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On the regularity of Ricci flows coming out of metric spaces

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Abstract. We consider smooth, possibly incomplete, n-dimensional Ricci flows $(M, g(t))_{t\in(0,T)}$ with $\mathrm{Ric}(g(t)) \geq -1$ and $|\mathrm{Rm}(g(t))| \leq c/t$ for all $t \in (0, T)$ *coming out* of metric spaces (M, d_0) in the sense that $(M, d(g(t)), x_0) \to (M, d_0, x_0)$ as $t \searrow 0$ in the pointed Gromov–Hausdorff sense. If $B_{g(t)}(x_0, 1) \Subset M$ for all $t \in (0, T)$ and $(B_{d_0}(x_0, 1), d_0)$ can be isometrically and compactly embedded in a smooth n-dimensional Riemannian manifold $(\Omega, d(\tilde{g}_0))$, then we show using the Ricci-harmonic map heat flow that there is a corresponding smooth solution $(\tilde{g}(t))_{t\in(0,T)}$ to the δ-Ricci–DeTurck flow on a Euclidean ball $\mathbb{B}_r(p_0) \subset \mathbb{R}^n$, for some small $0 < r < 1$, and $\tilde{g}(t) \to \tilde{g}_0$ smoothly as $t \to 0$. We further show that this implies that the original solution g can be extended locally to a smooth solution defined up to time zero, in view of the method of Hamilton.

Keywords. Ricci flow, metric spaces, regularity

Contents

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1. Introduction

1.1. Overview

In this paper, we investigate and answer in certain cases the following question.

Problem 1.1. *Let* $(M, g(t))_{t\in(0,T)}$ *be a (possibly incomplete) Ricci flow which satisfies*

$$|\mathrm{Rm}(\cdot, t)| \le c_0/t, \tag{1.1}$$

and for which $(M, d(g(t)))$ *Gromov–Hausdorff converges to a metric space* (X, d_0) *as* $t \searrow 0$.

What further assumptions on the regularity of (X, d_0) *and* $(M, g(t))_{t\in(0,T)}$ *guarantee that* $g(t)$ *converges locally smoothly (or continuously) to a smooth (or continuous) metric as t approaches zero?*

Remark 1.2. We recall that for a connected, open, not necessarily complete, Riemannian manifold (M, g), there is a metric $d(g)$ induced by g which makes (M, d) into a metric space. Note that the distance between two points $p, q \in M$ is not necessarily realised by a geodesic; nevertheless, for every $x \in M$ there exists $r > 0$ such $B_{d(g)}(x, r)$ is geodesically convex, and the distance between any two points in $B_{d(g)}(x, r)$ is uniquely realised by a smooth geodesic.

If we assume in Problem 1.1 that $(M, g(t))_{t\in(0,T)}$ is complete for each $t \in (0, T)$ and in addition to (1.1) we have

$$\mathrm{Ric}(\cdot, t) \ge -1 \tag{1.2}$$

for all $t \in (0, T)$, then X is homeomorphic to M and the topology of (M, d_0) agrees with that of $(M, g(t))$ for all $t \in (0, T)$. This is a consequence of the following estimate on the induced distances [18, Lemma 3.1]: If $d_t = d(g(t))$, then $d_t \to d_0$ for a metric d_0 on M and

$$e^t d_0 \ge d_t \ge d_0 - \gamma(n)\sqrt{c_0 t} \quad \text{for all } t \in (0, T). \tag{1.3}$$

This implies convergence of d_t to d_0 in the C^0 sense, which is stronger than Gromov–Hausdorff convergence. Since Gromov–Hausdorff limits are unique up to isometries, this implies: if $(M, d(g(t_i)), p) \to (X, d_X, x)$ in the Gromov–Hausdorff sense for a sequence of times $t_i \searrow 0$ then (X, d_X, x) is isometric to (M, d_0, p). Hence it is *not* possible that complete solutions satisfying (1.1) and (1.2) come out of metric spaces which are *not* manifolds.

Note that if $\mathrm{Ric}(\cdot, t) \ge -C$ for some $C > 1$, and (1.1) holds for all $t \in (0, T)$, then, since (1.1) is invariant under scaling, we can scale the solution so that it satisfies (1.2).

Estimate (1.3) can be localised as follows.

Lemma 1.3 (Simon–Topping, [18, Lemma 3.1]). *Let* $(M, g(t))_{t\in(0,T)}$, $T \le 1$, *be a smooth Ricci flow satisfying* $\mathrm{Ric}(\cdot,t) \ge -1$ *and* $|\mathrm{Rm}(\cdot,t)| \le c_0/t$, *where* M *is connected but* $(M, g(t))$ *is not necessarily complete. Assume furthermore that* $B_{g(t)}(x_0, 1) \Subset M$ *for all* $t \in (0, T)$. *Then* $X := \bigcap_{s\in(0,T)} B_{g(s)}(x_0, 1/2)$ *is non-empty and there is a well defined limiting metric* $d_t \to d_0$ *as* $t \searrow 0$, *where*

$$e^t d_0 \ge d_t \ge d_0 - \gamma(n)\sqrt{c_0 t} \quad \textit{for all } t \in [0,T) \textit{ on } X. \tag{1.4}$$

Furthermore, there exist $R = R(c_0, n) > 0$ *and* $S = S(c_0, n) > 0$ *such that* $B_{d_0}(x_0, r) \Subset \mathcal{X} \subseteq X$ *and* $B_{g(t)}(x_0, r) \Subset \mathcal{X} \subseteq X$ *for all* $r \le R$ *and* $t \le S$ *where* $\mathcal{X}$ *is the connected component of* X *which contains* x_0, *and the topology of* $B_{d_0}(x_0, r)$ *induced by* d_0 *agrees with that induced by the topology of* M.

Coming back to our initial question, assuming (1.1) and (1.2) hold and for some $r > 0$, $B_{g(t)}(x_0, r) \Subset M$ for all $t \in (0, T)$, the above result implies that the metric d_0 exists (locally) and the convergence is in the sense of (1.4).

Examples. We give examples of solutions satisfying the conditions (1.1) and (1.2).

(1) *Expanding gradient Ricci solitons coming out of non-negatively curved cones.* Consider a smooth Riemannian metric γ on $\mathbb{S}^{n-1}$ with eigenvalues of its curvature operator greater than or equal to 1 and the cone $C(\gamma) = (([0,\infty) \times \mathbb{S}^{n-1})/\sim, dr^2 \oplus r^2\gamma, O)$ where the equivalence relation $\sim$ identifies $O := (0, x) \sim (0, y)$. Note that the curvature operator of $C(\gamma)$ is non-negative away from the tip. In [14] it was shown that if $C(\gamma)$ arises as the tangent cone at infinity of a non-compact manifold M with non-negative and bounded curvature operator then there exists an expanding gradient soliton (M, g) such that its evolution under the Ricci flow, $(M, g(t))_{t\in(0,\infty)}$, has the property that $(M, d(g(t)), p) \to C(\gamma)$ in the pointed Gromov–Hausdorff sense as $t \searrow 0$. The construction in [14] guarantees that this convergence is in C^∞_{loc} away from the tip of the cone.

In [8] it was later shown that there always exists an expanding gradient soliton coming out of any such non-negatively curved cone $C(\gamma)$. The construction in [8] also guarantees that the convergence is in C^∞_{loc} away from the tip; the existence result is based on the Nash–Moser fixed point theorem. Problem 1.1 was partly motivated by the cost of using such a "black box". Indeed, the Nash–Moser fixed point theorem is not so sensitive to the nature of the non-linearities of the Ricci flow equation as long as the corresponding linearized operator satisfies the appropriate Fredholm properties. In particular, the use of the Nash–Moser fixed point theorem does not shed new light on the smoothing effect of the Ricci flow. Finally, we emphasize that uniqueness of such solutions is unknown within the class of asymptotically conical gradient Ricci solitons with positive curvature operator.

(2) *Ricci flows coming out of non-collapsed Ricci limit spaces.* Let $(M_i, g_i(0), x_i)_{i\in\mathbb{N}}$ be a sequence of smooth n-dimensional Riemannian manifolds with bounded curvature such that $\mathcal{R}(g_i(0)) + c \cdot \mathrm{Id}(g_i(0)) \in \mathcal{C}_{\mathcal{K}}$ and $\mathrm{Vol}(B_{g_i(0)}(x)) \ge v_0$ for all $x \in M_i$ for all $i \in \mathbb{N}$, for some $c, v_0 > 0$ where $\mathcal{R}$ is the curvature operator, Id is the identity operator of the sphere and $\mathcal{C}_{\mathcal{K}}$ is the cone of either (i) non-negative curvature operators, (ii) 2-non-negative curvature operators, (iii) weakly PIC_1 curvature operators, or (iv) weakly PIC_2

curvature operators. Then (by [17] for (i), (ii) in case $n = 3$ and [2] for (i)–(iv) for general $n \in \mathbb{N}$) there are solutions $(M_i, g_i(t), x_i)_{t\in[0,T(n,v_0,c)]}$ such that $\mathcal{R}(g_i(t))) + C \cdot \mathrm{Id} \in \mathcal{C}_{\mathcal{K}}$ (for some new $C > 0$). Note that this implies $\mathrm{Ric}(g_i(t)) \geq -c(n)C$. After scaling each solution we obtain a sequence of solutions satisfying (1.1) and (1.2). Taking a subsequential limit, we obtain a pointed Cheeger–Hamilton limit solution $(M^n, g(t), x_\infty)_{t\in(0,1)}$ which satisfies (1.1) and (1.2).

More generally, if $(M_i, g_i(t), x_i)_{t\in[0,1)}$ is a sequence of smooth complete solutions satisfying (1.1) and (1.2), and

$$\mathrm{Vol}(B_{g_i(t)}(x_i), 1) \geq v_0 \quad \text{for all } i \in \mathbb{N} \text{ and all } t \in [0, 1),$$

we obtain a pointed solution $(M^n, g(t), x_\infty)_{t\in(0,1)}$ as a subsequential Cheeger–Hamilton limit, which satisfies (1.1) and (1.2).

Local setting. Problem 1.1 can be considered locally in the context of the above examples as follows.

(1) Assume that $(M, g(t))_{t\in(0,\infty)}$ is a smooth self-similarly expanding solution with non-negative Ricci curvature coming out of a cone $(M^n, d_X) = (\mathbb{R}^+ \times S^{n-1}, dr^2 \oplus r^2\gamma)$, where γ is a smooth [continuous] Riemannian metric. Does the solution $(M, g(t))_{t\in(0,1)}$ come out smoothly [continuously]? That is, is the solution

$$(M \setminus \{p\}, g(t))_{t\in[0,1]}$$

smooth [continuous], where p is the tip of the cone, and g_0 is the cone metric on $M \setminus \{p\}$ at time zero?

If we replace the assumption that 'γ is smooth (continuous) on S^{n-1}' by 'γ is smooth [continuous] on an open set $V \subseteq S^n$', we ask: is

$$(\mathbb{R}^+ \times V, g(t)|_{\mathbb{R}^+\times V})_{t\in[0,1]}$$

smooth [continuous]?

(2) In the setting of Example (2) above let (M, d_0, x_∞) be the limit as $t \searrow 0$ of $(M, d(g(t)), x_\infty)$. Note that (M, d_0, x_∞) is isometric to the Gromov–Hausdorff limit of $(M_i, d(g_i(0)), x_i)$ as $i \to \infty$, in view of (1.4). Let $V \subseteq M$ be an open set such that (V, d_0) is isometric to a smooth [continuous] Riemannian manifold. Can $(V, g(t))_{t\in(0,1)}$ be extended smoothly [continuously] to $t = 0$, that is, does there exist a smooth [continuous] g_0 on V such that $(V, g(t))_{t\in[0,1)}$ is smooth [continuous]?

We will see in Theorem 1.6, that the answer to each of these questions in the smooth setting is **yes** if we measure the smoothness of the initial metric space appropriately. The answer to each of these questions is also **yes** in the continuous setting (see Theorem 1.7) if we measure the continuity of the initial metric appropriately *and* the convergence in the continuous setting is measured *up to diffeomorphisms*.

The smoothness (respectively continuity) of a metric space in this paper will be measured as follows. Let $0 < \varepsilon_0(n) < 100^{-1}$ be a small fixed positive constant depending only on n. We denote by $\mathbb{B}_r(x) \subset \mathbb{R}^n$ the (open) Euclidean ball with radius r, centred at x.

Definition 1.4. Let (X, d_0) be a metric space and let V be a set in X. We say (V, d_0) is *smoothly* (respectively *continuously*) *n-Riemannian* if for all $x_0 \in V$ there exist $0 < \tilde{r}, r$ with $\tilde{r} < \frac{1}{5}r$ and points $a_1, \dots, a_n \in B_{d_0}(x_0, r)$ such that the map

$$F_0(x) := (d_0(x, a_1), \dots, d_0(x, a_n)), \quad x \in B_{d_0}(x_0, r),$$

is a $(1 + \varepsilon_0)$-bi-Lipschitz homeomorphism on $B_{d_0}(x_0, 5\tilde{r})$ and the push-forward of d_0 via F_0 given by $\tilde{d}_0(\tilde{x}, \tilde{y}) := d_0((F_0)^{-1}(\tilde{x}), (F_0)^{-1}(\tilde{y}))$ on $\mathbb{B}_{4\tilde{r}}(F_0(x_0)) \Subset F_0(B_{d_0}(x_0, 5\tilde{r}))$ is induced by a smooth (respectively continuous) Riemannian metric: there exists a smooth (respectively continuous) Riemannian metric $\tilde{g}_0$ defined on $\mathbb{B}_{4\tilde{r}}(F_0(x_0))$ such that $\tilde{d}_0$ satisfies $\tilde{d}_0 = d(\tilde{g}_0)$ when restricted to $\mathbb{B}_{\tilde{r}}(F_0(x_0))$, where $d(\tilde{g}_0)$ is the distance on $(\mathbb{B}_{4\tilde{r}}(F_0(x_0)), \tilde{g}_0)$.

Since this definition might be slightly counter-intuitive at first reading, we give an alternative definition as well.

Definition 1.5. Let (X, d_0) be a metric space and let V be an open set in X. We say (V, d_0) is *smoothly n-Riemannian* if for all $x_0 \in V$ there exist a smooth n-dimensional connected Riemannian manifold $(U(x_0), \tilde{g}_0)$, a neighbourhood $V(x_0) \subseteq V$ of x_0, and an isometry $F_0 : (V(x_0), d_0) \to (F_0(V(x_0)), \tilde{d}_0) \subseteq (U(x_0), \tilde{d}_0)$, where $(U(x_0), \tilde{d}_0) = (U(x_0), d(\tilde{g}_0))$, and $d(\tilde{g}_0)$ is the distance on $(U(x_0), \tilde{g}_0)$.

Equivalence of definitions. The first definition clearly implies the second one. That the second definition implies the first may be seen as follows: Since $(U(x_0), \tilde{g}_0)$ is smooth, we may find $\tilde{a}_1, \dots, \tilde{a}_n \in F_0(V(x_0))$ such that $\tilde{F}_0(\tilde{x}) = (\tilde{d}_0(\tilde{a}_1, \tilde{x}), \dots, \tilde{d}_0(\tilde{a}_n, \tilde{x}))$ is $(1 + \varepsilon_0)$-bi-Lipschitz and smooth in a neighbourhood of $F(x_0)$. Defining $a_1 := F_0^{-1}(\tilde{a}_1), \dots, a_n := F_0^{-1}(a_n))$, and $\hat{g}_0 := (\tilde{F}_0)_*(\tilde{g}_0)$, we see that $\hat{F}_0 = \tilde{F}_0 \circ F_0$ satisfies $(\hat{F}_0)_*(d_0) := \hat{d}_0 = d(\hat{g}_0)$ and

$$\hat{F}_0(x) = \big(\tilde{d}_0(\tilde{a}_1, F_0(x)), \dots, \tilde{d}_0(\tilde{a}_n, F_0(x))\big) = (d_0(a_1, x), \dots, d_0(a_n, x))$$

is $(1 + \varepsilon_0)$-bi-Lipschitz, as required.

1.2. Main results

The first theorem gives a positive answer to the questions posed in the previous subsection in the smooth setting.

Theorem 1.6. *Let $(M, g(t))_{t \in (0,T]}$ be a smooth solution to the Ricci flow satisfying (1.1) and (1.2), assume $B_{g(t)}(x_0, 1) \Subset M$ for all $0 < t \le T$, and let (X, d_0) be the C^0 limit as $t \searrow 0$ of $(X, d(g(t)))$ established in Lemma 1.3, where $X := \bigcap_{s \in (0,T)} B_{g(s)}(x_0, 1/2)$. Assume further that $(B_{d_0}(x_0, r), d_0)$ is smoothly n-Riemannian in the sense of Definition 1.4, where $r \le R(c_0, n)$ and $R(c_0, n)$ is as in Lemma 1.3. Then there exists a smooth Riemannian metric g_0 on $B_{d_0}(x_0, s)$ for some $s \in (0, r)$ such that we can extend the smooth solution $(B_{d_0}(x_0, s), g(t))_{t \in (0,T)}$ to a smooth solution $(B_{d_0}(x_0, s), g(t))_{t \in [0,T)}$ by defining $g(0) = g_0$.*

As noted by Topping [20], this result was known for the Ricci flow of closed 2-manifolds by the results of Richard [13].

The second theorem is concerned with the corresponding question in the continuous setting.

Theorem 1.7. *Let $(M, g(t))_{t\in(0,T]}$ be a smooth solution to the Ricci flow satisfying* (1.1) *and* (1.2)*, assume $B_{g(t)}(x_0, 1) \Subset M$ for all $0 < t \le T$, and let (X, d_0) be the C^0 limit as $t \searrow 0$ of $(X, d(g(t)))$ established in Lemma* 1.3*, where $X := \bigcap_{s\in(0,T)} B_{g(s)}(x_0, 1/2)$. Assume further that $(B_{d_0}(x_0, r), d_0)$ is continuously n-Riemannian in the sense of Definition* 1.4*, where $r \le R(c_0, n)$ and $R(c_0, n)$ is as in Lemma* 1.3.

Then for any strictly monotone sequence $t_i \searrow 0$, there exists a radius $v > 0$ and a continuous Riemannian metric $\tilde{g}_0$, defined on $\mathbb{B}_v(p)$, $p \in \mathbb{R}^n$, and a family of smooth diffeomorphisms $Z_i : B_{d_0}(x_0, 2v) \to \mathbb{R}^n$ such that $(Z_i)_*(g(t_i))$ converges in the C^0 sense to $\tilde{g}_0$ as $t_i \searrow 0$ on $\mathbb{B}_v(p)$.

1.3. Metric space convergence and the conditions (1.1) *and* (1.2)

Assume we have a smooth complete solution $(M^n, g(t))_{t\in(0,1)}$ to the Ricci flow, satisfying (1.1) but not necessarily (1.2). Then there is no guarantee that a limit metric $d_0 = \lim_{t\to 0} d(g(t))$ exists. Similarly, if we have a sequence of smooth complete solutions $(M_i^n, g_i(t), x_i)_{t\in[0,1)}$, satisfying $|\mathrm{Rm}(g_i(t)))| \le c_0/t$ and $\mathrm{Vol}(B_{g_i(t)}(x)) \ge v_0 > 0$ for all $t \in (0, 1)$ and all $x \in M_i$ for some $c_0, v_0 > 0$, for all $i \in \mathbb{N}$, we obtain a limiting solution in the smooth Cheeger–Hamilton sense, $(M^n, g(t), p)_{t\in(0,1)}$, which satisfies $|\mathrm{Rm}(\cdot, t)| \le c_0/t$ for all $t \in (0, 1)$, but again there is *no* guarantee that a limit metric $d_0 = \lim_{t\to 0} d(g(t))$ exists. Furthermore, *if* a pointed Gromov–Hausdorff limit (M, d_0, p), as $t \to 0$, of $(M, d(g(t)), p)$ exists and *if* a Gromov–Hausdorff limit (X, d_X, y) in $i \in \mathbb{N}$ of $(M_i^n, d(g_i(0)), x_i)_{i\in\mathbb{N}}$ exists, then there is no guarantee that (X, d_X, y) is isometric to (M, d_0, p), or that (M, d_0) has the same topology as $d(g(t))$ for $t > 0$.

An example which considers the metric behaviour under limits of solutions with no uniform bound from below on the Ricci curvature but with $|\mathrm{Rm}(\cdot, t)| \le c_0/t$ is given in a recent work of Peter Topping [20]. There, he constructs examples of smooth solutions $(\mathbb{T}^2, g_i(t))_{t\in[0,1)}$ to the Ricci flow, satisfying $(\mathbb{T}^2, d(g_i(0))) \to (T^2, d(\delta))$ as $i \to \infty$, where δ is the standard flat metric on $\mathbb{T}^2$, and $|\mathrm{Rm}(g_i(t))| \le c/t$ for all $t \in (0, 1), i \in \mathbb{N}$ for some $c > 0$, but so that the limiting solution $(\mathbb{T}^2, g(t))_{t\in(0,1]}$ satisfies $(\mathbb{T}^2, d(g(t)))_{t\in(0,1)} = (\mathbb{T}^2, \hat{d})$, where $(\mathbb{T}^2, \hat{d})$ is isometric to $(T^2, d(2\delta))$. The initial smooth data $g_i(0)$ do not satisfy $\mathrm{Ric}(g_i(0)) \ge -k$ for some fixed $k > 0$ for all $i \in \mathbb{N}$, and so the arguments used to show that the Gromov–Hausdorff limit of the initial data is the same as the limit as $t \to 0$ of the limiting solution, are not valid.

1.4. Related results

We recall the basic setup for the initial trace problem for the scalar heat equation. Consider a smooth solution $u : \mathbb{R}^n \times (0, 1) \to \mathbb{R}$ of $\frac{\partial}{\partial t}u = \Delta u$, and assume that $u(\cdot, t) \to u_0(\cdot)$ in $C^0_{\mathrm{loc}}(\mathbb{B}_1(0))$ as $t \searrow 0$, where $u_0 \in C^\infty(\mathbb{B}_1(0))$. Then the solution can be locally extended to a smooth local solution $v : \mathbb{B}_{1/2}(0) \times [0,1) \to \mathbb{R}$ by defining $v(\cdot,0) = u_0(\cdot)$ on $\mathbb{B}_{1/2}(0)$, as we now explain: since $u \in C^0(\overline{\mathbb{B}_{3/4}(0)} \times [0,1])$, by standard theory there exists a solution $z \in C^0(\overline{\mathbb{B}_{3/4}(0)} \times [0,1]) \cap C^\infty(\mathbb{B}_{3/4}(0) \times [0,1])$ of the heat equation such that $z = u$ on the parabolic boundary $\partial\mathbb{B}_{3/4}(0) \times [0,1] \cup \overline{\mathbb{B}_{3/4}(0)} \times \{0\}$. The maximum principle then implies that $z \equiv u$ and hence u is smooth on $\mathbb{B}_{3/4}(0) \times [0,1]$, as required. Here the linear theory simplifies the situation. We have also assumed that $u(\cdot,t) \to u_0$ locally uniformly. In the Ricci flow setting, assuming (1.1) and (1.2), we saw above that the initial values must be taken on uniformly, albeit for the distance, and not necessarily for the Riemannian metric.

A non-linear setting closer to the one considered in this paper is as follows. In [1], Appleton considers (among other things) the δ-Ricci–DeTurck flow of metrics g_0 on $\mathbb{R}^n$ which are close to the standard metric δ, in the sense that $|g_0 - \delta|_\delta \le \varepsilon(n)$. In the work of Koch and Lamm [11, Theorem 4.3], it was shown that under this closeness condition there always exists a *weak solution* $(\mathbb{R}^n, g(t))_{t\in(0,\infty)}$. Weak solutions defined on $[0,T)$ ($T = \infty$ is allowed) are smooth for all $t > 0$ and $h(x,t) := g(x,t) - \delta(x)$ has bounded X_T norm, where

$$\|h\|_{X_T} := \sup_{0<t<T} \|h(t)\|_\delta + \sup_{x\in\mathbb{R}^n} \sup_{0<R^2<T} \left(R^{-n/2}\|\nabla h\|_{L^2(\mathbb{B}_R(x)\times(0,R^2))} + R^{\frac{2}{n+4}}\|\nabla h\|_{L^{n+4}(\mathbb{B}_R(x)\times(R^2/2,R^2))}\right).$$

If the initial values g_0 are continuous then they are attained in the C^0 sense, that is, $|g(t) - g_0|_\delta \to 0$ as $t \to 0$. Appleton [1, Theorem 4.5] showed that any weak solution $h(t) := g(t) - \delta$ which has $g_0 \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ and $|h_0|_\delta \le \varepsilon(n)$ must have $h(t) \in H^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\mathbb{R}^n \times [0,\infty))$. In particular the zeroth, first and second spatial derivatives of $h(t)$ locally approach those of h_0 as $t \searrow 0$. That is, for classical initial data $h_0 \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, any weak solution $h(t)$ restricted to Ω approaches $h(0)$ in the $C^{2,\alpha}(\Omega)$ norm on Ω for any precompact, open set Ω.

Hochard [10] established some results similar to some of those appearing in Sections 2 and 3 of the current paper. We received a copy of Hochard's thesis after a preprint version, including the relevant sections, of this paper was finished but not yet published. We have included references to the results of Hochard at the appropriate points throughout this paper. His approach differs slightly, as we explain at the relevant points.

1.5. Outline of paper

We outline the idea of the proof of the main theorems, Theorems 1.6 and 1.7. The idea is somewhat similar to the one we used above to show smoothness of solutions to the heat equation coming out of smooth initial data, which are smooth for positive times.

It is well known that since the Ricci flow is invariant under diffeomorphisms, it only represents a degenerate parabolic system. To be able to prove initial regularity, it is thus necessary to put the Ricci flow into a good gauge, transforming the Ricci flow into a strictly parabolic system. The strategy we follow here is to construct a local family of diffeomorphims solving the Ricci-harmonic map heat flow into $\mathbb{R}^n$ and push forward the Ricci flow solution to obtain a solution to the δ-Ricci–DeTurck flow.

More precisely, despite the low initial regularity, we show that there is a solution to the Ricci-harmonic map heat flow, $Z \in C^0(\overline{B_{d_0}(x_0,1)} \times [0,T);\mathbb{R}^n) \cap C^\infty(B_{d_0}(x_0,1) \times (0,T);\mathbb{R}^n)$, with initial and boundary values given by the map F_0, which represents distance coordinates at time zero. The a priori estimates we prove in Sections 2 and 3 help us to construct this solution, and from the Regularity Theorem 3.8, we see that there is $S(n) > 0$ and a small $\alpha(n) > 0$ such that the solution is $(1+\alpha(n))$-bi-Lipschitz for each $t \in (0, S(n)) \cap [0, T/2)$. The explicit construction of Z is carried out in Theorem 3.11.

Hence, we may consider the push-forward $\tilde{g}(t) := (Z_t)_*(g(t))$, which is by construction a solution to δ-Ricci–DeTurck, and $\tilde{g}(t)$ is $\alpha(n)$-close to δ in the C^0 sense. We first restrict to the case that the push-forward of d_0 with respect to F_0 is generated locally by a continuous Riemannian metric $\tilde{g}_0$. A further application of Theorem 3.8 shows that $\tilde{g}(t)$ converges locally to the continuous metric $\tilde{g}_0$. This is explained in detail in Theorem 4.3.

If we assume further that $\tilde{g}_0$ is smooth and sufficiently close to δ, then we consider the Dirichlet solution ℓ to the δ-Ricci–DeTurck flow on a Euclidean ball $\mathbb{B}_r(0) \times [0,T]$, with parabolic boundary data given by $\tilde{g}$. The existence of this solution is shown in Section 5, where Dirichlet solutions to the δ-Ricci–DeTurck flow with given parabolic boundary values C^0 close to δ are constructed. The L^2-Lemma (Lemma 6.1) tells us that the (weighted) spatial L^2 norm of the difference $g_1 - g_2$ of two solutions g_1, g_2 to the δ-Ricci–DeTurck flow defined on a Euclidean ball is non-increasing if g_1 and g_2 have the same values on the boundary of that ball, and are sufficiently close to δ for all $t \in [0,T]$. An application of the L^2-lemma then proves that $\ell = \tilde{g}$. The construction of ℓ carried out in Section 5 guarantees that ℓ is smooth on $\mathbb{B}_r(0) \times [0,T]$. Hence $\tilde{g}$ is smooth on $\mathbb{B}_r(0) \times [0,T]$. Section 7 completes the proof of Theorem 1.6: the smoothness of $\tilde{g}$ on $\mathbb{B}_r(0) \times [0,T]$ implies that one can extend g smoothly (locally) to $t = 0$. In Section 7 we discuss some of the consequences of Theorem 1.6 in the context of expanding gradient Ricci solitons with non-negative Ricci curvature.

1.6. An open problem

The lower bound on the Ricci curvature in (1.2) is crucial to obtaining the bound from above for d_t in (1.4). It is also used in Section 4 when showing that $\tilde{g}(t)$ converges to $\tilde{g}_0$ in the C^0 norm.

Problem 1.8. *Can the bound from below on the Ricci curvature in Section 3 and/or other sections be replaced by a weaker condition?*

We comment on this at various points in the paper.

1.7. Notation

We collect notation used throughout this paper.

(1) For a connected Riemannian manifold (M, g), $x, y \in M$, $r \in \mathbb{R}^+$:

(1a) $(M, d(g))$ refers to the associated metric space,

$$d(g)(x, y) = \inf_{\gamma \in G_{x,y}} L_g(\gamma),$$

where $G_{x,y}$ is the set of smooth regular curves $\gamma : [0, 1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$, and $L_g(\gamma)$ is the length of γ with respect to g.

(1b) $B_g(x, r) := B_{d(g)}(x, r) := \{y \in M \mid d(g)(y, x) < r\}$.

(1c) If g is locally in C^2 then $\mathrm{Ric}(g)$ is the Ricci tensor, $\mathrm{Rm}(g)$ is the Riemannian curvature tensor, and $\mathrm{R}(g)$ is the scalar curvature.

(2) For a one-parameter family $(g(t))_{t\in(0,T)}$ of Riemannian metrics on a manifold M, the distance induced by the metric $g(t)$ is denoted by $d(g(t))$ or d_t for $t \in (0, T)$.

(3) For a metric space (X, d), $x \in X$, $r \in \mathbb{R}^+$, $B_d(x, r) := \{y \in X \mid d(y, x) < r\}$.

(4) $\mathbb{B}_r(x)$ refers to the Euclidean ball with radius $r > 0$ and centre $x \in \mathbb{R}^n$.

2. The Ricci-harmonic map heat flow for functions with bounded gradient

In this section we prove some local results about the Ricci-harmonic map heat flow.

Hochard [10, Section II.3.2], in independent work, proved some results which are similar to those in this section. Hochard uses blow-up arguments to prove some of his estimates, whereas we use a more direct argument involving the maximum principle applied to various evolving quantities.

The first theorem we present is a local version of a theorem of Hamilton [9, p. 15] for solutions satisfying (1.1) and (1.2).

Theorem 2.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a smooth background solution to the Ricci flow satisfying (1.1) and (1.2) such that $B_{g(0)}(x_0, 2) \Subset M$ and $\partial B_{g(0)}(x_0, 1)$ is a smooth $(n-1)$-dimensional manifold. Let $Z_0 : \overline{B_{g(0)}(x_0, 1)} \to \mathbb{R}^n$ be a smooth map such that*

- $|\nabla Z_0|^2_{g(0)} \le c_1$,
- $Z_0(\overline{B_{g(0)}(x_0, 1)}) \subseteq \mathbb{B}_r(0)$ *for some* $r \le 2$.

Then there is a unique solution

$$Z \in C^\infty(B_{g(0)}(x_0, 1) \times [0, T]; \mathbb{R}^n) \cap C^0(\overline{B_{g(0)}(x_0, 1)} \times [0, T]; \mathbb{R}^n),$$

to the Dirichlet problem for the Ricci-harmonic map heat flow

$$\begin{aligned} &\tfrac{\partial}{\partial t} Z = \Delta_{g(t),\delta} Z, \\ &Z(\cdot, 0) = Z_0, \\ &Z(\cdot, t)|_{\partial B_{g(0)}(x_0,1)} = Z_0|_{\partial B_{g(0)}(x_0,1)} \quad \text{for } t \in [0, T], \end{aligned} \tag{2.1}$$

and constants $c(c_0, c_1, n), S(n, c_0) > 0$ *such that*

$$Z_t(\overline{B_{g(0)}(x_0, 1)}) \subseteq \mathbb{B}_r(0) \qquad \text{for all } t \leq T, \tag{2.2}$$

$$B_{g(t)}(x_0, 3/4) \subseteq B_{g(0)}(x_0, 1) \quad \text{for all } t \leq \min(T, S(n, c_0)), \tag{2.3}$$

$$|\nabla^{g(t)} Z(\cdot, t)|^2_{g(t)} \leq c(c_0, c_1, n) \quad \text{on } B_{g(t)}(x_0, 1/2) \text{ for all } t \leq \min(T, S(n, c_0)). \tag{2.4}$$

Proof. We first note that the system for Z actually decouples into n independent linear equations. Since $\overline{B_{g(0)}(x_0, 1)} \subset M$ is a compact set, and the solution $(M, g(t))_{t\in[0,T]}$ is smooth, by standard theory there is a unique solution

$$Z \in C^\infty(B_{g(0)}(x_0, 1) \times [0, T]; \mathbb{R}^n) \cap C^0(\overline{B_{g(0)}(x_0, 1)} \times [0, T]; \mathbb{R}^n)$$

to the Dirichlet problem (2.1).

For the sake of clarity, we omit the dependence of the Levi-Civita connections on the metrics $(g(t))_{t\in[0,T]}$.

The statement (2.2) follows from the maximum principle and the evolution equation for $|Z|^2 = \sum_{i=1}^n (Z^i)^2$:

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|Z|^2 = -2|\nabla Z|^2_{g(t)}. \tag{2.5}$$

Statement (2.3) follows from the distance estimates (1.4), which hold on $B_{g(0)}(x_0, 1)$ for any solution to the Ricci flow satisfying (1.1), (1.2) and $B_{g(0)}(x_0, 2) \Subset M$: see [19, Lemma 3.1].

Regarding (2.4), we first recall the following fundamental evolution equation satisfied by $|\nabla Z|^2_{g(t)}$:

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\nabla Z|^2_{g(t)} = -2|\nabla^2 Z|^2_{g(t)}. \tag{2.6}$$

Notice that the term $\mathrm{Ric}(g(t))(\nabla Z, \nabla Z)$ showing up in the Bochner formula applied to ∇Z cancels with the pointwise evolution equation of the squared norm of ∇Z along the Ricci flow.

In case the underlying manifold is closed, the use of the maximum principle would give us the expected result.

In order to localize this argument, we construct a Perelman type cut-off function $\eta : M \to [0, 1]$ with $\eta(\cdot, t) = 0$ on $M \setminus B_{g(t)}(x_0, 3/4)$ and $\eta(\cdot, t) = e^{-k(n,c_0)t}$ on $B_{g(t)}(x_0, 1/2)$ such that $\frac{\partial}{\partial t}\eta(\cdot, t) \leq \Delta_{g(t)}\eta(\cdot, t)$ everywhere, and $|\nabla\eta|^2_{g(t)} \leq c_3(n)\eta$ everywhere, as long as $t \leq \min(S(n, c_0), T)$; see, for example, [19, Section 7] for details.

We consider the function $W := \eta|\nabla Z|^2_g + c_2|Z|^2$ with $c_2 = 10c(n)c_3(n)$. The quantity W is less than $c_1 + 4c_2$ everywhere at time zero. We consider a first time and point where W becomes equal to $c_1 + 5c_2$ on $\overline{B_{g(0)}(x_0, 1)}$. This must happen in $B_{g(0)}(x_0, 1)$, since $\eta = 0$ on a small open set U containing $\partial B_{g(0)}(x_0, 1)$ and $c_2|Z|^2 < 4c_2$ by (2.2). At such a point and time (x, t) we have, by (2.5) and (2.6) together with the properties of η,

$$\begin{aligned} 0 &\le \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) W(x,t) \\ &\le -2c_2|\nabla Z|^2_{g(t)} - 2\eta|\nabla^2 Z|^2_{g(t)} - 2g(t)(\nabla\eta, \nabla|\nabla Z|^2_{g(t)}) \\ &\le -2c_2|\nabla Z|^2_{g(t)} - 2\eta|\nabla^2 Z|^2_{g(t)} + 4c(n)\frac{|\nabla\eta|^2_{g(t)}}{\eta}|\nabla Z|^2_{g(t)} + \eta|\nabla^2 Z|^2_{g(t)} \\ &< 0 \end{aligned}$$

by the choice of c_2, which yields a contradiction. Hence $W(x,t) \le c_1 + 5c_2$ for all $t \le S(n, c_0)$, which implies

$$e^{-kt}|\nabla Z|^2_{g(t)}(\cdot, t) \le e^{-kt}|\nabla Z|^2_{g(t)}(\cdot, t) + c_2|Z|^2(\cdot, t) \le c_1 + 5c_2$$

on $B_{g(t)}(x_0, 1/2)$ for all $t \le \min(S(n, c_0), T)$. This gives

$$|\nabla Z|^2_{g(t)}(\cdot, t) \le e^{kS}(c_1 + 5c_2)$$

on $B_{g(t)}(x_0, 1/2)$ for all $t \le \min(S(n, c_0), T)$ as required. ■

We aim to prove an estimate for the second covariant derivatives of a solution to the Ricci-harmonic map flow. In fact, once we have a solution to the Ricci-harmonic map heat flow with bounded gradient, the solution smoothes out the second derivatives in a controlled way, as the following theorem shows.

Theorem 2.2. *For all $c_1 > 0$ and $n \in \mathbb{N}$, there exists $\hat{\varepsilon}_0(c_1, n) > 0$ such that the following is true. Let $(M^n, g(t))_{t\in[0,T]}$ be a smooth solution to the Ricci flow such that*

$$\mathrm{Ric}(g(t)) \ge -1, \quad |\mathrm{Rm}(\cdot, t)| \le \varepsilon_0^2/t, \quad \text{for all } t \in (0, T],$$

where $\varepsilon_0 \le \hat{\varepsilon}_0$. Assume furthermore that $B_{g(0)}(x_0, 1) \Subset M$, and $Z : B_{g(0)}(x_0, 1) \times [0, T] \to \mathbb{R}^n$ is a smooth solution to the Ricci-harmonic map heat flow

$$\frac{\partial}{\partial t} Z(x,t) = \Delta_{g(t)} Z(x,t)$$

for all $(x, t) \in B_{g(0)}(x_0, 1) \times [0, T]$, such that $|\nabla^{g(t)} Z(\cdot, t)|^2_{g(t)} \le c_1$ on $B_{g(t)}(x_0, 1)$ for all $t \in [0, T]$. Then

$$t|\nabla^{g(t),2} Z(\cdot, t)|^2_{g(t)} \le c(n, c_1)$$

on $B_{g(t)}(x_0, 1/4)$ for all $t \le \min(S(n), T)$, where $S(n) > 0$ is a constant just depending on n.

Remark 2.3. The condition $|\mathrm{Rm}(\cdot,t)| \le \varepsilon_0^2/t$ where $\varepsilon_0 \le \hat{\varepsilon}_0(c_1,n)$ is sufficiently small is not necessary: $|\mathrm{Rm}(\cdot,t)| \le k/t$ with k arbitrary is sufficient for the argument, as can be seen by examining the proof, but then the conclusions remain only valid on the time interval $(0, S(c_1,k,n)]$, where $S(c_1,k,n) > 0$ is sufficiently small. We only consider small k, as this is sufficient for the setting of the following sections. A version of this theorem, with $c_1 = c(n)$ and the condition $|\mathrm{Rm}(\cdot,t)| \le k/t$, k arbitrary, was independently proven by R. Hochard using a contradiction argument: see [10, Lemma II.3.9].

Proof of Theorem 2.2. In the following, we denote constants $C(\varepsilon_0, c_1, n)$ simply by ξ_0 if $C(\varepsilon_0, c_1, n)$ goes to 0 as ε_0 tends to 0 and c_1 and n remain fixed. For example $c_1^2 n^4 \varepsilon_0$ and $b(c_1,n)\sqrt{\varepsilon_0}$ are denoted by ξ_0 if $b(c_1,n)$ is a constant depending on c_1 and n, and $d(c_1,n)\sqrt{\xi_0}$ may be replaced by ξ_0 if $d(c_1,n)$ is a constant depending on c_1 and n only.

For the sake of clarity in the computation to follow, we use ∇ to denote $\nabla^{g(t)}$ at a time t, Rm to denote $\mathrm{Rm}(g(t))$ at a time t, et cetera, although the objects in question do indeed depend on the evolving metric. By standard commutator identities for the second covariant derivatives of a tensor T, $\nabla^2 T(V,P,\cdot) = \nabla^2 T(P,V,\cdot) + (\mathrm{Rm} * T)(V,P,\cdot)$ (see for example [21]), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^2_{ij} Z^k &= \nabla^2_{ij}\left(\frac{\partial}{\partial t} Z\right)^k + (\nabla \mathrm{Ric} * \nabla Z)^k_{ij} \\
&= \nabla^2(\Delta Z)^k_{ij} + (\nabla \mathrm{Ric} * \nabla Z)^k_{ij} \\
&= \Delta(\nabla_i \nabla_j Z^k) + (\mathrm{Rm} * \nabla^2 Z)^k_{ij} + (\nabla \mathrm{Rm} * \nabla Z)^k_{ij}.
\end{aligned}$$

Shi's estimates and the distance estimates (1.4) guarantee that

$$|\nabla \mathrm{Rm}(g(t))| \le \xi_0 t^{-3/2} \qquad \text{for } t \le S(n)$$

on $B_{g(t)}(x_0, 3/4)$; see for example Lemma 3.2 (after scaling once by 400). On $B_{g(t)}(x_0, 3/4)$, we see that for $t \le S(n)$,

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^2 Z|^2_{g(t)} &\le \Delta_{g(t)}(|\nabla^2 Z|^2_{g(t)}) - 2|\nabla^3 Z|^2_{g(t)} + c(n)|\mathrm{Rm}(g(t))|_{g(t)}|\nabla^2 Z|^2_{g(t)} \\
&\quad + c(n)|\nabla \mathrm{Rm}(g(t))|_{g(t)}|\nabla^2 Z|_{g(t)}|\nabla Z|_{g(t)} \\
&\le \Delta_{g(t)}(|\nabla^2 Z|^2_{g(t)}) - 2|\nabla^3 Z|^2_{g(t)} + \frac{\xi_0}{t}|\nabla^2 Z|^2_{g(t)} \\
&\quad + \frac{\xi_0}{t^{3/2}}|\nabla^2 Z|_{g(t)}|\nabla Z|_{g(t)} \\
&\le \Delta_{g(t)}(|\nabla^2 Z|^2_{g(t)}) - 2|\nabla^3 Z|^2_{g(t)} + \frac{\xi_0}{t}|\nabla^2 Z|^2_{g(t)} + \frac{\xi_0}{t^2}|\nabla Z|^2_{g(t)}. \qquad (2.7)
\end{aligned}$$

For $a_0 \ge 1$, let

$$W := t(a_0 + |\nabla Z|^2_{g(t)})|\nabla^2 Z|^2_{g(t)}.$$

Using (2.6) together with (2.7), we see that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) W &= \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\big(t(a_0 + |\nabla Z|^2_{g(t)})|\nabla^2 Z|^2_{g(t)}\big) \\
&= (a_0 + |\nabla Z|^2_{g(t)})|\nabla^2 Z|^2_{g(t)} + t(\partial_t - \Delta_{g(t)})(|\nabla Z|^2_{g(t)}) \cdot |\nabla^2 Z|^2_{g(t)} \\
&\quad + t(a_0 + |\nabla Z|^2_{g(t)})(\partial_t - \Delta_{g(t)})|\nabla^2 Z|^2_{g(t)} - 2t g(t)\big(\nabla|\nabla Z|^2_{g(t)}, \nabla|\nabla^2 Z|^2_{g(t)}\big) \\
&\le (a_0 + |\nabla Z|^2_{g(t)})|\nabla^2 Z|^2_{g(t)} - 2t|\nabla^2 Z|^4_{g(t)} - 2a_0 t|\nabla^3 Z|^2_{g(t)} \\
&\quad + (1 + a_0)\xi_0|\nabla^2 Z|^2_{g(t)} + \frac{(1 + a_0)\xi_0}{t} - 2t g(t)\big(\nabla|\nabla Z|^2_{g(t)}, \nabla|\nabla^2 Z|^2_{g(t)}\big) \\
&\le \frac{W}{t} - 2t|\nabla^2 Z|^4_{g(t)} - 2a_0 t|\nabla^3 Z|^2_{g(t)} + a_0\xi_0|\nabla^2 Z|^2_{g(t)} \\
&\quad + \frac{a_0\xi_0}{t} + tc(n, c_1)|\nabla^3 Z|_{g(t)}|\nabla^2 Z|^2_{g(t)} \\
&\le \frac{W}{t} + (c(n, c_1) - 2a_0)t|\nabla^3 Z|^2_{g(t)} - t|\nabla^2 Z|^4_{g(t)} + a_0\xi_0|\nabla^2 Z|^2_{g(t)} + \frac{a_0\xi_0}{t} \\
&\le \frac{W}{t} + (c(n, c_1) - 2a_0)t|\nabla^3 Z|^2_{g(t)} - \frac{t}{2}|\nabla^2 Z|^4_{g(t)} + \frac{a_0^2\xi_0}{t},
\end{aligned}
$$

where $c(n, c_1)$ denotes a positive constant depending on the dimension n and the Lipschitz constant c_1, which may vary from line to line, and we have used Young's inequality freely. Thus

$$
\begin{aligned}
\partial_t W &\le \Delta_{g(t)} W + \frac{W}{t} - \frac{W^2}{2(a_0 + c_1)^2 t} + \frac{a_0^2\xi_0}{t} \\
&\le \Delta_{g(t)} W + \frac{W}{t} - \frac{W^2}{4a_0^2 t} + \frac{a_0^2\xi_0}{t}
\end{aligned} \tag{2.8}
$$

if a_0 is chosen sufficiently large such that $a_0 \ge c(n, c_1)$.

In case the underlying manifold is closed, the use of the maximum principle would give us the expected result: if there is a first time and point (x, t) where $W(x, t) = 10a_0^2$ for example, we obtain a contradiction. Hence we must have $W \le 10a_0^2$.

In order to localize this argument, we introduce, as in the proof of Theorem 2.1, a Perelman type cut-off function $\eta : M \to [0, 1]$ with $\eta(\cdot, t) = 0$ on $B^c_{g(t)}(x_0, 2/3)$ and $\eta(\cdot, t) = e^{-t}$ on $B_{g(t)}(x_0, 1/2)$ such that $\frac{\partial}{\partial t}\eta(\cdot, t) \le \Delta_{g(t)}\eta(\cdot, t)$ everywhere, and $|\nabla\eta|^2_{g(t)} \le c(n)\eta$ everywhere, as long as $t \le S(n) \le 1$; see for example [19, Section 7] for details.

We first derive the evolution equation of the function $\hat{W} := \eta W$ with the help of inequality (2.8) for $t \le S(n)$:

$$
\partial_t \hat{W} \le \Delta_{g(t)}\hat{W} - 2\langle\nabla\eta, \nabla W\rangle_{g(t)} + \frac{\hat{W}}{t} - \eta\frac{W^2}{4a_0^2 t} + \eta\frac{a_0^2\xi_0}{t}. \tag{2.9}
$$

Thus

$$\eta(\partial_t - \Delta_{g(t)})\hat{W} \le -2\eta\langle\nabla\eta, \nabla W\rangle_{g(t)} + \frac{\eta\hat{W}}{t} - \frac{\hat{W}^2}{4a_0^2 t} + \eta^2\frac{a_0^2\xi_0}{t}$$
$$\le -2\langle\nabla\eta, \nabla\hat{W}\rangle_{g(t)} + c(n)\hat{W} + \frac{\hat{W}}{t} - \frac{\hat{W}^2}{4a_0^2 t} + \frac{a_0^2\xi_0}{t},$$

where again $c(n)$ denotes a positive constant depending on the dimension only and which may vary from line to line. If the maximum of $\hat{W}$ at any time is larger than $100a_0^2$ then this value must be achieved at some first time and point (x, t) with $t > 0$, since $\hat{W}(\cdot, 0) = 0$. This leads to a contradiction if $t \le S(n) \le 1$. ∎

3. Almost isometries, distance coordinates and the Ricci-harmonic map heat flow

In this section we set up the problem we will be investigating in the remaining sections, collect some background material and give an outline of the further strategy.

For convenience we recall a slightly refined version of Lemma 1.3 of the introduction, where $\beta(n) > 0$ is the constant appearing in [18, Lemma 3.1].

Lemma 3.1 (Simon–Topping, [18, Lemma 3.1]). *Let $(M, g(t))_{t\in(0,T)}$, $T \le 1$, be a smooth Ricci flow where M is connected but $(M, g(t))$ is not necessarily complete. Assume that for some $\varepsilon_0 > 0$ and $R > 100\beta^2(n)\varepsilon_0^2 + 200$ we have $B_{g(t)}(x_0, 200R) \Subset M$ for all $t \in (0, T)$ as well as*

$$|\mathrm{Rm}(\cdot, t)| \le \frac{\varepsilon_0^2}{(1+\beta^2(n))t} \quad \text{on } B_{g(t)}(x_0, 200R) \Subset M \text{ for all } t \in (0, T), \tag{ã}$$
$$\mathrm{Ric}(g(t)) \ge -1 \quad \text{on } B_{g(t)}(x_0, 200R) \Subset M \text{ for all } t \in (0, T). \tag{b̃}$$

Then for all $r, s \in (0, T)$,

$$e^{t-r} d_r \ge d_t \ge d_r - \varepsilon_0\sqrt{t-r} \quad \text{on } B_{g(s)}(x_0, 50R) \text{ for all } t \in [r, T).$$

This motivates us to work with the setup where $(M, g(t))_{t\in(0,T)}$ is a smooth solution to the Ricci flow, with

$$|\mathrm{Rm}(\cdot, t)| \le \varepsilon_0^2/t \quad \text{on } B_{g(t)}(x_0, 200R) \Subset M \text{ for all } t \in (0, T), \tag{a}$$
$$\mathrm{Ric}(g(t)) \ge -1 \quad \text{on } B_{g(t)}(x_0, 200R) \Subset M \text{ for all } t \in (0, T), \tag{b}$$
$$e^{t-r} d_r \ge d_t \ge d_r - \varepsilon_0\sqrt{t-r} \quad \text{for all } t \in [r, T) \text{ on } B_{g(s)}(x_0, 50R) \text{ and all } r, s \in (0, T), \tag{3.1}$$
$$B_{g(s)}(x_0, 50R) \Subset M \quad \text{for all } s \in (0, T). \tag{3.2}$$

As in Lemma 1.3, if (3.1) and (3.2) hold, then there exists a unique limiting metric as $t \searrow 0$:

$$\begin{aligned}
&d_0 = \lim_{t\searrow 0} d_t \text{ exists, is unique and is a metric on } X := \bigcap_{s\in(0,T)} B_{g(s)}(x_0, 50R),\\
&B_{d_0}(x_0, 20R) \subseteq \mathcal{X} \subseteq B_{g(s)}(x_0, 50R) \Subset M \quad \text{ for all } s \in \Big(0, \min\big(\tfrac{R^2}{1+\varepsilon_0^2}, \log(2), T\big)\Big),\\
&\text{and the topology of } B_{d_0}(x_0, 20R) \text{ induced by } d_0 \text{ agrees with the topology}\\
&\text{induced by } M,
\end{aligned} \tag{3.3}$$

where $\mathcal{X}$ is the connected component of the interior of $\bigcap_{s\in(0,T)} B_{g(s)}(x_0, 50R)$ containing x_0, and the limit is obtained uniformly; the existence of a unique limit follows from the fact that $d_t(\cdot,\cdot)$ is Cauchy in t, and the inclusions and the statement about the topology follow from the triangle inequality and the definition of interior.

Shi's estimates yield the following time interior decay.

Lemma 3.2. *Let $(M, g(t))_{t\in(0,T)}$ be a smooth, not necessarily complete, solution to the Ricci flow satisfying* (a), (b), (3.1) *and* (3.2) *for some $R \geq 1$. Then*

$$\sum_{i=0}^{j} |\nabla^i \mathrm{Rm}(\cdot, t)|^2 \leq \frac{\beta(k, n, \varepsilon_0)}{t^{2+j}} \tag{3.4}$$

for all $x \in B_{d_0}(x_0, 10R)$ and all $t \in (0, \min((1 + \varepsilon_0^2)^{-1} R^2, \log 2, T))$, where $\beta(k, n, \varepsilon_0) \to 0$ for fixed k and n, as $\varepsilon_0 \to 0$, and d_0 is the metric described in (3.3).

Proof. We scale the solution $\tilde{g}(\cdot, \tilde{t}) := t^{-1} g(\cdot, \tilde{t}t)$, so that time t in the original solution scales to time $\tilde{t}$ equal to 1 in the new one. We now have $|\widetilde{\mathrm{Rm}}(\cdot, s)| \leq 2\varepsilon_0^2$ on $B_{\tilde{g}(1/2)}(x, 1)$ for all $s \in [1/2, 2]$ and all $x \in B_{d_0}(x_0, 10R)$, in view of (a) and the (scaled) distance estimates (3.1). Shi's estimates (see for example [7, Theorem 6.5]) imply that $\sum_{i=0}^{k} |\nabla^i \mathrm{Rm}(x, 1)|^2 \leq \beta(k, n, \varepsilon_0)$ where $\beta(k, n, \varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, as claimed. ∎

In the main theorem of this section, Theorem 3.8, we consider distance maps F_t at time t, points $m_0 \in B_{d_0}(x_0, 15R)$ and $\varepsilon_0 < \frac{1}{2}$ such that $F_t : B_{d_0}(m_0, 10) \to \mathbb{R}^n$ satisfies

$$|F_t(x) - F_t(y)| \in \big((1 - \varepsilon_0)d_t(x, y) - \varepsilon_0\sqrt{t}, (1 + \varepsilon_0)d_t(x, y) + \varepsilon_0\sqrt{t}\big) \tag{c}$$

for all $x, y \in B_{d_0}(m_0, 10)$. If such a map F_t exists for all $t \in (0, T)$, and we further assume that $\sup_{t\in(0,T)} |F_t(x_0)| < \infty$, then for any sequence $t_i > 0$ with $t_i \to 0$ as $i \to \infty$, we can, after taking a subsequence, find a limiting map F_0 which is the C^0 limit of F_{t_i}, $\sup_{x\in B_{d_0}(m_0,10)} |F_0(x) - F_{t_i}(x)| \to 0$ as $i \to \infty$, which satisfies

$$1 - \varepsilon_0 \leq \frac{|F_0(x) - F_0(y)|}{d_0(x, y)} \leq 1 + \varepsilon_0 \tag{3.5}$$

on $B_{d_0}(x_0, 10)$. Indeed, we first define F_0 on a dense, countable subset $D \subset B_{d_0}(x_0, 10)$ using a diagonal subsequence and the theorem of Heine–Borel, and then we extend F_0 uniquely, continuously to all of $B_{d_0}(x_0, 10)$, which is possible in view of the fact that the bi-Lipschitz property (3.5) is satisfied on D. The sequence $(F_{t_i})_i$ converges uniformly to F_0 in view of (3.1), (c) and (3.5).

Thus F_0 is a $(1+\varepsilon_0)$-bi-Lipschitz map between the metric spaces $(B_{d_0}(x_0, 10), d_0)$ and $(F_0(B_{d_0}(x_0, 10)), \delta)$. This is equivalent to $H_0(\cdot) = F_0(\cdot) - F_0(x_0)$ being $(1+\varepsilon_0)$-bi-Lipschitz between the metric spaces $(B_{d_0}(x_0, 10), d_0)$ and $H_0(B_{d_0}(x_0, 10))$ where $\mathbb{B}_5(0) \subseteq H_0(B_{d_0}(x_0, 10)) \subseteq \mathbb{B}_{20}(0)$.

In Theorem 3.8, we see that if we consider the Ricci-harmonic map heat flow of one of the functions F_t, and we assume that the solution Z satisfies a gradient bound, $|\nabla^{g(s)} Z(\cdot, s)|_{g(s)} \leq c_1$ for $s \in [t, T]$ on some ball, then after flowing for a time t, the resulting map will be $(1+\alpha_0)$-bi-Lipschitz on a smaller ball if ε_0 is less than a constant $\hat{\varepsilon}_0(n, \alpha_0, c_1) > 0$, and $t \leq \min(S(n, \alpha_0, c_1), T)$. This property continues to hold if we flow for a time s where $t \leq s \leq \min(S(n, \alpha_0, c_1), T)$.

For convenience, we introduce the following notation:

Definition 3.3. Let (W, d) be a metric space. We call $F : (W, d) \to \mathbb{R}^n$ an ε_0-*almost isometry* if

$$(1-\varepsilon_0)d(x, y) - \varepsilon_0 \leq |F(x) - F(y)| \leq (1+\varepsilon_0)d(x, y) + \varepsilon_0$$

for all $x, y \in W$; and $F : (W, d) \to \mathbb{R}^n$ is a $(1+\varepsilon_0)$-*bi-Lipschitz map* if

$$(1-\varepsilon_0)d(x, y) \leq |F(x) - F(y)| \leq (1+\varepsilon_0)d(x, y)$$

for all $x, y \in W$.

In the main applications of Theorems 3.8 and 3.11 in this section, we will assume

$$\begin{aligned} &\text{there are points } a_1, \ldots, a_n \in B_{d_0}(x_0, R) \text{ such that the map} \\ &F_0 : B_{d_0}(x_0, R) \to \mathbb{R}^n, \\ &\qquad x \mapsto (d_0(a_1, x), \ldots, d_0(a_n, x)) := ((F_0)_1(x), \ldots, (F_0)_n(x)), \\ &\text{is a } (1+\varepsilon_0)\text{-bi-Lipschitz homeomorphism on } B_{d_0}(x_0, 100). \end{aligned} \tag{$\hat{c}$}$$

The components of F_0 are referred to as *distance coordinates*. As a consequence of this assumption and the distance estimates (3.1), we see that the corresponding distance coordinates at time t, $F_t : B_{d_0}(x_0, 50) \to \mathbb{R}^n$, given by $F_t(\cdot) := (d_t(a_1, \cdot), \ldots, d_t(a_n, \cdot))$, are mappings satisfying property (c) for all $t \in (0, T)$. That is: in the main application, we begin with a $(1+\varepsilon_0)$-bi-Lipschitz map F_0 and find, as a first step, maps F_t which are ε_0-almost isometries for all $t \in (0, T)$.

Remark 3.4. Hochard also looked independently at some related objects in his PhD thesis, and some of the infinitesimal results he obtained are similar to those of this section (cf. [10, Theorem II.3.10]), as we explained in the introduction. Hochard considers points x_0 which are so called (m, ε) *explosions at all scales less than* R (only $m = n$ is relevant in this discussion). For $m = n$, this means that there exist points $p_1, \ldots, p_n$ such that for all x in the ball $B_{d_0}(x_0, R)$ and all $r < R$, there exists a Gromov–Hausdorff εr-approximation $\psi : B_{d_0}(x, r) \to \mathbb{R}^n$ such that the components ψ^i are each close to the components of distance coordinates $d(\cdot, p_i) - d(x, p_i)$ at scale r, in the sense that $|\psi^i(\cdot) - (d(\cdot, p_i) - d(x, p_i))| \leq \varepsilon r$ on $B_{d_0}(x, r)$. Our approach and our main conclusion differ slightly from the approach and main conclusions of Hochard. The condition (c) we consider above looks at the closeness of the maps F_t to being a bi-Lipschitz homeomorphism, and this closeness is measured at time t using the maps F_t; and our main conclusion is that the map will be a $(1 + \alpha_0)$-bi-Lipschitz homeomorphism after flowing for an appropriate time by the Ricci-harmonic map heat flow, if $t > 0$ is small enough. We make the assumption on the evolving curvature, that it is close to that of $\mathbb{R}^n$, after scaling in time appropriately. Nevertheless, the proof of Theorem 3.8 below and of [10, Theorem II.3.10] have a number of similarities, as do some of the concepts.

Outline of the section. The main application of this section is, assuming (a), (b) and that F_0 are distance coordinates which define a $(1 + \varepsilon_0)$-bi-Lipschitz homeomorphism, to show that it is possible to define a Ricci–DeTurck flow $(\tilde{g}(s))_{s\in(0,T]}$ starting from the metric $\tilde{d}_0 := (F_0)_* d_0$, on some Euclidean ball, which is obtained by pushing forward the solution $(g(s))_{s\in(0,T]}$ by diffeomorphisms. How this should be understood will be explained in more detail in the next subsection.

The strategy we adopt is as follows. Assuming (a), (b), we consider the distance maps F_{t_i} at time t_i defined above for a sequence of times $t_i > 0$ with $t_i \to 0$. We mollify each F_{t_i} at an appropriately small scale, so that they become smooth, but so that the essential property, (c), of the F_{t_i} is not lost (at least up to a factor of 2). Then we flow each F_{t_i} on $B_{d_0}(x_0, 100)$ by the Ricci-harmonic map heat flow, keeping the boundary values fixed. The existence of the solutions $Z_{t_i} : B_{d_0}(x_0, 100) \times [t_i, T) \to \mathbb{R}^n$ is guaranteed by Theorem 2.1. According to Theorem 3.8, the $Z_{t_i}(s)$ are then $(1 + \alpha_0)$-bi-Lipschitz maps (on a smaller ball) for all $s \in [2t_i, S(n, \varepsilon_0)] \cap (0, T/2)$ if $\varepsilon_0 = \varepsilon_0(\alpha_0, n)$ is small enough. If we take the push-forward of $g(s)$ with respect to $Z_{t_i}(s)$, $s \in [2t_i, S(n, \varepsilon_0)] \cap (0, T/2)$, (on a smaller ball), then after taking a limit of a subsequence in i, we obtain a solution $(\tilde{g}(s))_{s\in(0,S(n,\varepsilon_0))\cap(0,T/2)}$ to the δ-Ricci–DeTurck flow such that $(1 - \alpha_0)\delta \leq \tilde{g}(s) \leq (1 + \alpha_0)\delta$ for all $s \in (0, S(n, \varepsilon_0)) \cap (0, T/2)$ in view of the estimates of Theorem 3.8. The solution then satisfies $d(\tilde{g}(t)) \to \tilde{d}_0 := (F_0)_*(d_0)$ as $t \to 0$ and hence may be thought of as a solution to the Ricci–DeTurck flow coming out of $\tilde{d}_0$. This is explained in Theorem 3.11.

In the next section we examine the regularity properties of this solution, which depend on the regularity properties of $\tilde{d}_0$.

3.1. Almost isometries and the Ricci-harmonic map heat flow

In this subsection we provide some technical lemmas giving insight into the evolution of almost isometries under the Ricci-harmonic map heat flow. These results will be needed in the following subsection.

Lemma 3.5. *For all σ, there exists $0 < \gamma(\sigma) \leq \sigma$ small with the following property: if $L : \mathbb{B}_{\gamma^{-1}}(0) \to \mathbb{R}^n$ is a γ-almost isometry fixing 0, then there exists an $S \in O(n)$ such that $|L - S|_{L^\infty(\mathbb{B}_{\sigma^{-1}}(0))} \leq \sigma$.*

Proof. If not, then for some $\sigma > 0$, we have a sequence of maps $L_i : \mathbb{B}_{i^{-1}}(0) \to \mathbb{R}^n$, $i \in \mathbb{N}$, such that L_i is an i^{-1}-almost isometry fixing 0, but L_i is not σ-close in the L^∞ sense to any element $S \in O(n)$ on $\mathbb{B}_{\sigma^{-1}}(0)$. Let $N := 2\sigma^{-1}$ and $D \subseteq \mathbb{B}_N(0)$ be a dense subset of $\mathbb{B}_N(0)$. By taking a diagonal subsequence and using $|L_i|_{L^\infty(\mathbb{B}_N(0))} \leq N + 1$ in conjunction with the Bolzano–Weierstraß Theorem, we obtain a map $L : D \to \mathbb{R}^n$ satisfying $L(x) := \lim_{i\to\infty} L_i(x)$ for all $x \in D$. This map satisfies $|L(x) - L(y)| = |x - y|$ for all $x, y \in D$, and hence may be continuously extended to a map $L : B_N(0) \to \mathbb{R}^n$ which is an isometry, $|L(x) - L(y)| = |x - y|$ for all $x, y \in B_N(0)$. Using the facts that the L_i's are almost isometries, $L_i \to L$ poinwise on D, and L is an isometry, we see that in fact $L_i \to L$ uniformly on $B_N(0)$ for $i \to \infty$, which contradicts the assumption that L_i is not σ-close to any $S \in O(n)$ on $B_{\sigma^{-1}}(0)$. ∎

Lemma 3.6. *For all $c_1, \alpha_0 > 0$ and $n \in \mathbb{N}$ there exists $0 < \alpha(c_1, n, \alpha_0) \leq \alpha_0$ such that the following is true. Let $Z : \mathbb{B}_{\alpha^{-1}}(0) \times [0, 1] \to \mathbb{R}^n$ be a smooth solution to the harmonic map heat flow with an evolving background metric,*

$$\frac{\partial}{\partial t} Z(\cdot, t) = \Delta_{h(t)} Z(\cdot, t),$$

where $h(\cdot, t)_{t\in[0,1]}$ is smooth and $Z_0(0) = 0$, where $Z_0(\cdot) := Z(\cdot, 0)$. Assume that Z_0 is an α-almost isometry with respect to $h(0)$, and further that

$$\begin{aligned}
&|h_{ij} - \delta_{ij}|_{C^2(\mathbb{B}_{\alpha^{-1}}(0)\times[0,1])} \leq \alpha,\\
&|\nabla^{h(t)} Z(\cdot, t)|_{h(t)} \leq c_1 \qquad \text{on } \mathbb{B}_{\alpha^{-1}}(0),\\
&|\nabla^{h(t),2} Z(\cdot, t)|_{h(t)} \leq \frac{c_1}{\sqrt{t}} \qquad \text{on } \mathbb{B}_{\alpha^{-1}}(0),
\end{aligned}$$

for all $t \in [0, 1]$. Then

$$\begin{aligned}
&|dZ(\cdot, s)(v)| \in (1 - \alpha_0, 1 + \alpha_0)|v|_{h(s)}\\
&|Z(x, s) - Z(y, s)| \in ((1 - \alpha_0)d(h(s))(x, y), (1 + \alpha_0)d(h(s))(x, y))\\
&|Z_0(x) - Z(x, t)| \leq \alpha_0,
\end{aligned} \tag{3.6}$$

for all $s \in [1/2, 1]$, all $t \in [0, 1]$, all $x, y \in \mathbb{B}_{\alpha_0^{-1}}(0)$ and all $v \in T_x\mathbb{B}_{\alpha_0^{-1}}(0)$.

Proof. We omit the dependence of the Levi-Civita connections on the metrics $h(t)$, $t \in [0, 1]$. From Lemma 3.5, we know that there exists an $S \in O(n)$ such that

$$|Z(\cdot, 0) - S(\cdot)|_{C^0(\mathbb{B}_{\beta^{-1}}(0))} \leq \beta$$

where $0 < \alpha \leq \beta = \beta(n, c_1, \alpha)$, but still $\beta(n, c_1, \alpha) \to 0$ as $\alpha \to 0$.

We also know that $\left|\frac{\partial}{\partial t} Z(x, t)\right| = |\Delta_{h(t)} Z(x, t)| \leq c_1/\sqrt{t}$ and hence $|Z(x, t) - Z(x, 0)| \leq 2c_1$ for all $t \in [0, 1]$ and all $x \in \mathbb{B}_{\beta^{-1}}(0)$, which implies

$$|Z(x, t) - S(x)| \leq |Z(x, t) - Z(x, 0)| + |Z(x, 0) - S(x)| \leq 3c_1$$

(we can assume $\beta \leq c_1$) for all $t \in [0, 1]$ for all $x \in \mathbb{B}_{\beta^{-1}}(0)$.

Let $\eta : \mathbb{B}_{\beta^{-1}}(0) \to [0,1]$ be a smooth cut-off function such that $\eta(\cdot) = 1$ on $\mathbb{B}_{\beta^{-1}/2}(0)$, $\eta(\cdot) = 0$ on $\mathbb{R}^n \setminus \mathbb{B}_{\beta^{-1}}(0)$, and

$$|D^2\eta| + |D\eta|^2/\eta \le c(n)\beta^2, \quad |D\eta| \le c(n)\beta.$$

Since h is α-close to δ in the C^2 norm, we see that

$$|\Delta_h(S)(x,t)| = |h^{ij}(x,t)(\partial_i\partial_j S(x) - \Gamma(h)_{ij}^k(x,t)\partial_k S)| \le c(n)\alpha.$$

Hence, we have

$$\frac{\partial}{\partial t}(Z - S)(\cdot,t) = \Delta_{h(t)}(Z - S)(\cdot,t) + E,$$

where $|E| \le c(n)\alpha$. But then, using the cut-off function η on $\mathbb{B}_{\beta^{-1}}(0)$, we get

$$\begin{aligned}\frac{\partial}{\partial t}(|Z - S|^2\eta) &\le \Delta_{h(t)}(|Z - S)|^2\eta) - 2\eta|\nabla(Z - S)|^2 - |Z - S|^2\Delta_{h(t)}\eta \\ &\quad + c(n)\alpha|Z - S| - 2h(t)(\nabla|Z - S|^2, \nabla\eta) \\ &\le \Delta_{h(t)}(|Z - S)|^2\eta) + c(n)\beta|Z - S|^2 - \eta|\nabla(Z - S)|^2 + c(n)c_1\alpha \\ &\le \Delta_{h(t)}(|Z - S|^2\eta) + c(n, c_1)\beta.\end{aligned}$$

Hence, by the maximum principle,

$$|Z - S|^2(t) \le \alpha^2 + c(n, c_1)\beta \le c(n, c_1)\beta$$

for all $t \in [0,1]$ on $\mathbb{B}_{\beta^{-1}/2}(0)$.

Since $|\nabla(Z - S)|^2 + |\nabla^2(Z - S)|^2 \le c(n)c_1$ for $s \in [1/2, 1]$ on $\mathbb{B}_{\beta^{-1}/2}(0)$, we can use interpolation inequalities as in [6, Lemma B.1] to deduce that on $\mathbb{B}_{\beta^{-1}/4}(0)$,

$$|D(Z - S)|_\delta \le c(n, c_1)\beta^{1/4}.$$

Again, since $h(t)$ is α-close to δ, this implies that for α sufficiently small,

$$|dZ(\cdot, s)(v)| \in (1 - \alpha_0, 1 + \alpha_0)$$

for all $v \in T_y\mathbb{R}^n$ of length 1 with respect to $h(s)$ for $s \in [1/2, 1]$ and $y \in \mathbb{B}_{\beta^{-1}/4}(0)$. We also see that

$$\begin{aligned}&|Z(x,s) - Z(y,s)| \\ &\qquad = |DZ(p,s)(v_p)|\,|x - y| \in \big(d(h(s))(x,y)(1 - \alpha_0), d(h(s))(x,y)(1 + \alpha_0)\big),\end{aligned}$$

where by the mean value theorem, p is some point on the unit speed line between x and y, and v_p is a vector of length 1 with respect to δ and

$$|Z(x,t) - Z_0(x)| \le |Z(x,t) - S| + |Z_0(x) - S| \le c(n, c_1)\beta^{1/2} \le \alpha_0$$

for all $x, y \in \mathbb{B}_{\beta^{-1}/4}(0)$, and all $s \in [1/2, 1]$, $t \in [0,1]$ and α sufficiently small. ∎

Lemma 3.7. *For all $n, k \in \mathbb{N}$ and $L > 0$ there exists an $\varepsilon_0 = \varepsilon_0(n, k, L) > 0$ such that the following holds. Let M^n be a connected smooth manifold, and g and h be smooth Riemannian metrics on M with $B_h(y_0, L) \Subset M$ and $|d_h - d_g|_{C^0(B_h(y_0,L))} \le \varepsilon_0$. Assume further that*

$$\sup_{B_h(y_0,L)} (|\mathrm{Rm}(g)|_g + \cdots + |\nabla^{g,k+2}\mathrm{Rm}(g)|_g) \le \varepsilon_0$$

and that there exists a map $F : B_h(y_0, L) \to \mathbb{R}^n$ which is an ε_0-almost isometry with respect to h, that is,

$$(1 - \varepsilon_0)d_h(z, y) - \varepsilon_0 \le |F(z) - F(y)| \le (1 + \varepsilon_0)d_h(z, y) + \varepsilon_0$$

for all $z, y \in B_h(y_0, L)$, and $F(y_0) = 0$. Then $(B_h(y_0, L/2), g)$ is $1/L$-close to the Euclidean ball $(\mathbb{B}_{L/2}(0), \delta)$ in the C^k-Cheeger–Gromov sense.

Proof. Assume this is not the case. Then there is an $L > 0$ for which the theorem fails, so we can find sequences $g(i), h(i)$, $M(i)$, $F(i) : B_{h(i)}(y_i, L) \to \mathbb{R}^n$ satisfying the above conditions with $\varepsilon_0 := 1/i$ but so that the conclusion of the theorem is not correct. Using the almost isometry, we see that for any $\varepsilon > 0$ we can cover $B_{h(i)}(y_0(i), 6L/7)$ by $N(\varepsilon)$ balls (with respect to h) of radius ε, for all i. Hence, using

$$|d_{h(i)} - d_{g(i)}|_{C^0(B_h(y_0,L))} \le 1/i,$$

we see that the same is true for $B_{g(i)}(y_0(i), 5L/6) \subseteq B_{h(i)}(y_0(i), 6L/7)$ with respect to $g(i)$: we can cover $B_{g(i)}(y_0(i), 5L/6)$ by $N(\varepsilon)$ balls (with respect to $g(i)$) of radius ε, for all i. Hence, due to the compactness theorem of Gromov (see for example [3, Theorem 8.1.10]), there is a Gromov–Hausdorff limit

$$(X, d) = \lim_{i\to\infty} (B_{g(i)}(y_0(i), 4L/5), g(i)) = \lim_{i\to\infty} (B_{h(i)}(y_0(i), 4L/5), h(i)).$$

In particular, there must exist

$$G(i) : B_d(y_0, 3L/4) \to B_{h(i)}(y_0(i), 4L/5),$$

which are Gromov–Hausdorff $\varepsilon(i)$-approximations, where $\varepsilon(i) \to 0$ as $i \to \infty$. Using the maps $G(i)$ and the $1/i$-almost isometries $F(i)$, we see that there is a pointwise limit map, $H := \lim_{i\to\infty} F(i) \circ G(i)$,

$$H : (B_d(y_0, 3L/4), d) \to (\mathbb{B}_{4L/5}(0), \delta),$$

which is an isometry. Hence the volume of $B_{g(i)}(y_0(i), 2L/3)$ converges to $\omega_n(2L/3)^n$ (in particular the sequence is non-collapsing) as $i \to \infty$, since volume is convergent for spaces of bounded curvature (which are for example Aleksandrov spaces and spaces with Ricci curvature bounded from below). Hence $(B_{g(i)}(y_0(i), L/2), g(i))$ converges to $(\mathbb{B}_{L/2}(0), \delta)$ in the C^k norm in the Cheeger–Gromov sense, which leads to a contradiction if i is large enough. ∎

3.2. The Ricci-harmonic map heat flow of $(1+\varepsilon_0)$*-bi-Lipschitz maps and distance coordinates*

We begin with a regularity theorem for solutions to the Ricci-harmonic map heat flow whose initial values are sufficiently close to a $(1+\varepsilon_0)$-bi-Lipschitz map.

Theorem 3.8. *For all* $\alpha_0 \in (0,1), n \in \mathbb{N}$ *and* $c_1 \in \mathbb{R}^+$*, there exist* $\varepsilon_0(n, c_1, \alpha_0) > 0$ *and* $S(n, c_1, \alpha_0) > 0$ *such that the following holds. Let* $(M, g(t))_{t\in(0,T)}$ *be a smooth solution to the Ricci flow satisfying the conditions* (a), (3.1) *and* (3.2) *for some* $R > 100$*, where* d_0 *is the metric appearing in* (3.3)*. If* $m_0 \in B_{d_0}(x_0, 15R)$ *and* $Z : B_{d_0}(m_0, 10) \times [t, T) \to \mathbb{R}^n$ *is a solution to the Ricci-harmonic map heat flow*

$$\frac{\partial}{\partial s} Z = \Delta_{g(s)} Z$$

for some $t \in [0, T)$ *on* $B_{d_0}(m_0, 10)$ *for all* $s \in [t, T)$*, which satisfies* $|\nabla^{g(s)} Z|_{g(s)} \leq c_1$ *on* $B_{d_0}(m_0, 10)$ *for all* $s \in [t, T)$*, and the initial values of* Z *satisfy* (c)*, that is,*

$$|Z(x,t) - Z(y,t)| \in \big((1-\varepsilon_0)d_t(x,y) - \varepsilon_0\sqrt{t}, (1+\varepsilon_0)d_t(x,y) + \varepsilon_0\sqrt{t}\big) \tag{3.7}$$

for all $x, y \in B_{d_0}(m_0, 10)$*, then*

$$|\nabla^{g(s),2} Z(x,s)|^2_{g(s)} \leq \alpha_0 (s-t)^{-1}, \tag{3.8}$$

$$|Z(x,s) - Z(x,t)| \leq c(c_1, n)\sqrt{s-t}, \tag{3.9}$$

$$|dZ(x,s)(v)| \in (1-\alpha_0, 1+\alpha_0)|v|_{g(s)}, \tag{3.10}$$

$$|Z(x,s) - Z(y,s)| \in ((1-\alpha_0)d_s(x,y), (1+\alpha_0)d_s(x,y)), \tag{3.11}$$

for all $x, y \in B_{d_0}(m_0, 2)$*, all* $v \in T_x B_{d_0}(m_0, 2)$*, and all* $s \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2]$*. Furthermore, the maps* $Z(s) : B_{d_0}(m_0, 3/2) \to D_s := Z(s)(B_{d_0}(m_0, 3/2)) \subseteq \mathbb{R}^n$ *for* $s \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2]$ *are homeomorphisms and their images satisfy*

$$\mathbb{B}_{5/4}(Z(s)(m_0)) \subseteq D_s \subseteq \mathbb{B}_2(Z(s)(m_0))$$

for $Z(s) := Z_s(\cdot)$*.*

A direct consequence of (3.10) is the following corollary.

Corollary 3.9. *Assuming the set-up of Theorem* 3.8*, the following metric inequalities hold:*

$$(1-\alpha_0)^2 g(x,s) \leq (Z(s))^*\delta \leq (1+\alpha_0)^2 g(x,s), \tag{3.12}$$

$$(1-\alpha_0)^2 (Z(s))_* g(y,s) \leq \delta \leq (1+\alpha_0)^2 (Z(s))_* g(y,s), \tag{3.13}$$

for all $x \in B_{d_0}(m_0, 2)$*, all* $y \in \mathbb{B}_{5/4}(Z(s)(m_0)) \subseteq Z(s)(B_{d_0}(m_0, 2)) \subseteq \mathbb{R}^n$ *and all* $s \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2]$*.*

Proof of Theorem 3.8. From Theorem 2.2, we know that $|\nabla^2 Z(x,s)|_{g(s)} \le c(c_1,n)s^{-1/2}$ and hence

$$|Z(x,s) - Z(x,t)| \le c(c_1,n)\sqrt{s-t} \tag{3.14}$$

for all $s \in [t, S(n,c_1)] \cap [0,T)$ and $x \in B_{d_0}(m_0,8) \subseteq B_{g(s)}(m_0,9) \subseteq B_{d_0}(m_0,10)$. We show the rest of the estimates hold for all $r \in [2t, S(n,c_1,\alpha_0)] \cap [0,T/2]$ and $x \in B_{d_0}(m_0,4)$ if $S(n,c_1,\alpha_0)$ is chosen small enough. First we scale the solution to the Ricci-harmonic map heat flow and the metric $g(t)$ by $1/\sqrt{r}$ respectively $1/r$:

$$\tilde{Z}(z,s) := \frac{1}{\sqrt{r}} Z(z,sr) \quad \text{and} \quad \tilde{g}(\cdot,s) = \frac{1}{r} g(\cdot,rs)$$

for $z \in B_{d_0}(m_0,8)$. Then the solution is defined for $s \in [\tilde{t} := t/r, \tilde{T} := T/r]$ on $B_{\tilde{d}_0}(x, 1/\sqrt{r})$ for any $x \in B_{d_0}(m_0,7)$ where $\tilde{t} = t/r \le 1/2$ since $r \ge 2t$, and $\tilde{T} \ge 2$ since $r \le T/2$, and the radius $V := 1/\sqrt{r}$ satisfies $V \ge 1/\sqrt{S(n,c_1,\alpha_0)}$. Since before scaling we have $|\mathrm{Rm}(\cdot,s)| \le \varepsilon_0^2/s$, after scaling we still have $|\widetilde{\mathrm{Rm}}(\cdot,s)| \le \varepsilon_0^2/s$ on $B_{\tilde{d}_0}(x,V)$ for all $s \in [\tilde{t},2]$. The time r has been scaled to time 1. The property (3.7) scales to

$$\begin{aligned} |\tilde{Z}(z,\tilde{t}) - \tilde{Z}(y,\tilde{t})| &\in \big((1-\varepsilon_0)\,\tilde{d}_{\tilde{t}}(z,y) - \varepsilon_0\sqrt{\tilde{t}}, (1+\varepsilon_0)\,\tilde{d}_{\tilde{t}}(z,y) + \varepsilon_0\sqrt{\tilde{t}}\big) \\ &\subseteq \big((1-\varepsilon_0)\,\tilde{d}_{\tilde{t}}(z,y) - \varepsilon_0, (1+\varepsilon_0)\,\tilde{d}_{\tilde{t}}(z,y) + \varepsilon_0\big) \end{aligned} \tag{3.15}$$

for all $z, y \in B_{\tilde{d}_0}(x,V)$ since $\tilde{t} = t/r \le 1/2$. The inequality (3.14) scales to

$$|\tilde{Z}(z,\tilde{s}) - \tilde{Z}(z,\tilde{t})| \le c(c_1,n)\sqrt{\tilde{s}-\tilde{t}} \tag{3.16}$$

for all $\tilde{s} \in [\tilde{t},2]$ and $z \in B_{\tilde{d}_0}(x,V)$, and the gradient estimate is also scale invariant: $|\tilde{\nabla}\tilde{Z}(\cdot,\tilde{s})|_{\tilde{g}(\tilde{s})} \le c_1$ still holds on $B_{\tilde{d}_0}(x,V)$ for all $\tilde{s} \in [\tilde{t},2]$. We also have

$$|\tilde{Z}(\cdot,\tilde{\sigma}) - Z(\cdot,\tilde{t})| \le c(c_1,n)\sqrt{\sigma}$$

on $B_{\tilde{d}_0}(x,V)$, due to (3.16), for $\tilde{\sigma} := \sigma + \tilde{t}$ with $\sigma \in (0,1)$. Hence

$$\begin{aligned} |\tilde{Z}(z,\tilde{\sigma}) - \tilde{Z}(y,\tilde{\sigma})| &\ge |\tilde{Z}(z,\tilde{t}) - \tilde{Z}(y,\tilde{t})| - |\tilde{Z}(z,\tilde{t}) - \tilde{Z}(z,\tilde{\sigma})| - |\tilde{Z}(y,\tilde{t}) - \tilde{Z}(y,\tilde{\sigma})| \\ &\ge (1-\varepsilon_0)\tilde{d}_{\tilde{t}}(z,y) - \varepsilon_0 - c(c_1,n)\sqrt{\sigma} \\ &\ge (1-\sigma)\tilde{d}_{\tilde{\sigma}}(z,y) - 2c(c_1,n)\sqrt{\sigma} \end{aligned}$$

for σ fixed and $\varepsilon_0 \le \sigma^2$, and similarly

$$|\tilde{Z}(z,\tilde{\sigma}) - \tilde{Z}(y,\tilde{\sigma})| \le (1+\sigma)\tilde{d}_{\tilde{\sigma}}(z,y) + 2c(c_1,n)\sqrt{\sigma}$$

for $z, y \in B_{\tilde{d}_0}(x,V)$ if $\varepsilon_0 \le \sigma^2$. That is, $\tilde{Z}(\cdot,\tilde{\sigma})$ is an α^2-almost isometry on $B_{\tilde{d}_0}(x,V)$ if we choose $\sigma = \alpha^8$. At this point we fix $\alpha := \alpha(n,c_1,\alpha_0^3)$ where α is the function appearing in the statement of Lemma 3.6, and we set $\sigma := \alpha^8$. Without loss of generality, $\alpha \le \alpha_0 < c(c_1, n)$, and $(\alpha_0)^{-1} \ge 2c(c_1, n)$ for any given $c(c_1, n) \ge 1$. We also still assume $\varepsilon_0 \le \sigma^2 = \alpha^{16}$, so that the previous conclusion, that $\tilde{Z}(\cdot, \tilde{\sigma})$ is an α^2-almost isometry, and hence certainly an α-almost isometry, on $B_{\tilde{d}_0}(x, V)$, holds, as explained above.

The curvature estimate, $|\widetilde{\mathrm{Rm}}(\cdot, s)| \le \varepsilon_0^2/s$ for all $s \in [0, 2] \supseteq [\tilde{t}, 2]$, holds, as do the scaled distance estimates

$$\tilde{d}_\ell + \varepsilon_0 \sqrt{s - \ell} \ge \tilde{d}_s \ge \tilde{d}_\ell - \varepsilon_0 \sqrt{s - \ell} \tag{3.17}$$

on $B_{\tilde{d}_0}(x, V) \Subset M$, for all $0 \le \ell \le s \in [0, 2]$. Shi's estimates imply, as explained in Lemma 3.2, that at time $\tilde{\sigma} := \tilde{t} + \sigma = \tilde{t} + \alpha^8$,

$$|\widetilde{\mathrm{Rm}}|(\cdot, \tilde{\sigma}) + |\tilde{\nabla}\widetilde{\mathrm{Rm}}|^2(\cdot, \tilde{\sigma}) + \cdots + |\tilde{\nabla}^k \widetilde{\mathrm{Rm}}|(\cdot, \tilde{\sigma}) \le \frac{\beta(k, n, \varepsilon_0)}{\sigma^{k+2}} = \frac{\beta(k, n, \varepsilon_0)}{\alpha^{8k+16}}$$

on $B_{\tilde{d}_0}(x, 2V/3)$, where $\beta(k, n, \varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$ for fixed k and n. In particular

$$\sup_{B_{\tilde{d}_0}(x, 2V/3)} \big(|\widetilde{\mathrm{Rm}}|(\cdot, \tilde{\sigma}) + |\tilde{\nabla}\widetilde{\mathrm{Rm}}|^2(\cdot, \tilde{\sigma}) + \cdots + |\tilde{\nabla}^k \widetilde{\mathrm{Rm}}|(\cdot, \tilde{\sigma})\big) \to 0 \tag{3.18}$$

as $\varepsilon_0 \to 0$ for fixed c_1, k, n, α_0. Without loss of generality, $L = 1/\sqrt{S(n, c_1, \alpha_0)} \ge 10/\alpha$ and hence $B_{\tilde{d}_s}(x, 2\alpha^{-1}) \subseteq B_{\tilde{d}_0}(x, L/2) \subseteq B_{\tilde{d}_0}(x, 2V/3)$ for all $s \in [0, 2]$.

By (3.17), (3.18), and (3.15), we see using Lemma 3.7 with $h = \tilde{g}(\tilde{t})$ and $g = \tilde{g}(\tilde{\sigma})$, $L = \alpha^{-1}$ that $(B_{\tilde{d}_0}(x, \alpha^{-1}), \tilde{g}(\tilde{\sigma}))$ is α-close in the C^k norm to a Euclidean ball with the standard metric in the Cheeger–Gromov sense (that is, up to smooth diffeomorphism) if ε_0 is small enough.

Hence there are geodesic coordinates φ on the ball $(B_{\tilde{d}_0}(x, \alpha^{-1}), \tilde{g}(\tilde{\sigma}))$ such that the metric $\tilde{g}(\tilde{\sigma})$ written in these coordinates is α-close to δ in the C^k norm if we keep σ fixed and choose ε_0 small enough. Using (3.18) and the evolution equation $\frac{\partial}{\partial t}\tilde{g} = -2\mathrm{Ric}(\tilde{g})$ in the coordinates φ, we see that the evolving metric $h(\cdot) = \varphi_*(\tilde{g}(\cdot))$ in these coordinates is also, without loss of generality, α-close to δ for $t \in [\tilde{\sigma}, 2]$ in the C^2 norm.

Using the above, we see that

$$G(\cdot, t) := (\tilde{Z}(\cdot, t + \tilde{\sigma}) - \tilde{Z}(x, \tilde{\sigma})) \circ (\varphi)^{-1}(\cdot)$$

defined on $\mathbb{B}_{\alpha^{-1}}(0) \times [0, 3/2]$ sends 0 to 0 at $t = 0$, is Lipschitz with respect to δ with Lipschitz constant $2c(c_1, n)$ and is an α-almost isometry at time 0. Lemma 3.6 is then applicable to the function G and tells us that $G(\cdot, s)$ is an α_0^3-almost isometry at $s = 1 - \tilde{\sigma}$ on a ball of radius $(\alpha_0)^{-3}$ and the inequalities (3.6) hold. Hence,

$$\begin{aligned}
&|d\tilde{Z}(v)(1)| \in ((1 - \alpha_0^3)|v|_{\tilde{g}(1)}, (1 + \alpha_0^3)|v|_{\tilde{g}(1)}),\\
&|\tilde{Z}(z, 1) - \tilde{Z}(y, 1)| \in ((1 - \alpha_0^3)d_{\tilde{g}(1)}(z, y), (1 + \alpha_0^3)d_{\tilde{g}(1)}(z, y)),\\
&|\tilde{Z}(z, \tilde{t}) - \tilde{Z}(z, 1)| \le \alpha_0^3\sqrt{1 - \tilde{t}},
\end{aligned}$$

for all $z, y \in B_{\tilde{d}_0}(x, \alpha_0^{-3})$ and all $v \in T_z M$. This scales back to

$$
\begin{aligned}
&|dZ(v)(r)| \in ((1-\alpha_0^3)|v|_{g(r)}, (1+\alpha_0^3)|v|_{g(r)}),\\
&|Z(z,r)-Z(y,r)| \in ((1-\alpha_0^3)\, d_{g(r)}(z,y), (1+\alpha_0^3)\, d_{g(r)}(z,y)),\\
&|Z(z,t)-Z(z,r)| \le \alpha_0^3\sqrt{r-t},
\end{aligned}
$$

for all $z, y \in B_{d_0}(x, \sqrt{r}/\alpha_0^3)$ and all $v \in T_z M$.

For $z \in B_{d_0}(x, \sqrt{r}/(2\alpha_0^3))$ and $y \in (B_{d_0}(x, \sqrt{r}/\alpha_0^3))^c \cap B_{d_0}(m_0, 6)$, we show that the property (3.11) also holds. For such z, y, we have $d_0(z, y) \ge \sqrt{r}/(2\alpha_0^3)$ and hence

$$
\begin{aligned}
|Z(z,r)-Z(y,r)| &\ge |Z(z,t)-Z(y,t)| - c(c_1,n)\sqrt{r-t}\\
&\ge (1-\varepsilon_0)d_t(z,y) - \varepsilon_0\sqrt{t} - c(c_1,n)\sqrt{r}\\
&\ge (1-\varepsilon_0)d_0(z,y) + (1-\varepsilon_0)(d_t(z,y)-d_0(z,y)) - 2c(c_1,n)\sqrt{r}\\
&\ge (1-\varepsilon_0)d_0(z,y) - \varepsilon_0\sqrt{r} - 2c(c_1,n)\sqrt{r}\\
&\ge (1-\alpha_0^2)d_0(z,y) - 3c(c_1,n)\sqrt{r}\\
&\ge (1-10\alpha_0^2)d_0(z,y) + 9\alpha_0^2 d_0(z,y) - 3c(c_1,n)\sqrt{r}\\
&\ge (1-10\alpha_0^2)d_0(z,y) + 9\alpha_0^2\frac{\sqrt{r}}{2\alpha_0^3} - 3c(c_1,n)\sqrt{r}\\
&\ge (1-10\alpha_0^2)d_0(z,y) + 9c(c_1,n)\sqrt{r} - 3c(c_1,n)\sqrt{r}\\
&\ge (1-10\alpha_0^2)d_0(z,y) + \sqrt{r}\\
&\ge (1-10\alpha_0^2)d_r(z,y) - \varepsilon_0\sqrt{r} + \sqrt{r}\\
&\ge (1-\alpha_0)d_r(z,y)
\end{aligned}
$$

as required, where the first inequality follows from (3.14), the second from the condition (c), the seventh and eighth from $d_0(z, y) \ge \sqrt{r}/(2\alpha_0^3) \ge c(c_1, n)\sqrt{r}/\alpha_0^2$, and we have used $\varepsilon_0 < \alpha_0^4$, $c(c_1, n) \ge 1$, the distance estimates (3.1), and $\alpha_0 \le 1/c(c_1, n)$ freely.

It remains to show that

$$
\mathbb{B}_1(Z(s)(m_0)) \subseteq D_s \subseteq \mathbb{B}_2(Z(s)(m_0))
$$

for $D_s := Z(s)(B_{d_0}(m_0, 3/2)) \subseteq \mathbb{R}^n$, for $s \in (2t, S]$, $s \le T/2$, $Z(s) := Z(\cdot, s)$. Observe that

$$
Z(s) : B_{d_0}(m_0, 3/2) \to \mathbb{R}^n
$$

is smooth and satisfies (3.10). In particular, $Z(s)$ is a local diffeomorphism, due to the Inverse Function Theorem, and hence $\mathbb{B}_r(Z(s)(m_0)) \subset D_s$ for some maximal $r > 0$. Let $p_i := Z(s)(x_i) \in \mathbb{B}_r(Z(s)(m_0)) \cap D_s$ be such that $p_i \to p \in \partial\mathbb{B}_r(Z(s)(m_0))$ where $p \notin D_s$. Assume that $r \le 5/4$. Then $x_i \in B_{d_0}(m_0, 5/4 + C(n)\alpha)$. After taking a subsequence if necessary, $x_i \to x \in B_{d_0}(m_0, 5/4 + C(n)\alpha))$, and consequently $p_i = Z(s)(x_i) \to Z(s)(x) = p \in D_s$ as $i \to \infty$, which contradicts the definition of $r > 0$ if $r \le 5/4$. Hence $r \ge 5/4$, which implies $\mathbb{B}_{5/4}(Z(s)(m_0)) \subset D_s$, as claimed (the other inclusion follows immediately from the bi-Lipschitz property). ∎

With the help of the previous theorem, we now show that it is possible to construct a solution to the δ-Ricci–DeTurck flow coming out of $\tilde{d}_0 := (F_0)_* d_0$ using the harmonic map heat flow if we assume that ($\hat{c}$) is satisfied. First we show that by slightly mollifying the distance coordinates at time t, we obtain maps which satisfy (c).

Lemma 3.10. *Let $(M, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow satisfying ($\tilde{a}$) and (b) for some $R \geq \beta^2(n)\varepsilon_0^2 + 200$ and let d_0 be as defined in (3.3). Assume that there are points $a_1, \ldots, a_n$ such that $F_0 : B_{d_0}(x_0, R) \to \mathbb{R}^n$, $F_0(x) := (d_0(a_1, x), \ldots, d_0(a_n, x))$, satisfies ($\hat{c}$) on $B_{d_0}(x_0, 100)$, and let $F_t : B_{d_0}(x_0, R) \to \mathbb{R}^n$ be given by*

$$F_t(x) = (d_t(a_1, x), \ldots, d_t(a_n, x)).$$

Then by mollifying F_t at an appropriately small scale, as explained in the proof, we obtain a map $\hat{F}_t : B_{d_0}(x_0, R) \to \mathbb{R}^n$ which is smooth and satisfies

$$|\nabla^{g(t)} \hat{F}_t|_{g(t)} \leq c(n)$$

as well as (c) on $B_{d_0}(x_0, 50)$ (with ε_0 replaced by $2\varepsilon_0$) provided $t \leq \hat{T}(\varepsilon_0, R)$.

Proof. As already noted,

$$F_t|_{B_{d_0}(x_0,50)} : B_{d_0}(x_0, 50) \to \mathbb{R}^n$$

satisfies (c) in view of the distance estimates (3.1) if $t \leq \hat{T}(\varepsilon_0, R)$. Also, it is well known that the Lipschitz norm of any map F_t as defined above may be estimated by a constant depending only on n:

$$\sup_{x\neq y\in B_{d_0}(x_0,R)} \frac{|F_t(x) - F_t(y)|}{d_t(x, y)} \leq c(n),$$

in view of the triangle inequality. Hence, by mollifying the map F_t at an appropriately small scale, we obtain a map $\hat{F}_t : B_{d_0}(x_0, 50) \to \mathbb{R}^n$ which is smooth and satisfies (c) (with ε_0 replaced by $2\varepsilon_0$) and $|\nabla^{g(t)} \hat{F}_t|_{g(t)} \leq c(n)$. ∎

Theorem 3.11. *Let $(M, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow satisfying ($\tilde{a}$) and (b) for an $R \geq \beta^2(n)\varepsilon_0^2 + 200$ and let d_0 be as defined in (3.3). Assume that there are points $a_1, \ldots, a_n$ such that $F_0 : B_{d_0}(x_0, R) \to \mathbb{R}^n$, $F_0(x) := (d_0(a_1, x), \ldots, d_0(a_n, x))$ satisfies ($\hat{c}$) on $B_{d_0}(x_0, 100)$, and let $\hat{F}_{t_i}$ be the corresponding mollified functions from Lemma 3.10 for any sequence of times $t_i > 0$ with $t_i \to 0$ as $i \to \infty$. Let $\hat{S} := \min(S(n, c(n), \alpha), \hat{T}(\varepsilon_0, R), T)$, where $c(n), T(\varepsilon_0, R)$ come from Lemma 3.10, and S comes from Theorem 3.8, and let*

$$Z_{t_i} : B_{d_0}(x_0, 100) \times [t_i, \hat{S}] \to \mathbb{R}^n$$

be the Dirichlet solution to the Ricci-harmonic map heat flow with boundary and initial values given by $Z_{t_i}(\cdot, s)|_{\partial B_{d_0}(x_0,100)} = \hat{F}_{t_i}(\cdot)$ for all $s \in [t_i, \hat{S}]$ and $Z_{t_i}(\cdot, t_i) = \hat{F}_{t_i}(\cdot)$. Then, after taking a subsequence in i, the maps

$$Z_{t_i}(s) : B_{d_0}(x_0, 3/2) \to D_{s,i} := Z_{t_i}(s)(B_{d_0}(x_0, 3/2)) \subseteq \mathbb{R}^n$$

are homeomorphisms for all $s \in [2t_i, \hat{S}]$, *with* $\mathbb{B}_1(F_0(x_0)) \subseteq D_{s,i}$ *and* $(Z_{t_i})_* g \to \tilde{g}$ *smoothly as* $i \to \infty$ *on compact subsets of* $\mathbb{B}_1(F_0(x_0)) \times (0, \hat{S}]$, *where* $(\tilde{g}(s))_{s\in(0,\hat{S}]}$ *is a smooth family of metrics which solve the* δ*-Ricci–DeTurck flow and*

$$(1-\alpha_0)\delta \leq \tilde{g}(s) \leq (1+\alpha_0)\delta \tag{3.19}$$

for all $s \in (0, \hat{S})$ *provided* $\varepsilon_0 = \varepsilon_0(\alpha_0, n) > 0$ *from* (a) *and* ($\hat{c}$) *is small enough. The metric* $\tilde{d}(t) := d(\tilde{g}(t))$ *satisfies* $\tilde{d}(t) \to \tilde{d}_0 := (F_0)_* d_0$ *uniformly on* $\mathbb{B}_1(F_0(x_0))$ *as* $t \to 0$.

Remark 3.12. Examination of the proof of Theorem 3.11 shows that:

(i) We can remove condition (b) if we assume that the estimates (3.1) are satisfied.

(ii) If we remove condition ($\hat{c}$) and replace it by the assumption: there exists a sequence of times $t_i > 0$ with $t_i \to 0$ as $i \to \infty$, and maps $\hat{F}_{t_i} : B_{d_0}(x_0, 100) \to \mathbb{R}^n$ each of which satisfies (c), $\sup_{i\in\mathbb{N}} |\hat{F}_{t_i}(x_0)| < \infty$, and $|\nabla^{g(t_i)} \hat{F}_{t_i}|_{g(t_i)} \leq c_1$ then we can use those $\hat{F}_{t_i}$ in the above, instead of the slightly mollified distance functions, and the conclusions of the theorem still hold for $s \leq \hat{S} := \min(S(n, c_1, \alpha_0), T/2)$ if the $\varepsilon_0 = \varepsilon_0(n, c_1, \alpha_0)$ appearing in (a) and (c) is small enough. In this case, F_0 is the uniform C^0 limit of a subsequence of the F_{t_i} as $i \to \infty$ and satisfies (3.5). The existence of such an F_0 is always guaranteed in this setting, as explained directly after the introduction of condition (c).

Proof of Theorem 3.11. Theorem 3.8 tells us that the maps $Z_{t_i}(s) : B_{d_0}(x_0, 3/2) \to D_{s,i}$ are homeomorphisms with $\mathbb{B}_{5/4}(Z_{t_i}(s)(x_0)) \subseteq D_{s,i}$ for $s \in [2t_i, \hat{S}]$. Hence

$$\mathbb{B}_1(F_0(x_0)) \subseteq \mathbb{B}_{5/4}(Z_{t_i}(s)(x_0)) \subseteq D_{s,i}$$

for $s \in [2t_i, \hat{S}]$, in view of (3.9). We define $\tilde{g}_i(s) := (Z_{t_i})_* g(s)$ for $s \in [2t_i, \hat{S}]$ on $\mathbb{B}_{3/2}(F_{t_i}(x_0))$. This is well defined in view of Theorem 3.8. Then $\tilde{g}_i$ is a solution to the δ-Ricci–DeTurck flow on $\mathbb{B}_{3/2}(F_{t_i}(x_0))$ (see [9, Chapter 6] for instance) and satisfies the metric inequalities (3.19) for all $s \in (2t_i, \hat{S})$ in view of Corollary 3.9. Using [16, Lemma 4.2] we see that

$$|D^j \tilde{g}_i(s)| \leq \frac{c(j,n)}{(s-2t_i)^{p(j)}}$$

for all $j \in \mathbb{N}$ and all $s \in (2t_i, \hat{S})$ on $\mathbb{B}_1(F_{t_i}(0))$. Taking a subsequence in i we obtain the desired solution $(\tilde{g}(s))_{s\in(0,\hat{S})}$ on $\mathbb{B}_1(F_0(0))$ with

$$|D^j \tilde{g}(s)| \leq \frac{c(j,n)}{s^{p(j)}}.$$

The Z_{t_i} all satisfy the estimates stated in the conclusions of Theorem 3.8, and so there is a uniform $C^{1,\alpha}$ limit map $Z : B_{d_0}(x_0, 2) \times (0, \hat{S}) \to \mathbb{R}^n$, in view of the Arzelà–Ascoli Theorem. Furthermore, $Z(s) = Z(\cdot, s)$ satisfies

$$|Z(s) - F_0| \leq c_1\sqrt{s}$$

for $s \in (0, \hat{S})$ in view of the estimate (3.9). Let $v, w \in \mathbb{B}_1(0)$ be arbitrary, and x, y the corresponding points in $B_{d_0}(x_0, 2)$ at time s, that is, the unique points x, y with $Z(s)(x) = v$, $Z(s)(y) = w$. Then

$$\begin{aligned}\tilde{d}_s(v, w) = d_s(x, y) &\le d_0(x, y) + \varepsilon_0\sqrt{s} = \tilde{d}_0(F_0(x), F_0(y)) + \varepsilon_0\sqrt{s}\\ &\le \tilde{d}_0(Z(s)(x), Z(s)(y)) + \beta(s) + \varepsilon_0\sqrt{s} = \tilde{d}_0(v, w) + \beta(s) + \varepsilon_0\sqrt{s},\end{aligned}$$

where $\beta(s) \to 0$ as $s \to 0$. Here, we have used the fact that $\tilde{d}_0 := (F_0)_*(d_0)$ is continuous, and hence uniformly continuous on $\overline{\mathbb{B}_1(0)}$, and that

$$\sup_{B_{d_0}(x_0,2)} |Z(s)(\cdot) - F_0(\cdot)| \le c_1\sqrt{s}$$

for all $s \in (0, \hat{S})$ in the above. The continuity of $\tilde{d}_0 := (F_0)_* d_0$ with respect to the norm in $\mathbb{R}^n$ follows from the fact that $\tilde{d}_0$ is a metric, equivalent to the standard metric on $\mathbb{R}^n$ in view of the property (3.5). Similarly, $\tilde{d}_s(v, w) \ge \tilde{d}_0(v, w) - \beta(s) - \varepsilon_0\sqrt{s}$, as required. ∎

4. The Ricci-harmonic map heat flow in the continuous setting

We now assume, in addition to the assumptions (a), (b) and ($\hat{c}$) of the previous section, more regularity on d_0 and $\tilde{d}_0$. Namely, we assume that $\tilde{d}_0$ is generated by a continuous Riemannian metric $\tilde{g}_0$ on $\mathbb{B}_1(0)$. This assumption will guarantee for all $\varepsilon > 0$ the existence of local maps defined on balls of radius $r(\varepsilon)$, which are $(1+\varepsilon)$-bi-Lipschitz maps at $t = 0$. We explain this in the following lemma.

Lemma 4.1. *Let (X, d) be a metric space, $B_d(y_0, 10) \Subset X$ and $F : B_d(y_0, 1) \to \mathbb{R}^n$ be a $(1+\varepsilon_0)$-bi-Lipschitz homeomorphism with $F(y_0) = 0$, and assume that $\tilde{d} = (F)_* d$ is generated on $\mathbb{B}_{1/4}(0)$ by a continuous Riemannian metric $\tilde{g}$, which is defined on $\mathbb{B}_1(0)$. Then for all $\varepsilon > 0$, there exists an $r > 0$ such that for all $p \in B_d(y_0, 1/8)$ there exists a linear transformation, $A = A(p)$, $A : \mathbb{B}_r(F(p)) \to \mathbb{R}^n$ with $|A - \mathrm{Id}|_{C^0} \le 2$ such that $\hat{F} := A \circ F$ satisfies*

$$(1-\varepsilon)|\hat{F}(y) - \hat{F}(q)| \le d(y, q) \le (1+\varepsilon)|\hat{F}(y) - \hat{F}(q)|$$

for all $y, q \in B_d(p, r/2)$, and $\mathrm{Vol}(B_d(p, s)) \in ((1-\varepsilon)^n \omega_n s^n, (1+\varepsilon)^n \omega_n s^n)$ for all $s \le r$ and all such p.

Proof. The continuity of $\tilde{g}$ means: for any $\varepsilon > 0$ and any $x \in \mathbb{B}_1(0)$ we can find an $r > 0$ and a linear transformation $A : \mathbb{B}_r(x) \to A(\mathbb{B}_r(x))$ with $|A - \mathrm{Id}|_{C^0(\mathbb{B}_r(x))} \le c(n)\varepsilon_0$ such that $\hat{g} := A_*(\tilde{g})$ satisfies $|\hat{g} - \delta|_{C^0(\mathbb{B}_r(x))} \le \varepsilon$, and $\hat{g}(x) = \delta$. For the distance $\hat{d} := A_*\tilde{d}$ this means

$$(1-\varepsilon)|z - w| \le \hat{d}(z, w) \le (1+\varepsilon)|z - w|$$

for all $z, w \in \mathbb{B}_{r/2}(A(x))$. Returning to the original domain, we see that this means

$$(1-\varepsilon)|\hat{F}(y) - \hat{F}(q)| \le d(y, q) \le (1+\varepsilon)|\hat{F}(y) - \hat{F}(q)|$$

for all $y, q \in B_d(p, r/2)$ where $\hat{F}(p) = A(x)$, and $\hat{F} = A \circ F$. This means in particular, in view of the existence of the $(1+\varepsilon)$-bi-Lipschitz map $\hat{F}$, that $(1 + c(n)\varepsilon)^n \omega_n s^n \geq \mathrm{Vol}(B_d(p, s)) \geq (1 - c(n)\varepsilon)^n \omega_n s^n$ for all $s \leq r$. ∎

This implies that we can replace condition (a) by

$$|\mathrm{Rm}(\cdot, t)| \leq \varepsilon(t)/t \text{ on } B_{d_0}(x_0, 1/10) \quad \text{for all } t \in (0,1) \text{ where}$$
$$\varepsilon : [0,1] \to [0,1] \text{ is a continuous non-decreasing function with } \varepsilon(0) = 0, \tag{â}$$

as we show in the following lemma.

Lemma 4.2. *Assume $(M, g(t))_{t \in (0,T)}$ is a solution to the Ricci flow satisfying (ã) and (b) for some $R \geq \beta^2(n)\varepsilon_0^2 + 200$, and assume that $F_0 : B_{d_0}(x_0, 1) \to \mathbb{R}^n$ is a bi-Lipschitz map and $\tilde{d}_0 = (F_0)_* d_0$ and F_0 satisfy the assumptions of Lemma 4.1, where d_0 is defined by (3.3). Then (â) holds on $B_{d_0}(x_0, 1/10)$.*

Proof. Let $\sigma > 0$ be given, and assume that there are $t_i \to 0$ and $p_i \in B_{d_0}(x_0, 1/100)$ with $|\mathrm{Rm}(p_i, t_i)| = \sigma / t_i$. We scale the solution $(g(t))_{t \in (0,T)}$ so that the time t_i scales to time 1, i.e. we define a sequence of solutions to the Ricci flow as follows:

$$g_i(s) := t_i^{-1} g(t_i s) \quad \text{for } s \in (0, T/t_i).$$

The distance estimates (3.1) and Shi's estimates (3.4) hold, and $\mathrm{Vol}(B_{g_i(s)}(p_i, 1)) \geq \omega_n/2$ for all $s \in (0, 100)$ in view of Corollary 3.9. Hence, after taking a subsequence, we obtain a smooth solution $(\Omega, \ell(t), p_0)_{t \in (0,10]}$ with $|\mathrm{Rm}(p_0, 1)| = \sigma$, $\mathrm{Ric} \geq 0$ everywhere, and $|\mathrm{Rm}(\cdot, t)| \leq \varepsilon_0^2/t$ everywhere. Furthermore, writing $d_0(i) := t_i^{-1/2} d_0$, we see, in view of the distance estimates, that $d(\ell(t)) \to \hat{d}_0$ as $t \to 0$, where $(B_{d_0(i)}(p_i, 1/\sqrt{t_i}), d_0(i), p_i) \to (\Omega, \hat{d}_0, p_0)$ in the Gromov–Hausdorff sense as $i \to \infty$. But $(\Omega, \hat{d}_0)$ must be isometric to $(\mathbb{R}^n, \delta)$ since there are $(1+\varepsilon)$-bi-Lipschitz maps $F(i) : (B_{d_0(i)}(p_i, r(\varepsilon)/\sqrt{t_i}), d_0(i)) \to \mathbb{R}^n$ for all $\varepsilon > 0$, in view of Lemma 4.1. Using a similar argument to the one used in [22] and [14], we see that the asymptotic volume ratio of $(\Omega, \ell(t))$ must be ω_n, as we now explain. Take a sequence $R_i \to \infty$ at time $s > 0$. Scale $\ell_i(s) := R_i^{-2} \ell(s R_i^2)$. Since $(\Omega, R_i^{-1} \hat{d}_0)$ is isometric to $(\mathbb{R}^n, \delta)$ for all $i \in \mathbb{N}$, we must have $(B_{\hat{\ell}_i}(x_0, 1), \hat{\ell}_i) \to (\mathbb{B}_1(0), \delta)$ in the Gromov–Hausdorff sense as $i \to \infty$ for $\hat{\ell}_i := \ell_i(1/R_i^2)$ $(= R_i^{-2} \ell(1))$ in view of the scaled distance estimates (3.1). Hence, $\mathrm{Vol}(B_{\hat{\ell}_i}(x_0, 1), \hat{\ell}_i) \to \omega_n$ as $i \to \infty$, in view of the theorem of Cheeger–Colding on volume convergence [5, Theorem 5.9]. But this means that the asymptotic volume ratio of $(\Omega, \ell(s), p_0)$ is ω_n, and hence $(\Omega, \ell(s), p_0)$ is isometric to $(\mathbb{R}^n, \delta)$, in view of the Bishop–Gromov comparison principle (the case of equality). This contradicts the fact that $|\mathrm{Rm}(p_0, 1)| = \sigma$. ∎

Note that we obtain better distance estimates in the setting of this lemma:

$$d_r + \varepsilon(t)\sqrt{t-r} \geq d_t \geq d_r - \varepsilon(t)\sqrt{t-r} \quad \text{for all } t \in [r, 1)$$
$$\text{on } B_{d_0}(x_0, 1/20) \subseteq B_{g(s)}(x_0, 1) \Subset M \tag{4.1}$$

for all $r \geq 0$, where ε is without loss of generality the same function that appears in condition ($\hat{a}$).

Theorem 4.3. *Assume $(M, g(t))_{t\in(0,T]}$ is a solution to the Ricci flow satisfying ($\tilde{a}$) and (b), and that there are points $a_1, \ldots, a_n$ such that $F_0 : B_{d_0}(x_0, R) \to \mathbb{R}^n$, $F_0(\cdot) := (d_0(a_1,\cdot), \ldots, d_0(a_n,\cdot))$ satisfies ($\hat{c}$) on $B_{d_0}(x_0, 100)$, and $\tilde{d}_0 = (F_0)_* d_0$ and F_0 satisfy the assumptions of Lemma 4.1, where d_0 is as defined in (3.3). Then the solution $(\mathbb{B}_{1/2}(0), \tilde{g}(s))_{s\in(0,\min T, S(n,\alpha_0,c_1)]}$ to the δ-Ricci–DeTurck flow constructed in Theorem 3.11 satisfies $|\tilde{g}(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{1/20}(0))} \to 0$ as $s \to 0$.*

Proof. Using Lemma 4.1 we see the following: for any $\varepsilon > 0$ and any $p_0 \in B_{d_0}(x_0, 1/20)$ we can find an $r > 0$ and a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(1-\varepsilon)|\hat{F}_0(y) - \hat{F}_0(q)| \leq d_0(y,q) \leq (1+\varepsilon)|\hat{F}_0(y) - \hat{F}_0(q)| \tag{4.2}$$

for all $y, q \in B_{d_0}(p_0, r)$, with $\hat{F}_0 = A \circ F_0$. We define $z_0 := F_0(p_0)$ and $\hat{z}_0 := \hat{F}_0(p_0)$. Now since A is a linear transformation with $|A - \mathrm{Id}|_{C^0} \leq 2$, and (4.1) holds, we have $|\hat{F}_t - \hat{F}_0|_{C^0(B_{d_0}(x_0,1/20))} \leq \varepsilon(t)\sqrt{t}$ for $\hat{F}_t = A \circ F_t$, and hence

$$(1-\varepsilon)d_t(v,q) - \varepsilon\sqrt{t} \leq |\hat{F}_t(y) - \hat{F}_t(q)| \leq (1+\varepsilon)d_t(v,q) + \varepsilon\sqrt{t}$$

on $B_{d_0}(p_0, r)$ for all $t \leq T(\varepsilon)$, where we have also used (4.2). Let $Z_{t_i} : B_{d_0}(x_0, 1/2) \times [t_i, S(n,\alpha_0)) \to \mathbb{R}^n$ be the solutions to the Ricci-harmonic map heat flow constructed in Theorem 3.11. Then $\hat{Z}_{t_i} = A \circ Z_{t_i}$ is also a solution to the Ricci-harmonic map heat flow. Using the regularity theorem, Theorem 3.8, we see that we must have

$$\begin{aligned}
&(1-\sigma(\varepsilon))d_s(z,w) \leq |\hat{Z}_{t_i}(z,s) - \hat{Z}_{t_i}(w,s)| \leq (1+\sigma(\varepsilon))d_s(z,w),\\
&|\nabla^{g(s)}\hat{Z}_{t_i}(v)| \in \big((1-\sigma(\varepsilon))|v|_{g(s)}, (1+\sigma(\varepsilon))|v|_{g(s)}\big),
\end{aligned}$$

for all $z, w \in B_{d_0}(p_0, r/5)$, all $v \in T_z B_{d_0}(p_0, r/5)$, and all $s \in (2t_i, S(n,\varepsilon))$ where $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence Corollary 3.9 tells us that the push-forward $\hat{g}_i(s) := (\hat{Z}_i)_*(g(s))_{s\in(2t_i,S(n,\varepsilon))}$ satisfies $|\hat{g}_i(s) - \delta|_{C^0(\mathbb{B}_{r/5}(\hat{z}_0))} \leq \sigma(\varepsilon)$. Transforming back with A^{-1} we see that this means $|\tilde{g}_i(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{r/5}(z_0))} \leq \sigma(\varepsilon)$ for all $s \in (2t_i, S(n,\varepsilon))$, and hence $|\tilde{g}(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{r/5}(z_0))} \leq \sigma(\varepsilon)$ for all $s \in (0, S(n,\varepsilon))$. As $p_0 \in \mathbb{B}_{d_0}(x_0, 1/20)$ was arbitrary, we see by letting $\varepsilon \to 0$ that $|\tilde{g}(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{1/20}(0))} \to 0$ as $s \to 0$, as required. ∎

The estimates of the previous theorem allow us to give a proof of the second main theorem of the introduction:

Proof of Theorem 1.7. Let $a_1, \ldots, a_n \in B_{d_0}(x_0, r)$ be as in the statement of Theorem 1.7. That is, $F_0(\cdot) := (d_0(a_1,\cdot), \ldots, d_0(a_n,\cdot))$ is $(1+\varepsilon_0)$-bi-Lipschitz on $B_{d_0}(x_0, 5\tilde{r})$ for some $\tilde{r} \leq r/5$. The metric $\tilde{d}_0(\tilde{x}, \tilde{y}) := d_0((F_0)^{-1}(\tilde{x}), (F_0)^{-1}(\tilde{y}))$ defined on $B_{\tilde{r}}(F_0(x_0))$ is generated by a continuous (with respect to the standard topology on $\mathbb{R}^n$) Riemannian metric $\tilde{g}_0$ defined on $B_{4\tilde{r}}(F_0(x_0)) \subseteq \mathbb{R}^n$. By scaling everything once, that is, $\hat{g}(t) = \tilde{r}^{-1}g(t)$ and $\hat{d}_0 = \tilde{r}^{-1/2}d_0$, we see that we are in the setting of Theorem 4.3 (choosing $R = 1/\sqrt{\tilde{r}}$ in conditions (a), (b), ($\hat{c}$)). The conclusions of that theorem, when scaled back, imply the conclusions of Theorem 1.7. ∎

5. Existence and estimates for the Ricci–DeTurck flow with C^0 boundary data

In this section we construct solutions ℓ to the Dirichlet problem for the δ-Ricci–DeTurck flow on a Euclidean ball which are smooth up to the boundary at time zero, and have C^0 parabolic boundary values. These solutions are constructed as a limit of smooth solutions ℓ_α whose parabolic boundary values converge to those of ℓ.

Recall that the δ-Ricci–DeTurck flow equation for a smooth family of metrics ℓ is given by (see [9, p. 15] and/or [15, Lemma 2.1])

$$\begin{aligned}\frac{\partial}{\partial t}\ell_{ij} = \ell^{ab}\partial_a\partial_b\ell_{ij} + \tfrac{1}{2}\ell^{ab}\ell^{pq}\big(&\partial_i\ell_{pa}\partial_j\ell_{qb} + 2\partial_a\ell_{jp}\partial_q\ell_{ib} \\ &- 2\partial_a\ell_{jp}\partial_b\ell_{iq} - 2\partial_j\ell_{pa}\partial_b\ell_{iq} - 2\partial_i\ell_{pa}\partial_b\ell_{jq}\big). \end{aligned} \tag{5.1}$$

First we estimate the closeness of smooth solutions to δ in the C^0 norm, assuming C^0 closeness on the spatial boundary and a bound on the C^2 norm at time zero.

Lemma 5.1. *Let ℓ be an $H^{2+\alpha,1+\alpha/2}(\mathbb{B}_R(0)\times[0,T]) \cap C^0(\overline{\mathbb{B}_R(0)}\times[0,T])$ solution to the δ-Ricci–DeTurck flow such that $\ell(\cdot,0)=\ell_0$, $\|\ell(\cdot,t)-\ell_0\|_{L^\infty(\partial\mathbb{B}_R(0))} \le \beta$ for $t\in[0,T]$ and $\|\ell_0-\delta\|_{C^2(\mathbb{B}_R(0))} \le \varepsilon(n)$, where $\beta \le \varepsilon(n)$. Then*

$$\begin{aligned}\|\ell(\cdot,t)-\delta\|_{L^\infty(\mathbb{B}_R(0))} &\le c(n)\varepsilon(n) \qquad \text{for all } t\in[0,T],\\ \|\ell(\cdot,t)-\ell_0\|_{L^\infty(\mathbb{B}_R(0))} &\le \beta + \varepsilon(n)t \qquad \text{for all } t\in[0,T].\end{aligned}$$

Proof. We will denote by $|\cdot|$ all norms induced by the metric δ. From smoothness and the boundary conditions, we know that ℓ is a smooth invertible metric for a small time interval $[0,\tau]$ with $|\ell-\delta|^2 \le \varepsilon(n)$ during that interval. By (5.1) we can compute

$$\begin{aligned}\frac{\partial}{\partial t}|\ell-\delta|^2 &\le 2(\ell-\delta)^{ij}\ell^{ab}\partial_a\partial_b(\ell-\delta)_{ij} + c(n)|\ell-\delta|\,|D\ell|^2\\ &\le \ell^{ab}\partial_a\partial_b|\ell-\delta|^2 - 2(D\ell,D\ell)_\ell + c(n)|\ell-\delta||D\ell|^2\\ &\le \ell^{ab}\partial_a\partial_b|\ell-\delta|^2 - |D\ell|^2 + c(n)\varepsilon(n)|D\ell|^2\\ &\le \ell^{ab}\partial_a\partial_b|\ell-\delta|^2\end{aligned}$$

for all $t\in[0,\tau]$ if $\varepsilon(n)$ is sufficiently small. Thus by the maximum principle, $|\ell-\delta|^2 \le \varepsilon(n)$ remains true as long as this is true on the boundary. Thus we can take $\tau = T$.

We perform a similar calculation for $|\ell-\ell_0|^2$. By the above estimate, we can freely use $\frac{1}{2}\delta \le \ell \le 2\delta$ for all $t\in[0,T]$. We also use $|D\ell_0| + |D^2\ell_0| \le \epsilon(n)$ due to the assumptions. We have

$$\begin{aligned}\frac{\partial}{\partial t}|\ell-\ell_0|^2 &\le 2(\ell-\ell_0)^{ij}\ell^{ab}\partial_a\partial_b\ell_{ij} + c(n)|\ell-\ell_0|\,|D\ell|^2\\ &= 2(\ell-\ell_0)^{ij}\ell^{ab}\partial_a\partial_b(\ell-\ell_0)_{ij} + 2(\ell-\ell_0)^{ij}\ell^{ab}\partial_a\partial_b(\ell_0)_{ij} + c(n)\varepsilon(n)|D\ell|^2\\ &\le \ell^{ab}\partial_a\partial_b|\ell-\ell_0|^2 - |D(\ell-\ell_0)|^2 + \varepsilon(n) + c(n)\varepsilon(n)|D\ell|^2\\ &\le \ell^{ab}\partial_a\partial_b|\ell-\ell_0|^2 - \tfrac{1}{2}|D\ell|^2 + c(n)\varepsilon(n)\end{aligned}$$

if $\varepsilon(n)$ is small enough. Hence,

$$\frac{\partial}{\partial t}(|\ell-\ell_0|^2 - c(n)\varepsilon(n)t) \leq \ell^{ab}\partial_a\partial_b(|\ell-\ell_0|^2 - \varepsilon(n)c(n)t),$$

and consequently

$$|\ell-\ell_0|^2 \leq \beta + \varepsilon(n)c(n)t$$

for $t \leq T$, in view of the fact that

$$|\ell-\ell_0|^2 \leq \beta \leq \beta + \varepsilon(n)c(n)t \quad \text{ on } \partial\mathbb{B}_R(0)\times\{t\}$$

for $t \leq T$, and $|\ell-\ell_0|^2 = 0$ for $t = 0$ on $\overline{\mathbb{B}_R(0)}$. ■

We now consider the problem of constructing solutions to the Dirichlet problem for the δ-Ricci–DeTurck flow with boundary data h given on the parabolic boundary P of $\underline{B_R(0)\times(0,T)}$. We will assume that the boundary data is given as the restriction of $h \in C^\infty(\overline{\mathbb{B}_R(0)}\times[0,T])$ which is $\varepsilon(n)$-close in the C^0 norm to δ on $\overline{\mathbb{B}_R(0)}\times[0,T]$. We now explain how to construct a solution to this Dirichlet problem if the compatibility conditions of the first type are satisfied.

Definition 5.2. Let $h \in C^\infty(\overline{\mathbb{B}_R(0)}\times[0,T])$ be a smooth family of Riemannian metrics on $\overline{\mathbb{B}_R(0)}$. We say h satisfies Comp_k, or h *satisfies the compatibility conditions of the k-th order*, if

$$\frac{\partial^l}{\partial t^l}h(x,0) = L_l(h(x,0)) \quad \text{ for } l = 1,\ldots,k$$

for all $x \in \partial\mathbb{B}_R(0)$, where L_l is the differential operator of order $2l$ which one obtains by differentiating (5.1) l-times with respect to t, and inserting iteratively the formulas obtained for the m-th derivative in time for $m = 1,\ldots,l-1$. For example

$$\begin{aligned}L_1(h)_{ij} &= h^{ab}\partial_a\partial_b h_{ij} + \tfrac{1}{2}h^{ab}h^{pq}(\partial_i h_{pa}\partial_j h_{qb} + 2\partial_a h_{jp}\partial_q h_{ib}\\ &\qquad -2\partial_a h_{jp}\partial_b h_{iq} - 2\partial_j h_{pa}\partial_b h_{iq} - 2\partial_i h_{pa}\partial_b h_{jq}),\\ L_2(h)_{ij} &= -h^{ak}h^{bm}L_1(h)_{km}\partial_a\partial_b h_{ij} + h^{ab}\partial_a\partial_b L_1(h)_{ij} + \cdots.\end{aligned}$$

We are now prepared to derive the following existence result.

Theorem 5.3. *Let $h \in C^\infty(\overline{\mathbb{B}_R(0)}\times[0,T])$, $T \leq 1$ and assume $|h(\cdot,t)-\delta| \leq \varepsilon(n)$ for all $t \in [0,T]$, and $|h_0-\delta|_{C^{2,\alpha}(\mathbb{B}_R(0))} \leq \varepsilon(n)$ and h satisfies* Comp_1. *Then there exists a solution $\ell \in C^0(\overline{\mathbb{B}_R(0)}\times[0,T]) \cap H^{2+\alpha,1+\alpha/2}(\overline{\mathbb{B}_R(0)}\times[0,T])$ to the δ-Ricci–DeTurck flow such that $\ell|_P = h|_P$ on the parabolic boundary P. Furthermore,*

$$|\ell(x,s)-h(x,s)| \leq C(n,R,\alpha)K(s^{-1},\|h\|_{C^{2,1}(\mathbb{B}_R(0)\times[s,T])})(R-|x|)^\alpha \tag{5.2}$$

for all $s \in (0,T]$ and $x \in B_R(0)$, and any given $\alpha \in (0,1)$, where $K : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ is an increasing function with respect to each of its arguments.

Proof. The first part of the proof follows closely the proof of the existence result given in [15, Chapter 3]. Assume for the moment that a solution ℓ exists, and set $S := \ell - h$. Then $S = 0$ on the parabolic boundary, and the evolution equation for S is

$$\begin{aligned}\frac{\partial}{\partial t} S(x,t) &= \left(\frac{\partial}{\partial t}\ell - \frac{\partial}{\partial t}h\right)(x,t)\\ &= \left(\ell^{ij}\partial_i\partial_j\ell + \ell^{-1} * \ell^{-1} * D\ell * D\ell - \frac{\partial}{\partial t}h\right)(x,t)\\ &= \left(\ell^{ij}\partial_i\partial_j S + \ell^{ij}\partial_i\partial_j h + \ell^{-1} * \ell^{-1} * D\ell * D\ell - \frac{\partial}{\partial t}h\right)(x,t)\\ &= a^{ij}(S(x,t),x,t)\partial_i\partial_j S(x,t) + b(S(x,t), DS(x,t), x, t)\end{aligned}$$

where $a^{kl}(z,x,t) := (h(x,t)+z)^{kl}$ is the inverse of $h(x,t)+z$ (which is well defined as long as $h(x,t)+z$ is invertible) and b is defined similarly:

$$\begin{aligned}b(z,p,x,t) &:= (h(x,t)+z)^{-1} * (h(x,t)+z)^{-1} * (p + Dh(x,t)) * (p + Dh(x,t))\\ &\quad - \frac{\partial}{\partial t}h(x,t) + (h(x,t)+z)^{kl}\partial_k\partial_l h(x,t),\end{aligned}$$

which again is well defined as long as $z + h(x,t)$ is invertible. Note that if $|z|$ is sufficiently small then $z + h(x,t)$ is invertible. Since ℓ is assumed to be a solution, and $\ell = h$ on P, where $|h-\delta|_{C^0} \le \varepsilon(n)$, and $|\ell_0 - \delta|_{C^2} = |h_0 - \delta|_{C^2} \le \varepsilon(n)$, we deduce that $|\ell(\cdot,t) - \ell_0|_{C^0(\mathbb{B}_R(0))} \le \varepsilon(n)$ for all $t \in [0,T]$ in view of Lemma 5.1, and consequently $|S(t)|_{C^0(\mathbb{B}_R(0))} \le \varepsilon(n)$ for all $t \in [0,T]$.

We divide S by a small number $\delta(n) > 0$, and call the quotient $\tilde{S}$, i.e.

$$\tilde{S} := \delta^{-1}(n)S,$$

where we assume $\varepsilon(n) \ll \delta(n)$, for example we choose $\varepsilon(n) = \delta^3(n)$. Hence $|\tilde{S}(t)|_{C^0(\mathbb{B}_R(0))}$ is still small for all times $t \in [0,T]$, and we have $|\tilde{S}| \le \sqrt{\varepsilon(n)}$. The evolution of $\tilde{S}$ may be written as

$$\frac{\partial}{\partial t}\tilde{S}(x,t) = \tilde{a}^{ij}(\tilde{S}(x,t),x,t)\partial_i\partial_j\tilde{S}(x,t) + \tilde{b}(\tilde{S}(x,t), D\tilde{S}(x,t), x, t), \tag{5.3}$$

where $\tilde{a}^{ij}(\tilde{z},x,t) = (h(x,t) + \delta\tilde{z})^{ij}$ is the inverse of $h(x,t) + \delta\tilde{z}$, and

$$\begin{aligned}\tilde{b}(\tilde{z},\tilde{p},x,t) &:= \delta(h(x,t)+\delta\tilde{z})^{-1} * (h(x,t)+\delta\tilde{z})^{-1} * (\tilde{p} + \delta^{-1}Dh(x,t))\\ &\quad * (\tilde{p} + \delta^{-1}Dh(x,t))\\ &\quad - \frac{1}{\delta}\frac{\partial}{\partial t}h(x,t) + (h(x,t)+\delta\tilde{z})^{kl}\delta^{-1}\partial_k\partial_l h(x,t).\end{aligned}$$

In the setting we are examining, we see, defining $C_1(h,n) := 10|h|_{C^{2,1}(\mathbb{B}_R(0)\times[0,1])}$, that

$$|\tilde{b}(\tilde{z},\tilde{p},x,t)| \le \delta(n)(C_1(h)\delta^{-2}(n) + |\tilde{p}|^2)$$

for the $\tilde{z} = \tilde{S}(x,t)$ we are considering, since we have $|\tilde{S}|_{C^0} \le \sqrt{\varepsilon(n)}$. As long as $|\tilde{S}(\cdot,t)|_{C^0} \le \sqrt{\varepsilon(n)}$ for $t \in [0,1]$, we obtain $2\delta_{ij} \ge \tilde{a}^{ij}(\tilde{S}(x,t),x,t) \ge \frac{1}{2}\delta_{ij}$ and $|\tilde{b}(\tilde{S}(x,t), D\tilde{S}(x,t),x,t)| \le \delta(n)(C_1(h)\delta^{-2}(n) + |D\tilde{S}(x,t)|^2)$ in this case. We write this as

$$|\tilde{b}(\tilde{S}(x,t), D\tilde{S}(x,t),x,t)| \le Q(|D\tilde{S}(x,t)|,|\tilde{S}(x,t)|)(1+|D\tilde{S}(x,t)|^2)$$

where Q is a smooth function with

$$Q(|p|,|u|) = 2\delta(n)\eta(|p|) + 2\delta(n)\big(1-\eta(|p|)(1+100C_1(h)\delta^{-2}(n))\big)$$

where η is a smooth cut-off function with $\eta(r) = 0$ for $r \le 100C_1(h)\delta^{-2}(n)$ and $\eta(r) = 1$ for $r \ge 200C(h,n)\delta^{-1}(n)$.

Hence, from the general theory of non-linear parabolic equations of second order (see for example [12, Theorem 7.1, Chapter VII]) we see that equation (5.3) with zero parabolic boundary values has a solution $\ell \in H^{2+\alpha,1+\alpha/2}(\overline{\mathbb{B}_R(0)} \times [0,T])$ for all times $t \in [0,T]$, as long as $|\tilde{S}(\cdot,t)|_{C^0} \le \sqrt{\varepsilon(n)}$ for $t \in [0,T]$.

Writing $\ell = h + \delta\tilde{S}$ and using the arguments above, we see that this will not be violated for $t \in [0,T]$, $T \le 1$, and ℓ solves the δ-Ricci–DeTurck equation and $\ell|_P = h|_P$.

This proves the existence of the solution. It remains to prove the Hölder boundary estimate, (5.2). For ease of reading, we assume $R = 1$.

For q, r fixed consider the functions

$$\varphi_+ := (\ell_{qr} - h_{qr}) + \lambda|\ell - h|^2 \quad \text{and} \quad \varphi_- := -(\ell_{qr} - h_{qr}) + \lambda|\ell - h|^2,$$

where $\lambda = \lambda(n)$ is a sufficiently large constant such that

$$\partial_t\varphi_\pm - \ell^{ij}\partial_i\partial_j\varphi_\pm \le C(n,h|_{[s,T]})$$

on $\mathbb{B}_1(0) \times [s,T]$, where $C(n,h) = C(n,\|h\|_{C^{2,1}(\mathbb{B}_1(0)\times[s,T])})$.

We consider the functions

$$\psi_+ := \eta(t)\varphi_+ - 2M\rho^\alpha \quad \text{and} \quad \psi_- := \eta(t)\varphi_- - 2M\rho^\alpha,$$

for some $0 < \alpha < 1$, where $\rho(x) = 1 - |x|$ and η is a non-negative cut-off function in time with $\eta(t) = 0$ for $0 \le t \le s$ and $\eta(t) = 1$ for $3s/2 \le t$ such that $|\eta'(t)|^2 \le cs^{-2}\eta(t)$ for some positive constant c. A direct calculation shows

$$\begin{aligned}
\ell^{ij}\partial_i\partial_j\rho^\alpha &= \ell^{ij}\partial_i\left(\alpha\rho^{\alpha-1}\frac{-x_j}{|x|}\right)\\
&= (\alpha-1)\alpha\rho^{\alpha-2}\ell^{ij}\frac{x_ix_j}{|x|^2} - \alpha\rho^{\alpha-1}\frac{\ell^{ii}}{|x|} + \alpha\rho^{\alpha-1}\frac{\ell^{ij}x_ix_j}{|x|^3}\\
&\le -\frac{\alpha(n-1)}{2}\frac{\rho^{\alpha-1}}{|x|} - \frac{\alpha|1-\alpha|}{2}\rho^{\alpha-2} \le -1 - \frac{\alpha|1-\alpha|}{2}\rho^{\alpha-2}
\end{aligned}$$

for all $|x| \in (1-\delta_0(\alpha,n),1)$. We note that $\psi_\pm$ cannot be zero very close to $\partial\mathbb{B}_R(0)$, since $|D(\ell - h)|$ is bounded by *some* constant according to [12, Theorem 7.1, Chapter VII], and $\ell - h = 0$ on $\partial\mathbb{B}_R(0)$. Also, by choosing $M = M(\alpha)$ large enough we have, without loss of generality, $\psi_\pm(x,\cdot) < 0$ for $|x| = 1 - \delta_0(\alpha, n)$. That is, $\psi_\pm(x,\cdot) < 0$ for $|x| = 1 - \delta_0(\alpha, n)$ and $|x| = 1 - \varepsilon$ for all $\varepsilon > 0$, $\varepsilon \ll \delta_0$ sufficiently small. We also observe that $\psi_\pm(\cdot, s) < 0$ for all $|x| \in [1 - \delta_0(\alpha, n), 1 - \varepsilon]$ for $t \leq s$. Hence if $\psi_\pm(x,t) = 0$ for some $|x| \in [1 - \delta_0(\alpha, n), 1 - \varepsilon]$ for some $t \geq s$, there must be a first time t for which this happens, and this must happen at an interior point x of $\mathbb{B}_{1-\varepsilon}(0) \setminus \overline{\mathbb{B}_{1-\delta_0}(0)}$. At such a point (x,t), we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_\pm &\leq \ell^{ij}\partial_i\partial_j\psi_\pm + C(n,h) + \eta'\varphi_\pm - 2M - M\alpha|1-\alpha|\rho^{\alpha-2} \\
&\leq \ell^{ij}\partial_i\partial_j\psi_\pm + C(n,h) + \frac{c(n)}{s}|\eta\varphi_\pm|^{1/2} - 2M - M\alpha|1-\alpha|\rho^{\alpha-2} \\
&\leq \ell^{ij}\partial_i\partial_j\psi_\pm + C(n,h) - 2M + \frac{c(n)}{s}(2M)^{1/2}\rho^{\alpha/2} - M\alpha|1-\alpha|\rho^{\alpha-2} \\
&< 0
\end{aligned}$$

for $|x| \in (1-\delta_0, 1-\varepsilon)$ if $M > C(C(n,h), \alpha, s)$ also holds. This is a contradiction, leading to the desired estimate close to the boundary. For points $|x| \in (0, 1-\delta_0)$ the estimate follows immediately from the fact that ℓ and h are $\varepsilon(n)$-close to δ and hence bounded, and $(1-|x|)^\alpha \geq \delta_0^\alpha$ for $|x| \in (0, 1-\delta_0)$. ∎

If we assume that higher order compatibility conditions are satisfied, we obtain more regularity of the solution.

Theorem 5.4. *Let $h \in C^\infty(\overline{\mathbb{B}_R(0)} \times [0,T])$, $T \leq 1$ and assume $|h(\cdot,t) - \delta| \leq \varepsilon(n)$ for all $t \in [0,T]$, and $|h_0 - \delta|_{C^{2,\alpha}(\mathbb{B}_R(0))} \leq \varepsilon(n)$ and h satisfies* Comp$_k$. *Then there exists a solution $\ell \in C^0(\overline{\mathbb{B}_R(0)} \times [0,T]) \cap H^{k+\alpha,k/2+\alpha/2}(\overline{\mathbb{B}_R(0)} \times [0,T])$ to the δ-Ricci–DeTurck flow with the values given by h on the parabolic boundary, that is, $\ell|_P = h|_P$. Furthermore,*

$$|\ell(x,s) - h(x,s)| \leq C(n,R,\alpha)K(s^{-1}, \|h\|_{C^{2,1}(\mathbb{B}_R(0)\times[s,T])})(R-|x|)^\alpha \tag{5.4}$$

for all $s \in (0,T]$ and $x \in B_R(0)$, and any given $\alpha \in (0,1)$, where $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is an increasing function with respect to each of its arguments.

Proof. The proof is the same, except that at the step where we used [12, Theorem 7.1, Chapter VII] to obtain a solution in $H^{2+\alpha,1+\alpha/2}$, we now obtain a solution $\ell \in H^{k+\alpha,k+\alpha/2}$, in view of the fact that the $\tilde{S}$ satisfies the compatibility condition of k-th order. ∎

We now explain how to construct a δ-Ricci–DeTurck flow for parabolic boundary values given by h which do not necessarily satisfy compatibility conditions of the first order, but are smooth at $t = 0$, smooth on $\overline{\mathbb{B}_R(0)} \times (0,T]$ and continuous on $\overline{\mathbb{B}_R(0)} \times [0,T]$. This is done by modifying the boundary values, so that the first (or higher order) compatibility conditions are satisfied, and then taking a limit.

Theorem 5.5. *Let $h \in C^\infty(\overline{\mathbb{B}_R(0)} \times (0,T]) \cap C^0(\overline{\mathbb{B}_R(0)} \times [0,T])$, $T \le 1$, be such that $h(\cdot,0) \in C^\infty(\overline{\mathbb{B}_R(0)})$ and assume that $|h(\cdot,t) - \delta| \le \varepsilon(n)$ for all $t \in [0,T]$, and $|h_0 - \delta|_{C^{2,\alpha}(\mathbb{B}_R(0))} \le \varepsilon(n)$. Then there exists a solution*

$$\ell \in C^0(\overline{\mathbb{B}_R(0)} \times [0,T]) \cap C^\infty(\mathbb{B}_R(0) \times (0,T])$$

to the δ-Ricci–DeTurck flow with the values given by h on the parabolic boundary P, that is, $\ell|_P = h|_P$. Furthermore,

$$|\ell|_{C^s(\mathbb{B}_{R'}(0)\times[0,T])} \le c(|R-R'|, |h_0|_{C^s(\mathbb{B}_{R'}(0))})$$

for any $R' < R$ and any $s \in \mathbb{N}$.

Proof. Let $\xi : \mathbb{R} \to \mathbb{R}$ be a non-increasing smooth function whose image is contained in $[0,1]$ such that ξ is equal to 1 on $[0,1/2]$ and to 0 on $[1,\infty)$.

For each $\tau \in [0,1]$, let $h(\tau)$ be the smooth Riemannian metric defined as follows:

$$h(\tau)|_{\mathbb{B}_R(0)\times[\tau,T]} := h,$$
$$h(\tau)|_{\mathbb{B}_R(0)\times[0,\tau]} := \xi\left(\frac{t}{\tau}\right)h_0(x) + \left(1 - \xi\left(\frac{t}{\tau}\right)\right)h(x,t) + t\xi\left(\frac{t}{\tau}\right)L_1(h_0)(x),$$

where L_1 is as in Definition 5.2, i.e.

$$L_1(h_0) := (h_0)^{ij}\partial_i\partial_j h_0 + (h_0)^{-1} * (h_0)^{-1} * Dh_0 * Dh_0.$$

Using $|h(t) - \delta| \le \varepsilon(n)$ and $|L_1(h_0)| \le c(n)$, we see that $|h(\tau)(t) - \delta| \le 2\varepsilon(n) = \varepsilon(n)$ if τ is sufficiently small. Furthermore,

$$\frac{\partial}{\partial t}h(\tau)(x,0) = L_1(h_0)(x,0) = L_1(h(\tau))(x,0),$$

that is, $h(\tau)$ satisfies $\underline{\mathrm{Comp}_1}$. Consequently, we may use Theorem 5.3 to obtain a solution $\ell(\tau) \in H^{2+\alpha,1+\alpha/2}(\overline{\mathbb{B}_R(0)} \times [0,T])$ to the δ-Ricci–DeTurck flow with parabolic boundary data given by $h(\tau)$. From the definition of $h(\tau)$, we see that on $\partial\mathbb{B}_R(0)$,

$$|\ell(\tau)(\cdot,t) - h_0)| = |h(\tau)(\cdot,t) - h_0(\cdot)| \le |h(\cdot,t) - h_0(\cdot)| + t\varepsilon(n) \le C(t,h,n)$$

where $C(t,h,n)$ is a function (independent of τ) such that $C(t,h,n) \le \varepsilon(n)$ and $C(t,h,n) \to 0$ as $t \searrow 0$ for n and h fixed. Lemma 5.1 then tells us that $|\ell(\tau)(\cdot,t) - h_0)| \le \hat{C}(t,h,n)$ on *all* of $\overline{\mathbb{B}_R(0)}$ for all $t \in [0,T]$, where $\hat{C}(t,h,n) \to 0$ as $t \searrow 0$ for n and h fixed (independent of τ). The boundary Hölder estimate (5.2) of Theorem 5.3 and the smoothness of $h = h(\tau)$ for $t \ge 2\tau$ also tell us that for any $\varepsilon > 0$ and $s > 0$ there exists a $\sigma > 0$ such that $|\ell(\tau)(x,t) - h(x,t)| \le \varepsilon$ for all $x \in \mathbb{B}_R(0) \setminus \mathbb{B}_{R-\sigma}(0)$ for all $t \in [s,T]$, where $\sigma = \sigma(\varepsilon,s,h,n) > 0$ is independent of τ if τ is sufficiently small. These two C^0 estimates imply that we have uniform (in τ) C^0 bounds

$$|\ell(\tau)(t) - h(t)|_{P_\varepsilon} \le C(\varepsilon,h,n)$$

where $C(\varepsilon,h,n) \to 0$ as $\varepsilon \to 0$ (for fixed h and n) and $P_\varepsilon = (\overline{\mathbb{B}_R(0)} \setminus \mathbb{B}_{R-\varepsilon}(0)) \times [0,1] \cup \overline{\mathbb{B}_R(0)} \times [0,\varepsilon]$ for τ sufficiently small.

If we define

$$h(\tau) := \xi(t/\tau)h_0(x) + (1-\xi(t/\tau))h(x,t) + t\xi(t/\tau)L_1(h_0)(x) + \frac{t^2}{2}L_2(h_0),$$

then we still have $|\ell(\tau)(\cdot,t) - h_0)| \leq C(t,h,n)$ and hence $|\ell(\tau)(\cdot,t) - h_0)| \leq \hat{C}(t,h,n)$ on *all* of $\overline{\mathbb{B}_R(0)}$ for all $t \in [0,T]$, where $\hat{C}(t,h,n) \to 0$ is independent of τ if τ is sufficiently small, in view of Lemma 5.1.

The Hölder estimates still hold: for any $\varepsilon > 0$ and $s > 0$, $|\ell(\tau)(x,t) - h(x,t)| \leq \varepsilon$ for all $x \in \mathbb{B}_R(0) \setminus \mathbb{B}_{R-\sigma}(0)$ and all $t \in [s,T]$, where $\sigma = \sigma(\varepsilon,s,h,n) > 0$ is independent of τ if τ is sufficiently small. Thus, the uniform C^0 estimates $|\ell(\tau)(t) - h(t)|_{P_\varepsilon} \leq C(\varepsilon,h,n)$ where $C(\varepsilon,h,n) \to 0$ as $\varepsilon \to 0$ (for fixed h and n) still hold. Continuing in this way, we can assume $\ell(\tau) \in H^{k+\alpha,\frac{k}{2}+\frac{\alpha}{2}}(\overline{\mathbb{B}_R(0)} \times [0,T])$ and $|\ell(\tau)(\cdot) - \delta|_{C^0(\mathbb{B}_R(0)\times[0,T])} \leq \varepsilon(n)$ and $\ell(\tau)(\cdot,0) = h_0$, and the uniform C^0 estimates hold, $|\ell(\tau)(t) - h(t)|_{P_\varepsilon} \leq C(\varepsilon)$ where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $P_\varepsilon = \overline{\mathbb{B}_R(0)} \setminus \mathbb{B}_{R-\varepsilon}(0) \times [0,T] \cup \overline{\mathbb{B}_R(0)} \times [0,\varepsilon]$. The proof of the interior estimates, explained in [16, Section 4], can be used here to show that

$$|\ell(\tau)(\cdot,t)|_{C^s(\mathbb{B}_{R'}(0))} \leq c(|R'-R|, |h_0|_{C^s(\mathbb{B}_R(0))})$$

for all $t \in [0,T]$ and all $R' < R$. By Arzelà–Ascoli's Theorem, one is able to take a limit: up to a subsequence, we obtain a solution ℓ with the desired properties. ∎

6. An L^2 estimate for the Ricci–DeTurck flow and applications

In this section we prove a lemma which estimates the change in the L^2 distance between two solutions to the δ-Ricci–DeTurck flow. Lemma 6.1 considers smooth solutions which are $\varepsilon(n)$-close in the L^2 norm at time zero and agree on the boundary at all times. If we weight the L^2 distance at time t of two smooth solutions appropriately, then this quantity is non-increasing in time. The weight has the property that it is uniformly bounded between 1 and 2, and hence the unweighted L^2 distance at time t of the two solutions can only increase by a factor of at most 2. With the help of the L^2-Lemma, we prove some uniqueness theorems for solutions to the δ-Ricci–DeTurck flow.

Lemma 6.1 (L^2-Lemma). *Let g_1, g_2 be two smooth solutions to the δ-Ricci–DeTurck flow on $\mathbb{B}_R(0) \times (S,T)$ such that $g_l \in C^\infty(\overline{\mathbb{B}_R(0)} \times (S,T))$ for $l = 1,2$ and $g_1 = g_2$ on $\partial\mathbb{B}_R(0) \times (S,T)$. Let $h := g_1 - g_2$ and*

$$v := |h|^2\big(1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)\big).$$

Assume that $|g_1(\cdot,t) - \delta| + |g_2(\cdot,t) - \delta| \leq \varepsilon(n)$ for all $t \in (S,T)$. Then for $\lambda \geq \hat{\lambda}(n)$ and $\varepsilon \leq \hat{\varepsilon}(n)$, where $\hat{\varepsilon}(n)$ is sufficiently small and $\hat{\lambda}(n)$ sufficiently large, and for $t \in (S,T)$, we have

$$\frac{\partial}{\partial t}\int_{\mathbb{B}_R(0)} v\, dx \leq 0.$$

Before proving the lemma, we state and prove two corollaries of this estimate.

Corollary 6.2. *Let g_1, g_2 be smooth solutions to the δ-Ricci–DeTurck flow on $\mathbb{B}_R(0) \times (0, T)$ such that $g_l \in C^\infty(\overline{\mathbb{B}_R(0)} \times (0, T)) \cap C^0(\overline{\mathbb{B}_R(0)} \times [0, T))$ for $l = 1, 2$ and $g_1 = g_2$ on $\partial\mathbb{B}_R(0) \times [0, T)$ and $g_1(\cdot, 0) = g_2(\cdot, 0)$. Assume further that $|g_1(\cdot, t) - \delta| + |g_2(\cdot, t) - \delta| \le \varepsilon(n)$ for all $t \in (0, T)$. Then $g_1 = g_2$ on $\mathbb{B}_R(0) \times [0, T)$.*

Remark 6.3. [7, Proposition 7.51] gives a uniqueness result for the Ricci–DeTurck flow with a background metric g with bounded curvature for solutions $(g(t))_{t\in(0,1)}$ that behave as follows: there exists a positive constant A such that $A^{-1}g \le g(t) \le Ag$ and $|\nabla^g g(t)| + \sqrt{t}|\nabla^{g,2} g(t)| \le A$ for all $t \in (0, 1)$. The proof is based on the maximum principle. Corollary 6.2 assumes a stronger condition on the closeness to the background Euclidean metric but it does not assume any a priori bounds on the first and second covariant derivatives; the proof is based on energy estimates.

Proof of Corollary 6.2. Let $h := g_1 - g_2$ and

$$v := |h|^2\big(1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)\big).$$

From the assumptions, we know that $|v(\cdot, 0)| = 0$ on $\overline{\mathbb{B}_R(0)}$ and hence

$$\int_{\mathbb{B}_R(0)} v(x, \tau)\, dx \le \sigma(\tau),$$

where $\sigma(\tau)$ tends to 0 as $\tau \searrow 0$, in view of the continuity of v. We also have $g_1(\cdot, t) = g_2(\cdot, t)$ on $\partial\mathbb{B}_R(0)$ for all $t \in [0, T]$, and so Lemma 6.1 implies $\int_{\mathbb{B}_R(0)} v(x, t)\, dx \le \sigma(\tau)$ for all $t \in (\tau, T)$. Letting $\tau \searrow 0$, we see $\int_{\mathbb{B}_R(0)} v(x, t)\, dx = 0$ for all $t \in (0, T)$. This implies $g_1(\cdot, t) = g_2(\cdot, t)$ for all $t \in [0, T)$ as required. ∎

By slightly modifying the previous proof, we can also show the following uniqueness statement.

Corollary 6.4. *Let h be a smooth solution to the δ-Ricci–DeTurck flow on $\mathbb{B}_R(0) \times (0, T)$ such that $h \in C^\infty(\overline{\mathbb{B}_R(0)} \times (0, T)) \cap C^0(\overline{\mathbb{B}_R(0)} \times [0, T))$, assume $h_0 \in C^\infty(\mathbb{B}_R(0))$, and let ℓ be the solution constructed in Theorem 5.5 with parabolic boundary data defined by $h|_P$. Assume further that $|h(\cdot, t) - \delta| \le \varepsilon(n)$ for all $t \in [0, T)$. Then $\ell = h$.*

Proof. Let $h(\tau)$ be the modified metric defined in the proof of Theorem 5.5, and $\ell(\tau)$ the solutions defined there. Let

$$v(\tau) := |h(\tau) - \ell(\tau)|^2\big(1 + \lambda(|h(\tau) - \delta|^2 + |\ell(\tau) - \delta|^2)\big).$$

From the construction of $h(\tau)$ we know that $|v(\tau)(\cdot, \tau)| \le \sigma(\tau)$ on $\mathbb{B}_R(0)$, where $\sigma(\tau) \to 0$ with $\tau \searrow 0$. This implies

$$\int_{\mathbb{B}_R(0)} v(\tau)(x, \tau)\, dx \le \sigma(\tau),$$

where $\sigma(\tau) \to 0$ with $\tau \searrow 0$. Since $h = h(\tau) = \ell(\tau)$ on $\partial\mathbb{B}_R(0)$ for all $t \in [\tau, T]$ and the solution $\ell(\tau)$ is C^k (in space and time) up to and including the boundary, we can use Lemma 6.1 to conclude that $\int_{\mathbb{B}_R(0)} v(\tau)(x,t)\,dx \le \sigma(\tau)$ for all $t \in (\tau, T)$. Letting $\tau \searrow 0$, we see $\int_{\mathbb{B}_R(0)} v(x,t)\,dx \le 0$ for all $t \in (0, T)$ with $v := |h-\ell|^2(1+\lambda(|h-\delta|^2 + |\ell-\delta|^2))$. This implies $h(\cdot,t) = \ell(\cdot,t)$ for all $t \in [0,T)$ as required. ■

Proof of Lemma 6.1. In the following, we will assume that $\varepsilon(n)$ is a small positive constant, and $\lambda(n) = 1/\sqrt{\varepsilon(n)}$ is a large constant which satisfies $\lambda(n)\varepsilon(n) = \sqrt{\varepsilon(n)} =: \sigma(n)$ per definition.

From (5.1) we have, for $l \in \{1,2\}$,

$$\frac{\partial}{\partial t} g_l = g_l^{ab}\partial_a\partial_b g_l + g_l^{-1} * g_l^{-1} * Dg_l * Dg_l,$$

and

$$\frac{\partial}{\partial t}|g_l - \delta|^2 \le g_l^{ab}\partial_a\partial_b|g_l-\delta|^2 - \frac{2}{1+\varepsilon}|Dg_l|^2.$$

Summing over $l = 1,2$, and writing $\tilde{h}^{ab} := \frac{1}{2}(g_1^{ab} + g_2^{ab})$ and $\hat{h}^{ab} := \frac{1}{2}(g_1^{ab} - g_2^{ab})$, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}(|g_1-\delta|^2 + |g_2-\delta|^2)\\
&\quad\le g_1^{ab}\partial_a\partial_b|g_1-\delta|^2 + g_2^{ab}\partial_a\partial_b|g_2-\delta|^2 - 2(1-\varepsilon(n))|Dg_1|^2 - 2(1-\varepsilon(n))|Dg_2|^2\\
&\quad= \tilde{h}^{ab}\partial_a\partial_b(|g_1-\delta|^2 + |g_2-\delta|^2) + \hat{h}^{ab}\partial_a\partial_b(|g_1-\delta|^2 - |g_2-\delta|^2)\\
&\qquad - 2(1-\varepsilon(n))|Dg_1|^2 - 2(1-\varepsilon(n))|Dg_2|^2\\
&\quad= \tilde{h}^{ab}\partial_a\partial_b(|g_1-\delta|^2 + |g_2-\delta|^2) + \sum_{l=1}^{2}(\hat{h} * (g_l-\delta) * D^2 g_l + \hat{h} * Dg_l * Dg_l)\\
&\qquad - 2(1-\varepsilon(n))|Dg_1|^2 - 2(1-\varepsilon(n))|Dg_2|^2\\
&\quad\le \tilde{h}^{ab}\partial_a\partial_b(|g_1-\delta|^2 + |g_2-\delta|^2) + \sum_{l=1}^{2}(\hat{h} * (g_l-\delta) * D^2 g_l)\\
&\qquad - 2(1-2\varepsilon(n))|Dg_1|^2 - 2(1-2\varepsilon(n))|Dg_2|^2,
\end{aligned}$$

in view of the fact that g_l is ε-close to δ for $l = 1,2$.

For the difference $h = g_1 - g_2$ we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} h &= g_1^{ab}\partial_a\partial_b g_1 + g_1^{-1} * g_1^{-1} * Dg_1 * Dg_1\\
&\quad - g_2^{ab}\partial_a\partial_b g_2 - g_2^{-1} * g_2^{-1} * Dg_2 * Dg_2\\
&= \tfrac{1}{2}(g_1^{ab} + g_2^{ab})\partial_a\partial_b h + \tfrac{1}{2}(g_1^{ab} - g_2^{ab})\partial_a\partial_b(g_1+g_2)\\
&\quad + (g_1^{-1} - g_2^{-1}) * g_1^{-1} * Dg_1 * Dg_1 + g_2^{-1} * (g_1^{-1} - g_2^{-1}) * Dg_1 * Dg_1\\
&\quad + g_2^{-1} * g_2^{-1} * Dh * Dg_1 + g_2^{-1} * g_2^{-1} * Dg_2 * Dh,
\end{aligned}$$

which we can write as

$$\frac{\partial}{\partial t}h = \tilde{h}^{ab}\partial_a\partial_b h + \hat{h}^{ab}\partial_a\partial_b(g_1 + g_2) + \hat{h} * g_1^{-1} * Dg_1 * Dg_1 + g_2^{-1} * \hat{h} * Dg_1 * Dg_1 + g_2^{-1} * g_2^{-1} * Dh * Dg_1 + g_2^{-1} * g_2^{-1} * Dg_2 * Dh.$$

This implies that

$$\frac{\partial}{\partial t}|h|^2 \leq \tilde{h}^{ab}\partial_a\partial_b|h|^2 - \frac{2}{1+\varepsilon}|Dh|^2 + h * \hat{h} * D^2(g_1 + g_2) + h * \hat{h} * g_1^{-1} * Dg_1 * Dg_1 + h * g_2^{-1} * \hat{h} * Dg_1 * Dg_1 + h * g_2^{-1} * g_2^{-1} * Dh * Dg_1 + h * g_2^{-1} * g_2^{-1} * Dg_2 * Dh,$$

in view of the fact that $\tilde{h}$ is ε-close to δ. We now consider the test-function

$$v := |h|^2\big(1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)\big).$$

We obtain

$$\begin{aligned}\frac{\partial}{\partial t}v \leq{}& \tilde{h}^{ab}\partial_a\partial_b v - 2\lambda\tilde{h}^{ab}\partial_a|h|^2\partial_b(|g_1 - \delta|^2 + |g_2 - \delta|^2) \\ & -2\lambda(1-2\varepsilon(n))|h|^2(|Dg_1|^2 + |Dg_2|^2) \\ & -2(1-2\varepsilon(n))\big(1+\lambda(|g_1-\delta|^2+|g_2-\delta|^2)\big)|Dh|^2 \\ & +\big(1+\lambda(|g_1-\delta|^2+|g_2-\delta|^2)\big)\Big(h*\hat{h}*D^2(g_1+g_2)+h*\hat{h}*g_1^{-1}*Dg_1*Dg_1 \\ & +h*g_2^{-1}*\hat{h}*Dg_1*Dg_1+h*g_2^{-1}*g_2^{-1}*Dh*Dg_1+h*g_2^{-1}*g_2^{-1}*Dg_2*Dh\Big) \\ & +\lambda|h|^2\sum_{l=1}^{2}(\hat{h}*(g_l-\delta)*D^2g_l).\end{aligned}$$

Using Young's inequality and the fact that g_l is ε-close to δ for $l = 1, 2$, as well as $|h * \hat{h}| \leq c(n)|h|^2$, we see that the first order terms appearing in the large brackets may be absorbed by the two negative first order gradient terms which appear just before the large brackets, if $\lambda \geq \Lambda(n)$, where $\Lambda(n)$ is sufficiently large. That is, we have

$$\begin{aligned}\frac{\partial}{\partial t}v \leq{}& \tilde{h}^{ab}\partial_a\partial_b v - 2\lambda\tilde{h}^{ab}\partial_a|h|^2\partial_b(|g_1 - \delta|^2 + |g_2 - \delta|^2) \\ & - \tfrac{3}{2}\lambda|h|^2(|Dg_1|^2 + |Dg_2|^2) - \tfrac{3}{2}|Dh|^2 \\ & + \big(1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)\big)(h * \hat{h} * D^2(g_1 + g_2)) \\ & + \lambda|h|^2\sum_{l=1}^{2}(\hat{h} * (g_l - \delta) * D^2 g_l).\end{aligned} \tag{6.1}$$

The second term on the right-hand side of (6.1) can be estimated as follows:

$$
\begin{aligned}
\big|2\lambda\tilde{h}^{ab}\partial_a|h|^2\partial_b(|g_1-\delta|^2+|g_2-\delta|^2)\big| \\
&\le \lambda\big|\tilde{h}*Dh*h*(|g_1-\delta|+|g_2-\delta|)*(Dg_1+Dg_2)\big| \\
&\le c(n)\varepsilon(n)\lambda|h|\,|Dh|(|Dg_1|+|Dg_2|).
\end{aligned}
$$

Therefore, it can also be absorbed by the negative terms just before the big brackets, in view of the fact that $\varepsilon(n)\lambda \le \sqrt{\varepsilon(n)}$. This leads to

$$
\begin{aligned}
\frac{\partial}{\partial t}v \le{}& \tilde{h}^{ab}\partial_a\partial_b v - \lambda|h|^2(|Dg_1|^2+|Dg_2|^2) - |Dh|^2 \\
&+ \big(1+\lambda(|g_1-\delta|^2+|g_2-\delta|^2)\big)(h*\hat{h}*D^2(g_1+g_2)) \\
&+ \lambda|h|^2\sum_{l=1}^{2}(\hat{h}*(g_l-\delta)*D^2g_l).
\end{aligned}
$$

In order to estimate the second order terms appearing in (6.1), we integrate over $\mathbb{B} := \mathbb{B}_R(0)$:

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{B}} v \le{}& \int_{\mathbb{B}}\tilde{h}^{ab}\partial_a\partial_b v - \lambda\int_{\mathbb{B}}|h|^2(|Dg_1|^2+|Dg_2|^2) - \int_{\mathbb{B}}|Dh|^2 \\
&+ \int_{\mathbb{B}}\big(1+\lambda(|g_1-\delta|^2+|g_2-\delta|^2)\big)(h*\hat{h}*D^2(g_1+g_2)) \\
&+ \int_{\mathbb{B}}\lambda\sum_{l=1}^{2}|h|^2(\hat{h}*(g_l-\delta)*D^2g_l).
\end{aligned}
$$

Since Dv and h are zero on the boundary of $\mathbb{B}$, we obtain no boundary terms when integrating by parts the first and the last two terms on the right-hand side above. Doing so, we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{B}} v \le{}& -\int_{\mathbb{B}}\partial_a\tilde{h}^{ab}\partial_b v - \lambda\int_{\mathbb{B}}|h|^2(|Dg_1|^2+|Dg_2|^2) - \int_{\mathbb{B}}|Dh|^2 \\
&+ \int_{\mathbb{B}} D\big((1+\lambda(|g_1-\delta|^2+|g_2-\delta|^2))*h*\hat{h}\big)*D(g_1+g_2) \\
&+ \int_{\mathbb{B}}\sum_{l=1}^{2}\lambda D\big(|h|^2(\hat{h}*(g_l-\delta))\big)*Dg_l \\
=:{}& A+B+C+D+E.
\end{aligned}
$$

Note also that $|\hat{h}| \le c(n)|h|$. We estimate the integrand of A as follows:

$$
\begin{aligned}
|\partial_a(\tilde{h})^{ab}\partial_b v| &\le c(n)(|Dg_1|+|Dg_2|)|Dv| \\
&= c(n)(|Dg_1|+|Dg_2|)\big|D(|h|^2(1+\lambda(|g_1-\delta|^2+|g_2-\delta|^2)))\big| \\
&\le c(n)(|Dg_1|+|Dg_2|)\big(2|h||Dh|+|h|^2\lambda\varepsilon(n)(|Dg_1|+|Dg_2|)\big) \\
&\le c(n)(|Dg_1|+|Dg_2|)|h||Dh|+|h|^2\lambda\varepsilon(n)(|Dg_1|+|Dg_2|)^2,
\end{aligned}
$$

and hence the integral A can be absorbed by the integrals B and C. In estimating the integral D, we will use

$$|D\hat{h}| \le c(n)(|Dh| + |h|\,|Dg_1| + |h|\,|Dg_2|),$$

the validity of which can be seen by writing $\hat{h} = \frac{1}{2}((g_1)^{-1} - (g_2)^{-1}) = \frac{1}{2}g_1^{-1}(g_2 - g_1)g_2^{-1} = -\frac{1}{2}g_1^{-1}hg_2^{-1}$, differentiating, and keeping in mind that g_1 and g_2 are ε-close to δ.

We estimate the integrand of D as follows:

$$\begin{aligned} D\big((1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)) * h * \hat{h}\big) * D(g_1 + g_2) \\ &= D\big((1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2))\big) * h * \hat{h} * D(g_1 + g_2) \\ &\quad + \big((1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2))\big) D(h * \hat{h}) * D(g_1 + g_2) \\ &\le c(n)\varepsilon(n)\lambda|h|^2(|Dg_1|^2 + |Dg_2|^2) \\ &\quad + c(n)\big(|Dh|\,|h| + |h|^2(|Dg_1| + |Dg_2|)\big)(|Dg_1| + |Dg_2|), \end{aligned}$$

and hence the integral D can also be absorbed by the integrals B and C. We estimate the final integral E in a similar way: the integrand of E can be estimated by

$$\begin{aligned} \Big|\sum_{l=1}^{2} \lambda D(|h|^2(\hat{h} * (g_l - \delta))) * Dg_l\Big| &\le \lambda c(n)|h|^2|Dh|\varepsilon(n)(|Dg_1| + |Dg_2|) \\ &\quad + \lambda c(n)|h|^3(|Dg_1|^2 + |Dg_2|^2) \end{aligned}$$

and hence the integral E can also be absorbed by the integrals B and C, in view of the fact that $|h| \le \varepsilon(n)$. The result is

$$\frac{\partial}{\partial t}\int_{\mathbb{B}} v \le 0,$$

as required. ∎

7. Smoothness of solutions coming out of smooth metric spaces

Let $(M, g(t))_{t\in(0,T)}$ be a smooth solution to the Ricci flow satisfying

$$|\mathrm{Rm}(\cdot, t)| \le c_0^2/t, \qquad \mathrm{Ric}(g(t)) \ge -1 \tag{7.1}$$

for all $t \in (0, T)$.

We first give an example for an application of Theorem 1.6 in the setting of expanding gradient Ricci solitons. As explained in the introduction, expanding gradient Ricci solitons coming out of smooth cones $(\mathbb{R}^n, d_X, o) = (\mathbb{R}^+ \times S^{n-1}, dr^2 \oplus r^2\gamma, o)$ where γ is a Riemannian metric on the sphere which is smooth and whose curvature operator has eigenvalues larger than 1, are examples of solutions which satisfy the estimates above. In [14] examples are constructed, and in [8] it is shown that there is always a solution which comes out smoothly, in the sense that the convergence is in the C^∞_{loc} sense away from the tip. The uniqueness of such solitons is unknown. Below, we make precise the meaning of an expanding gradient Ricci soliton which comes out of a metric cone.

Recall that an expanding gradient Ricci soliton is a triple $(M^n, g, \nabla^g f)$ where M is an n-dimensional Riemannian manifold with a complete Riemannian metric g and a smooth potential function $f : M \to \mathbb{R}$ satisfying

$$\mathrm{Ric}(g) - \mathrm{Hess}\, f = -g/2. \tag{7.2}$$

Also, to each expanding gradient Ricci soliton one may associate a self-similar solution of the Ricci flow. Indeed, let $(\psi_t)_{t>0}$ be the flow generated by $-\nabla^g f/t$ such that $\psi_{t=1} = \mathrm{Id}_M$ and define $g(t) := t\psi_t^* g$ for $t > 0$. Then $(M, g(t))_{t>0}$ defines a Ricci flow thanks to (7.2). Next, we notice that if an expanding gradient Ricci soliton $(M^n, g(t), p)_{t>0}$ admits a limit as t tends to 0 in the pointed Gromov–Hausdorff sense for some point p that lies in the critical set of the potential function f, then this limit must be the asymptotic cone in the sense of Gromov since $(M^n, g(t), p)$ is isometric to $(M^n, tg, \psi_t(p) = p)$ for $t > 0$ as pointed metric spaces. Therefore, there is a space-time dictionary for expanding gradient Ricci solitons: the initial condition can be interpreted as the asymptotic cone at spatial infinity and vice versa in case the potential function has a critical point.

Returning to the general setting, we assume further that we are in the setting of Lemma 4.1. That is, d_0 written in distance coordinates F_0 near a point x_0 is generated by a continuous Riemannian metric $\tilde{g}_0$ on a Euclidean ball. Then, using Lemma 4.2, we see that we may assume that $|\mathrm{Rm}(\cdot, t)| \le \varepsilon(t)/t$ for all $t \in (0, T)$, where $\varepsilon : [0, 1] \to \mathbb{R}_0^+$ is a non-decreasing function with $\varepsilon(0) = 0$, and that the improved distance estimates

$$d_r + \varepsilon(t)\sqrt{t-r} \ge d_t \ge d_r - \varepsilon(t)\sqrt{t-r} \quad \text{for all } t \in [r, T]$$
$$\text{on } B_{d_0}(x_0, v) \Subset B_{g(s)}(x_0, 2v) \text{ for all } s \in [0, T] \tag{7.3}$$

hold on some fixed ball.

We now make the further restriction that the metric $\tilde{g}_0$ is smooth on some Euclidean ball containing $F_0(x_0)$, in the sense of Definition 1.4.

Theorem 1.6 shows in this case that the original Ricci flow solution comes out smoothly from some smooth initial data, if we restrict to a small enough neighbourhood of x_0.

Proof of Theorem 1.6. We consider

$$(\mathbb{B}_{\tilde{r}}(0), \tilde{g}(t))_{t\in[0,T)\cap[0,S(n,r,\varepsilon_0)]},$$

the solution to the δ-Ricci–DeTurck flow of Theorem 4.3, which is smooth on $\overline{\mathbb{B}_{\tilde{r}}(0)} \times (0, T) \cap (0, S(n, r, \varepsilon_0))$ and continuous on $\overline{\mathbb{B}_{\tilde{r}}(0)} \times [0, T] \cap [0, S(n, r, \varepsilon_0))$ and satisfies $\tilde{g}(\cdot, 0) = \tilde{g}_0 \in C^\infty(\overline{\mathbb{B}_{\tilde{r}}(0)})$. Let

$$\ell \in C^\infty(\mathbb{B}_{\tilde{r}}(0) \times [0, 1]) \cap C^0(\overline{\mathbb{B}_{\tilde{r}}(0)} \times [0, 1])$$

be the solution to the δ-Ricci–DeTurck flow that we obtain from Theorem 5.5 if we use $h := \tilde{g}$ to define the parabolic boundary values. Corollary 6.4 tells us that $\ell = \tilde{g}$ and hence $\tilde{g} \in C^\infty(\mathbb{B}_{\tilde{r}}(0) \times [0, 1])$.

By the smoothness of $\tilde{g}$, we see that

$$\sup_{\mathbb{B}_{3\tilde{r}/4}(0)\times[0,1]} \left(|\mathrm{Rm}(\tilde{g}(t))|^2 + |\nabla \mathrm{Rm}(\tilde{g}(t))|^2 + \cdots + |\nabla^k \mathrm{Rm}(\tilde{g}(t))|^2\right) \leq C_k \tag{7.4}$$

for all $t \in [0, 1]$.

By the original construction of $\tilde{g}$, we have $\tilde{g}(t) = \lim_{i\to\infty}(Z_i(t))_*(g(t))$ for all $t > 0$ where the $Z_i(t)$ are smooth diffeomorphisms for all $i \in \mathbb{N}$, and the limit is in the smooth sense on any compact subset of $\mathbb{B}_{\tilde{r}}(0) \times (0, 1]$. Hence, we must have

$$\sup_{B_{d_0}(x_0,\tilde{r}/2)\times(0,1]} \left(|\mathrm{Rm}(g(t))|^2 + |\nabla \mathrm{Rm}(g(t))|^2 + \cdots + |\nabla^k \mathrm{Rm}(g(t))|^2\right) \leq C_k. \tag{7.5}$$

Using the method of Hamilton [9, Section 6], we can extend the solution smoothly back to time 0: there exists a smooth Riemannian metric g_0 defined on $B_{d_0}(x_0, \tilde{r}/4)$ such that $(B_{d_0}(x_0, \tilde{r}/4), g(t))_{t\in[0,1]}$ with $g(0) = g_0$ is smooth. ■

We return to the expanding gradient Ricci soliton examples provided by [14] and [8], discussed at the beginning of this section. By construction, they have non-negative Ricci curvature and bounded curvature at time $t = 1$, which amounts to saying that the corresponding Ricci flows satisfy (7.1).

We make a small digression to show that if an expanding gradient Ricci soliton satisfies (7.1) then it must have non-negative Ricci curvature. Indeed, let $(M, g(t) = t\varphi_t^* g)_{t\in(0,\infty)}$ be an expanding gradient Ricci soliton satisfying (7.1) for all $t \in (0, \infty)$. This clearly means that $\mathrm{Ric}(g(t)) \geq 0$: if this were not the case, say $\mathrm{Ric}(g)(x)(v, v) = -L < 0$ for some $x \in M$ and some vector $v \in T_x M$ of unit length with respect to g, then we must have

$$\mathrm{Ric}(g(t))(x_t)(v_t, v_t) = -\frac{L}{t} g(t)(x_t)(v_t, v_t)$$

for all $t > 0$ where $x_t := \varphi_t^{-1}(x)$ and $v_t := (d_{x_t}\varphi_t)^{-1}(v)$. Consequently, $\mathrm{Ric}(g(t))(x_t) < -1$ for $t > 0$ small enough, a contradiction. So without loss of generality, $\mathrm{Ric}(g(t)) \geq 0$ and hence the asymptotic volume ratio

$$\mathrm{AVR}(g(t)) := \lim_{r\to\infty} \frac{\mathrm{Vol}(B_{g(t)}(x, r))}{r^n}$$

is well defined for all time $t > 0$ and all points $x \in M$ by Bishop-Gromov's Theorem. Moreover, Hamilton [7, Proposition 9.46] has shown that $\mathrm{AVR}(g(t))$ is positive for all positive times t. Using the non-negativity of the Ricci curvature together with the soliton equation (7.2), one can show that the potential function is a proper strictly convex function. In particular, it admits a unique critical point p in M which is a global minimum. Since we are considering expanding gradient Ricci solitons, we know that $(M, g(t), p)$ is isometric to $(M, tg(1), p)$ as pointed metric spaces, and hence the asymptotic volume ratio $\mathrm{AVR}(g(t))$ is a constant independent of time $t > 0$. Let (M, d_0, o) be the well defined limit of $(M, d(g(t)), p)$ as $t \to 0$, the existence of which is explained in the introduction and guaranteed by [18, Lemma 3.1]. The theorem of Cheeger–Colding on volume convergence now guarantees that the asymptotic volume ratio of (M, d_0, o) is also $\mathrm{AVR}(g(1))$ and that (M, d_0, o) is a volume cone. In fact, it is also a metric cone, due to [4, Theorem 7.6] and the fact that (M, d_0, o) is the Gromov–Hausdorff limit of $(M, t d(g(1)), p)$ for any sequence $t \to 0$. If $x_0 \in M$ is a point where d_0 is locally smooth, in the sense explained in Definition 1.4, then $(B_{g(0)}(x_0, r), g(t)) \to (B_{g(0)}(x_0, r), g(0))$ smoothly for some small $r > 0$ as $t \to 0$, where $g(0)$ is the local (near x_0) smooth extension of $g(t)$ to time zero.

In particular, if (M, d_0, o) is a smooth cone away from the tip o, in the sense that locally distance coordinates introduce a smooth structure near x_0 for any x_0 in M not in the tip of the cone, then the solution comes out smoothly from the cone away from the tip.

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