\ell} \\
&\le c(n)\Big(1+\sum_{\alpha=1}^{n} r_\alpha^2\Big)|S|^2_{u,\ell} \le c(n)\big(1+|u|^2_h\big)|S|^2_{u,\ell}. \qquad \blacksquare
\end{aligned}
$$

Corollary D.2. *Let $T = (T_{ij})$, respectively $N = (N^{ij})$, be a zero-two, respectively two-zero, tensor defined on a manifold Ω, and g, ℓ-metrics on Ω. Then for all $p \in [1,\infty)$ there exists a $c(n,p)$ such that*

$$
\begin{aligned}
\int_\Omega |T|^p_g\, dg &\le c(n,p)\Big(\int_\Omega |\ell|^{2p}_g\, dg\Big)^{1/2}\Big(\int_\Omega |T|^{4p}_\ell\, d\ell\Big)^{1/4}\Big(\int_\Omega |g|^{n/2}_\ell\, dg\Big)^{1/4}, \\
\int_\Omega |N|^p_g\, dg &\le c(n,p)\Big(\int_\Omega |g|^{2p}_\ell\, dg\Big)^{1/2}\Big(\int_\Omega |N|^{4p}_\ell\, d\ell\Big)^{1/4}\Big(\int_\Omega |g|^{n/2}_\ell\, dg\Big)^{1/4}.
\end{aligned}
$$

Proof. In the following, dg/dl is the well-defined function on Ω given locally by $dg/dl(x) = \sqrt{\det(g(x))}/\sqrt{\det(l(x))}$. We have

$$
\begin{aligned}
\int_\Omega |T|^p_g\, dg &\le c(n,p)\int_\Omega |\ell|^p_g |T|^p_\ell\, dg \\
&\le c(n,p)\Big(\int_\Omega |\ell|^{2p}_g\, dg\Big)^{1/2}\Big(\int_\Omega |T|^{2p}_\ell\, dg\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= c(n,p)\Big(\int_\Omega |\ell|_g^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |T|_\ell^{2p}\frac{dg}{d\ell}\,d\ell\Big)^{1/2}\\
&\le c(n,p)\Big(\int_\Omega |\ell|_g^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |T|_\ell^{4p}\,d\ell\Big)^{1/4}\Big(\int_\Omega \Big(\frac{dg}{d\ell}\Big)^2\,d\ell\Big)^{1/4}\\
&\le c(n,p)\Big(\int_\Omega |\ell|_g^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |T|_\ell^{4p}\,d\ell\Big)^{1/4}\Big(\int_\Omega \frac{dg}{d\ell}\,dg\Big)^{1/4}\\
&\le c(n,p)\Big(\int_\Omega |\ell|_g^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |T|_\ell^{4p}\,d\ell\Big)^{1/4}\Big(\int_\Omega |g|_\ell^{n/2}\,dg\Big)^{1/4},
\end{aligned}
$$

and its analog:

$$
\begin{aligned}
\int_\Omega |N|_g^p\,dg &\le c(n,p)\int_\Omega |g|_\ell^p|N|_\ell^p\,dg\\
&\le c(n,p)\Big(\int_\Omega |g|_\ell^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |N|_\ell^{2p}\,dg\Big)^{1/2}\\
&= c(n,p)\Big(\int_\Omega |g|_\ell^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |N|_\ell^{2p}\frac{dg}{d\ell}\,d\ell\Big)^{1/2}\\
&\le c(n,p)\Big(\int_\Omega |g|_\ell^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |N|_\ell^{4p}\,d\ell\Big)^{1/4}\Big(\int_\Omega \Big(\frac{dg}{d\ell}\Big)^2\,d\ell\Big)^{1/4}\\
&\le c(n,p)\Big(\int_\Omega |g|_\ell^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |N|_\ell^{4p}\,d\ell\Big)^{1/4}\Big(\int_\Omega \frac{dg}{d\ell}\,dg\Big)^{1/4}\\
&\le c(n,p)\Big(\int_\Omega |g|_\ell^{2p}\,dg\Big)^{1/2}\Big(\int_\Omega |N|_\ell^{4p}\,d\ell\Big)^{1/4}\Big(\int_\Omega |g|_\ell^{n/2}\,dg\Big)^{1/4},
\end{aligned}
$$

which concludes the proof. ■

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