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Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature

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The purpose of this paper is to evolve non-smooth Riemannian metric tensors by the dual Ricci-Harmonic map flow. This flow is equivalent (up to a diffeomorphism) to the Ricci flow. One application will be the evolution of metrics which arise in the study of spaces whose curvature is bounded from above and below in the sense of Aleksandrov, and whose curvature operator (in dimension three Ricci curvature) is non-negative. We show that such metrics may always be deformed to a smooth metric having the same properties in a strong sense.

1. Introduction and statement of results.

Let (M^n, D) be an n-dimensional manifold with a smooth (C^∞) structure D. We say that a tensor S on a smooth manifold (M, D) is C^k or S is in $C^k((M, D))$ if in local co-ordinates (which come from the structure), $S = \{S_{ij}\}$, is C^k. To avoid any confusion we will fix the differentiable structure D of M and do not consider other structures $(M, \tilde{D})$. For this reason we will suppress the D in (M, D).

When considering a Riemannian metric tensor $g = \{g_{ij}\}$ on a compact manifold M we often assume g is C^2. This allows us to define the Riemannian curvature tensor which is then continuous. Given a C^∞ Riemannian metric g_0 on a compact manifold M, we can always find a $T > 0$ and a 1-parameter family of C^∞ Riemannian metrics $\{g(t)\}_{t\in[0,T]}$ on M, denoted $(M, g(t))$, such that

$$\begin{aligned} \tfrac{\partial}{\partial t} g(t) &= -2\mathrm{Ricci}(g(t)), \text{ for all } t \in [0,T] \\ g(0) &= g_0, \end{aligned} \tag{1.1}$$

where g is $C^\infty(M \times [0,T])$ (C^∞ on the manifold $(M \times [0,T])$ with the induced structure), and $\mathrm{Ricci}(g(t))$ is the Ricci curvature of the Riemannian manifold $(M, g(t))$. Notice that (1.1) makes no sense if g is not twice differentiable in space for all $t \in [0,T]$. The family $(M, g(t))_{t\in[0,T]}$ is called a solution to the Ricci flow with initial value g_0. Ricci flow was invented, and used by Hamilton to prove that every compact three manifold which admits a C^∞

Riemannian metric g_0 with Ricci$(g_0) > 0$ also admits a metric g_∞ of constant positive sectional curvature [Ha 1]. The flow was constructed in such a way that various geometrical conditions are preserved by the flow, and so that it is 'nearly' a gradient flow for the Yamabe quotient

$$E(g) = \frac{\int_M \mathrm{R}(g) vol_g}{(\int_M vol_g)^{\frac{(n-2)}{n}}},$$

where $\mathrm{R}(g)$ is the scalar curvature of (M, g) and vol_g is the volume form with respect to g on M. Many metric tensors on manifolds arise from Riemannian metric tensors which are not smooth. For example the geometric object obtained by cupping a two dimensional cylinder off with two hemispheres ([Pe] example 1.8) is a nice geometrical object sitting in $\mathbf{R}^3$. As a manifold it is simply topologically S^2, and we give this S^2 the standard differentiable structure. It inherits a natural Riemannian metric g from the ambient space R^3 (along the joins we define g by continuity). This metric g on S^2 is $C^{1,1}$ with respect to the standard differentiable structure of S^2, but *not* C^2. The curvature is defined away from the join and is bounded from above and below, but has a discontinuity at the join. This manifold with metric tensor is a well known example of a 'metric space with curvature bounded from above and below' studied initially by Aleksandrov [Al] in connection with his investigation of the intrinsic geometry of convex surfaces, and later for it's own sake by Aleksandrov and his followers (see [BN] for an overview of the theory and a good bibliography). Here the curvature bound from below is zero.

If we take two copies of a two dimensional truncated cone imbedded in $\mathbf{R}^3$ and join them at their boundary we obtain a nice geometrical object (as a manifold it is topologically equivalent to the infinite cylinder $\mathbf{R} \times S^1$). The metric g inherited from the ambient space $\mathbf{R}^3$ may be defined on the join by continuity and is then $C^{0,1}$ (Lipschitz continuous), but not C^1. Note that if we approximate this metric g by a family of metrics $\{\overset{\alpha}{g}\}_{\alpha \in \{1,2,\ldots\}}$ with $\overset{\alpha}{g} \to g$ as $\alpha \to \infty$ in the C^0 norm, then $\sup_{x \in M} |\mathrm{Riem}(\overset{\alpha}{g})| \to \infty$ as $\alpha \to \infty$. In this sense g has infinite curvature at the join, and (M, g) is not a manifold with curvature bounded from above and below.

The third example is the cone. Let us consider the two dimensional cone sitting in $\mathbf{R}^3$ as a graph over $\mathbf{R}^2$. This cone then inherits a metric g from the ambient space $\mathbf{R}^3$. Clearly g is C^∞ with respect to the standard coordinates in $\mathbf{R}^2$ away from the point corresponding to the tip of the cone (for simplicity let this point be $\tilde{0} = (0,0)$), but g cannot be continuously extended to this point. We see this as follows. The cone C is a graph over

$\mathbf{R}^2$, $C = \{(\tilde{x}, \alpha|\tilde{x}|), \tilde{x} \in \mathbf{R}^2\}$, where $\alpha > 0$ is some fixed constant, and hence using the formula for the metric of a graph, we obtain

$$g_{ij} = \delta_{ij} + \alpha^2 \frac{\partial}{\partial x^i}|\tilde{x}| \frac{\partial}{\partial x^j}|\tilde{x}| = \delta_{ij} + \alpha^2 \frac{x_i x_j}{|x|^2}.$$

Clearly $\tilde{x}_\varepsilon = (\varepsilon, \varepsilon) \to \tilde{0}$ as $\varepsilon \to 0$ and $\lim_{\varepsilon \to 0} g_{12}(\tilde{x}_\varepsilon) = \frac{\alpha^2}{2}$. Also $\tilde{y}_\varepsilon = (\varepsilon, -\varepsilon) \to \tilde{0}$ as $\varepsilon \to 0$, but $\lim_{\varepsilon \to 0} g_{12}(\tilde{y}_\varepsilon) = -\frac{\alpha^2}{2}$. Hence there is no way to continuously extend g to the point $\tilde{0}$. Note however that

$$\delta_{ij} \leq g_{ij} \leq (1 + \alpha^2)\delta_{ij}, \text{ for all } x \in M - (\tilde{0}),$$

in the sense of tensors. Later we shall see that metrics which fulfill such estimates, with $0 < \alpha^2 \leq \varepsilon(n)$ small, can nevertheless be flown.

We would like to have a way of evolving C^0 metrics g_0 by something like Ricci flow, so that for all times t bigger than zero, the solution $g(t)$ is smooth, and as time approaches zero from above, the metric $g(t)$ approaches g_0 uniformly on all compact subsets of M. The flow should also preserve various curvature conditions.

Non-regular Example. *Let $M = S^1 \times N$, where N is a compact manifold which admits a positive Einstein metric γ, g_0 be the warped product metric on M given by $g_0(x, q) = h_0(x) \oplus \gamma(q)$, where h_0 is a Riemannian metric on S^1. Then the Ricci flow has the solution $g(x, q, t) = h_0(x) \oplus (1 - 2kt)\gamma(q)$, which has for all times $t \geq 0$ the same regularity as the regularity of h_0.*

This means clearly that we cannot hope that the Ricci flow will 'smooth metrics out' on M with respect to the fixed differentiable structure.

It is well known that if a metric is a C^2 Einstein metric then one may introduce Harmonic co-ordinates for which the metric is C^∞ ([DK]). Such co-ordinates are only $C^{2,\alpha}$ compatible with our fixed structure D on M, and so not admissible as smooth (C^∞) co-ordinates for (M, D). We note that in example one, if we introduce Harmonic co-ordinates (change the structure) the metric will never be C^2 (otherwise we could apply the result of [DK] mentioned above, and introduce Harmonic co-ordinates which make the metric C^∞ which contradicts the fact that the scalar curvature has a discontinuity at the join).

In this paper we shall consider the dual Ricci-Harmonic Map flow (see section 6. [Ha 3]). This leads to a more general version of the Ricci DeTurck flow, considered initially by DeTurck in [DeT]. In the paper [Bem] the authors use Ricci flow to smooth out C^2 metrics by introducing harmonic co-ordinates at appropriately chosen times.

We give here a short introduction to the the dual Ricci-Harmonic and the Ricci DeTurck flow.

Let $g(t)$, $t \in [0,T]$ be an arbitrary one parameter family of smooth metrics, and $\phi_t : M \to M$ an arbitrary one parameter family of smooth diffeomorphisms. Then the metric $\hat{g}(t)$ defined by $\hat{g}(t) = \phi_t^* g(t)$ satisfies

$$\frac{\partial}{\partial t}\hat{g}_{ij}(t) = (\phi_t^* \frac{\partial}{\partial t} g(t))_{ij} + {}^t\hat{\nabla}_i \hat{V}_j + {}^t\hat{\nabla}_j \hat{V}_i,$$

where $(\hat{V})_i = (\phi_t^* V)_i$, $V_\alpha(p) = (\frac{\partial}{\partial t}\phi_t(p))^\beta g_{\beta\alpha}$, and ${}^t\hat{\nabla}$ is the co-variant derivative with respect to the metric $\hat{g}$ (see [Si], proposition 1.4). In particular if $g(t)$, $t \in [0,T]$ is a solution to Ricci flow, then

$$\begin{array}{rll} & \frac{\partial}{\partial t}\hat{g}_{ij}(t) & = -2\mathrm{Ricci}(\hat{g}(t)) + {}^t\hat{\nabla}_i \hat{V}_j + {}^t\hat{\nabla}_j \hat{V}_i, \\ \text{where} & (\hat{V})_j(p,t) & = \frac{\partial}{\partial x^\alpha}\phi^i(p,t) V^\alpha((\phi_t(p)),t)\hat{g}_{ij}, \\ \text{and} & \frac{\partial}{\partial t}(\phi_t(p)) & = V(\phi_t(p),t). \end{array} \tag{1.2}$$

We have now the freedom to choose the diffeomorphism ϕ. If g_0 is already Einstein, then the solution to the Ricci flow $g(t)$ is given by $g(t) = (1 - 2kt)g_0$,is also Einstein and has the same regularity as g_0. Hence $\hat{g}(t) = \phi_t^* g(t)$ is also Einstein. We want to choose ϕ_t so that $\hat{g}(t)$ will be regular for $t > 0$. As mentioned before, in harmonic co-ordinates an Einstein metric is regular. To this end we let $\phi = f^{-1} : (M \times [0,T]) \to (M \times [0,T])$, where f is the solution to the Harmonic map heat flow equation:

$$\begin{aligned} \frac{\partial}{\partial t} f(p,t) &= ({}^{g(t),h}\Delta f)(f(p,t)), \\ f(p,0) &= \mathrm{Id}(p), \end{aligned}$$

where h is some fixed smooth background metric. For an arbitrary function $f : (M,g) \to (N,h)$ between two Riemannian manifolds, the Laplacian of f is then a vector field in TN defined in co-ordinate form by

$$\begin{aligned} ({}^{g,h}\Delta f^i)(y) &= g^{\alpha\beta}(x)(\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} f^i(x) - {}^g\Gamma^\eta_{\alpha\beta}(x)\frac{\partial}{\partial x^\eta} f^i(x)) \\ &+ {}^h\Gamma^i_{jk}(y) g^{\alpha\beta}(x)\frac{\partial}{\partial x^\alpha} f^j(x)\frac{\partial}{\partial x^\beta} f^k(x), \end{aligned}$$

where $f(x) = y$. Since $\hat{V}(y,t) = -\frac{\partial}{\partial t} f(x,t) = -\Delta f(y,t)$, we obtain

$$\hat{V}(y,t) = \hat{g}^{\beta\delta}(y,t)({}^{\hat{g}(t)}\Gamma^\alpha_{\beta\delta} - {}^h\Gamma^\alpha_{\beta\delta})(y,t),$$

where $\hat{g} = f_* g$, in view of the way Christoffel symbols and tensors change under a co-ordinate transformation. So we see that the system (1.2) may be written

$$\begin{aligned} \frac{\partial}{\partial t}\hat{g}_{ij}(t) &= -2\mathrm{Ricci}(\hat{g}(t)) + {}^t\hat{\nabla}_i \hat{V}_j + {}^t\hat{\nabla}_j \hat{V}_i, \text{ where} \\ \hat{V}(x,t)_i &= \hat{g}_{ij}(x,t)\hat{g}^{kl}(x,t)({}^{\hat{g}(t)}\Gamma^j_{kl} - {}^h\Gamma^j_{kl})(x,t). \end{aligned} \tag{1.3}$$

The reader is referred to [Ha 3] section 6 for further discussion of the system (1.3) which is called the dual Ricci harmonic map heat flow, or [ES], [St] for further information about harmonic map heat flow. It is shown in [Ha 3] section 6 , that the evolution equation (1.3) for $\hat{g}(t)$ is a strictly parabolic system of equations. In particular if we choose $h = g_0$, then the evolution equation for $\hat{g}(t)$ in (1.3) is the Ricci-DeTurck flow, which was first introduced in [De] to prove the short time existence for Ricci flow on a compact manifold using standard parabolic techniques (short time existence for Ricci flow on a compact manifold was first proved by Hamilton [Ha 1] and relied upon the sophisticated machinery of the Nash-Moser Inverse function Theorem).

The evolution equation for ϕ_t in (1.2) may be written as a first order evolution equation in terms of $\hat{g}$. That is,

$$\begin{aligned} \frac{\partial}{\partial t}\phi^\alpha(p,t) &= (\phi_{t*}\hat{V})^\alpha(\phi(p,t)) = \frac{\partial}{\partial x^i}\phi^\alpha(p,t)\hat{g}^{jk}({}^{\hat{g}(t)}\Gamma^i_{jk} - {}^h\Gamma^i_{jk})(p,t), \\ \phi(p,0) &= \mathrm{Id}(p), \end{aligned} \tag{1.4}$$

in view of the derivation of $\hat{V}$ given above. If we can solve the evolution equation for $\hat{g}(t), t \in [0,T]$ in (1.3), and the solution $\hat{g}(t)$ is sufficiently regular, then we may solve (1.4) and then define $g(t)$ to be $g(t) = (\phi_t^{-1})^*(\hat{g}(t))$, which is then a solution to the Ricci flow. We say that $\hat{g}(t)$ solves the h Ricci flow or h flow of g_0. Many geometric quantities that are preserved by Ricci flow will also be preserved by h flow.

In Shi's paper [Sh], the Ricci-DeTurck flow was written term by term to obtain the evolution equation for solutions to (1.3) in co-ordinate form. We present here the evolution equation, in co-ordinate form, for metrics which solve (1.3) for an arbitrary smooth fixed background metric h. For the rest of the paper we shall be chiefly concerned with solutions of (1.3) and not solutions of Ricci flow. For this reason we will use the notation $g(t), t \in [0,T]$ to refer to a solution of (1.3). Let $g(t), t \in [0,T]$ be a solution to (1.3). Then $g(t)$, $t \in [0,T]$ solves the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t}g_{ab} = \;& g^{cd}\tilde{\nabla}_c\tilde{\nabla}_d g_{ab} - g^{cd}g_{ap}\tilde{g}^{pq}\tilde{R}_{bcqd} - g^{cd}g_{bp}\tilde{g}^{pq}\tilde{R}_{acqd} \\ & +\tfrac{1}{2}g^{cd}g^{pq}(\tilde{\nabla}_a g_{pc}\cdot\tilde{\nabla}_b g_{qd} + 2\tilde{\nabla}_c g_{ap}\cdot\tilde{\nabla}_q g_{bd} \\ & -2\tilde{\nabla}_c g_{ap}\cdot\tilde{\nabla}_d g_{bq} - 4\tilde{\nabla}_a g_{pc}\cdot\tilde{\nabla}_d g_{bq}), \\ g(0) = \;& g_0, \end{aligned} \tag{1.5}$$

where $\tilde{R}_{abcd} = \text{Riem}(h)_{abcd}$ and $\tilde{\nabla}$ is the co-variant derivative with respect to h. Note that if h is not twice differentiable, then (1.5) makes no sense, since then $\tilde{R}_{abcd} = \text{Riem}(h)_{abcd}$ is not defined. If we choose $h = g_0$, that is we wish to examine the Ricci DeTurck flow, and g_0 is not twice differentiable, then we cannot make sense of the above equation. For this reason we will always choose a smooth h not equal to g_0 (but close to g_0 in some to be specified C^0 sense) when examining (1.5).

The first part of this paper is concerned with finding a sensible solution to the h flow for initial data g_0 which is non-smooth. Theorem 1.1 (below) is the target theorem of this section.

Definition 1.1. *Let M be a complete manifold and g a C^0 metric, and $1 \leq \delta < \infty$ a given constant. A metric h is said to be a δ fair background metric for g, or 'δ fair to g', if h is C^∞ and there exists a constant k_0 with*

$$\sup_{x \in M} {}^h|\text{Riem}(h)(x)| = k_0 < \infty, \tag{1.6}$$

and

$$\frac{1}{\delta} h(p) \leq g(p) \leq \delta h(p) \text{ for all } p \in M. \tag{1.7}$$

Remark 1. *By the result of Shi [Shi], if g is Riemannian metric and h a smooth Riemannian metric satisfying (1.6) and (1.7) then there exists a smooth metric h' which is 2δ fair to g, and*

$$\sup_{x \in M} {}^h|{}^h\nabla^j \text{Riem}(h)(x)| = k_j < \infty,$$

where ${}^h\nabla^j$ is the jth covariant derivative with respect to h. We will assume (without loss of generality) that our h always fulfills such estimates.

Remark 2. *Let M be a compact manifold, and g a C^0 metric on M for which (M, g) is complete. Then for every $0 < \varepsilon < 1$ there exists a metric $h(\varepsilon)$, for which $h(\varepsilon)$ is $1 + \varepsilon$ fair to g.*

Proof (of Remark 2): We may use de Rham regularisation [deR], or a locally finite partition of unity and Sobolev averaging (see section on mollifiers in [GT]) to obtain a C^∞ metric h which is C^0 as close as we like to g. A bound on the curvature follows from the compactness of M. $\diamondsuit$

Theorem 1.1. *Let g_0 be a complete metric and h a complete metric on M which is $1 + \varepsilon(n)$ fair to g_0, $\varepsilon(n)$ as in Lemma 2.4. There exists a $T = T(n, k_0) > 0$ and a family of metrics $g(t), t \in (0, T]$ in $C^\infty(M \times (0, T])$ which solves h flow for $t \in (0, T]$,h is $(1 + 2\varepsilon)$ fair to $g(\cdot, t)$, for $t \in (0, T]$*

and

$$\lim_{t\to 0}\sup_{x\in\Omega'} {}^{h}|g(\cdot,t) - g_0(\cdot)| = 0,$$

$$\sup_{x\in M} |{}^{h}\nabla^{i} g|^2 \leq \frac{c_i(n,k_0,\ldots,k_i)}{t^i}, \text{ for all } t\in(0,T], i\in\{1,2,\ldots\},$$

where Ω' is any open set satisfying $\Omega' \subset\subset \Omega$, where Ω is any open set on which g_0 is continuous (see Theorem 5.2).

Remark 3. *As a consequence of Theorem 1.1, we see that if the metric g_0 is continuous except for a set $I \subseteq M$ of isolated points, then the distance function $\rho(t) : M \times M \to \mathbf{R}$, defined by $\rho(t)(x,y) = dist(g(t))(x,y)$ is lipschitz, and smooth almost everywhere, for all $t > 0$, and satisfies* $\lim_{t\to 0}\rho(t)(\cdot,\cdot) = \rho(0)(\cdot,\cdot)$ *uniformly on any compact subset of $M - I$.*

Remark 4. *If M is not compact, g is C^0 on M, and g is a 'metric of curvature bounded from above an below' (see below) outside some compact set $\bar{\Omega}$, and satisfies the global bound*

$$\sup_{x\in M-\bar{\Omega}} |\mathrm{Riem}(g)(x)|^2 \leq k_0,$$

for some constant $k_0 < \infty$, then for every $0 < \varepsilon < 1$ there exists an $h(\varepsilon)$ so that h is $1+\varepsilon$ fair to g.

Proof (of Remark 4): We mollify g as in the proof of remark two to obtain a metric h which is C^0 as close as we like to g. One needs to check that $\sup_M \mathrm{Riem}(h) < \infty$. On Ω this follows by compactness. Outside of Ω this is true because a metric with bounded curvature also has bounded curvature after it is mollified (see Lemma 6.1).

The second section of the paper is concerned with flowing metrics g_0 of bounded curvature from above and below (initially studied by Aleksandrov [Al], see [BN] for a good overview), or locally Lipschitz metrics which satisfy (for example) $\mathrm{Ricci}(g_0) \geq 0$, to obtain a smooth metric g which satisfies $\mathrm{Ricci}(g) \geq 0$. The main theorem of this section is as follows. Let $\mathcal{R}(g)$ be the curvature operator of g, and $G(g) : \Lambda^2(M)\otimes\Lambda^2(M) \to \mathbf{R}$ be the operator defined by

$$G(g)(\phi,\psi) = \phi^{ij}\psi^{kl}g_{ik}g_{jl}, \tag{1.8}$$

where $\Lambda^2(M)$ is the space of smooth two forms on M. $\mathcal{I}(g)$ will refer to the Isotropic curvature in the case that $M^n = M^4$ (see the proof of Theorem 6.7 for an overview of Isotropic curvature, and the discussion before Theorem 6.6 for an overview of the curvature operator).

Theorem 1.2. *Let $\mathcal{M}(n, k_0, d, v)$ be the set of (M^n, g) such that M^n is an n-dimensional compact manifold and g is a metric with curvature $K(M, g)$ bounded from above and below which satisfies*

$$-k_0 \leq K(M,g) \leq k_0, vol(M,g) \geq v, diam(M,g) \leq d.$$

There exists an $\varepsilon_1(3, k_0, d, v) > 0$, $\varepsilon_2(n, k_0, d, v) > 0$, $\varepsilon_3(4, k_0, d, v) > 0$ and $\varepsilon_4(n, k_0, d, v) > 0$ with the following properties. If (M^3, g) is an element of $\mathcal{M}(3, k_0, d, v)$ and satisfies $\mathrm{Ricci}(g) \geq -\varepsilon_1 g$, then there exists a smooth Riemannian metric g' on M^3 where (M^3, g') has non-negative Ricci-curvature. If (M, g) is an element of $\mathcal{M}(n, k_0, d, v)$ and satisfies $\mathcal{R}(g) \geq -\varepsilon_2 G(g)$, then there exists a smooth Riemannian metric g' on M where (M, g') has non-negative curvature operator. If (M, g) is an element of $\mathcal{M}(4, k_0, d, v)$ and satisfies $\mathcal{I}(g) \geq -\varepsilon_3$, then there exists a smooth Riemannian metric g' on M where (M, g') has non-negative Isotropic curvature. If (M, g) is an element of $\mathcal{M}(n, k_0, d, v)$ and satisfies $R(g) \geq -\varepsilon_4$, (scalar curvature), then there exists a smooth Riemannian metric g' on M where (M, g') has non-negative scalar curvature (see Theorem 6.8).

Remark 5. *In dimension three non-negative curvature operator is equivalent to non-negative sectional curvature. In dimensions bigger than three, non-negative curvature operator implies non-negative sectional curvature.*

Theorem 1.2 is proved by an application of Cheeger's finiteness Theorem and Gromov's compactness Theorem for metrics in $\mathcal{M}(n, k_0, d, v)$ and a contradiction argument, and an application of the following theorem.

Theorem 1.3. *Let M^n be a manifold (compact or not compact) which admits a complete metric g_0 of bounded curvature from above and below. If $\mathcal{R}(g_0) \geq 0$ then M^n admits a smooth Riemannian metric g satisfying $\mathcal{R}(g) \geq 0$. If $\mathrm{R}(g_0) \geq 0$ then M^n admits a smooth Riemannian metric g satisfying $\mathrm{R}(g) \geq 0$. If $n = 3$ and $Ricci(g_0) \geq 0$ then M^3 admits a smooth Riemannian metric g satisfying $Ricci(g) \geq 0$. If $n = 4$ and $\mathcal{I}(g_0) \geq 0$, then M^4 admits a smooth Riemannian metric g satisfying $\mathcal{I}(g) \geq 0$ (see Theorem 6.2, 6.6 and 6.7).*

Theorem 1.3 is proved by flowing the metric g_0 with the $hflow$ from Theorem 1.1, and showing that the smooth solution $g(t)$ also satisfies the curvature bounds from below.

We may slightly weaken the hypotheses of theorem 1.3 in the Ricci curvature case. We replace the bound on the curvature from above by a Lipschitz condition.

Theorem 1.4. *Let M^3 be a three manifold, and g_0 be a complete locally Lipschitz metric on M which satisfies $\mathrm{Ricci}(g_0) \geq 0$, in the weak sense of* *definition (6.4). Then the solution $g(x,t), t \in (0,T]$ to h flow of g_0 exists (for some smooth metric h) and satisfies* $\text{Ricci}(g(x,t)) \geq 0$ *for all $t \in (0,T]$ in the usual smooth Riemannian sense (see Theorem 6.5).*

2. A priori parabolicity.

Let g_0 be C^0, and h be δ fair to g_0. We define the function $\phi_0 : M \to M$ as follows,

$$\phi_0(p) = g_0^{ij}(p) h_{ij}(p).$$

We may always choose local co-ordinates around a fixed point p, so that at p we have $h_{ij}(p) = \delta_{ij}$, and $g_{ij}(p) = \delta_{ij}\lambda_i(p)$, and hence

$$\phi_0(p) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_n},$$

and hence, δ fairness implies that $\sup_{p \in M} \phi_0(p) \leq \frac{n}{\delta} < \infty$. We will use similar techniques to those of Shi [Sh] to obtain a priori estimates for a priori smooth solutions to the $hflow$ with initial C^∞ data g_0, where h is a metric δ fair to g_0 $(0 < \delta < \infty)$.

Lemma 2.1. *Let D be a compact region in M, Let g_0 be a $C^\infty(D)$ metric and h a metric on M which satisfies*

$$g_0 \geq (1-\delta)h. \tag{2.1}$$

Let $g(t), t \in [0,T]$ be a $C^\infty(D \times [0,T])$ solution to the h flow with Dirichlet boundary conditions $g|_{\partial D}(\cdot,t) = g_0(\cdot)$, $g(0) = g_0$. For every $\sigma > 1$ there exists an $S = S(n, k_0, \delta, \sigma)$ such that

$$g(t) \geq (1-\sigma)(1-\delta)h, \forall t \in [0,S] \cap [0,T].$$

Proof : We define the function ϕ by

$$\phi(x,t) = g^{j_1 i_1}(x,t) h_{i_1 j_2} g^{j_2 i_2}(x,t) h_{i_2 j_3} \cdots g^{j_m i_m}(x,t) h_{i_m j_1},$$

and note that it satisfies

$$\sup_{(x,t) \in D \times \{0\} \cup \partial D \times [0,T]} \phi(x,t) = \sup_{x \in D} \phi_0(x) \leq \frac{n}{(1-\delta)^m}, \tag{2.2}$$

due to (2.1). Using (1.5) as in [Sh] Lemma 2.2 we see that

$$\frac{\partial}{\partial t}\phi \leq g^{ij}\tilde{\nabla}_i \tilde{\nabla}_j \phi + k_0 \phi^{1+\frac{1}{m}},$$

and hence

$$\frac{\partial}{\partial t}(\phi)^{-\frac{1}{m}} \geq g^{ij}\tilde{\nabla}_i\tilde{\nabla}_j(\phi^{-\frac{1}{m}}) - (m+1)\phi^{\frac{1}{m}}\tilde{\nabla}_i\phi^{-\frac{1}{m}}\tilde{\nabla}_j\phi^{-\frac{1}{m}} - \frac{1}{m}k_0.$$

This implies

$$\begin{aligned}(\phi)^{-\frac{1}{m}} + \frac{1}{m}kt &\geq \inf_{(x,t)\in D\times\{0\}\cup\partial D\times[0,T]}(\phi)^{-\frac{1}{m}} \\ &\geq \frac{1}{\sup_{x\in D\times\{0\}}\phi^{\frac{1}{m}}} \geq \frac{(1-\delta)}{n^{\frac{1}{m}}},\end{aligned}$$

in view of (2.2) and the parabolic maximum principle. Rewriting the above inequality, we obtain $\frac{1}{\phi} \geq \left(\frac{(1-\delta)}{n^{\frac{1}{m}}} - \frac{1}{m}kt\right)^m$, which implies that

$$\phi \leq \Big(\frac{(1-\delta)}{n^{\frac{1}{m}}} - \frac{1}{m}kt\Big)^{-m},$$

which may be rewritten in co-ordinate form as

$$\frac{1}{{\lambda_1}^m} + \frac{1}{{\lambda_2}^m} + \ldots + \frac{1}{{\lambda_n}^m} \leq \Big(\frac{(1-\delta)}{n^{\frac{1}{m}}} - \frac{1}{m}kt\Big)^{-m}.$$

Since all the terms on the left hand side of the above equation are positive, we see that $\left(\frac{1}{\lambda_i}\right)^m \leq \left(\frac{(1-\delta)}{n^{\frac{1}{m}}} - \frac{1}{m}kt\right)^{-m}$, for fixed $i \in \{1, 2, \ldots n\}$ which implies that

$$\lambda_i(x,t) \geq \Big(\frac{(1-\delta)}{n^{\frac{1}{m}}} - \frac{1}{m}kt\Big).$$

This means for any given $\sigma > 0$, we may find an $S = S(k_0, n, \sigma)$, such that

$$\lambda_i \geq (1-\delta)(1-\sigma), \forall t \in [0,S]\cap[0,T].$$

♢

We wish also to obtain bounds from above for g in terms of h.

Lemma 2.2. *Let D be a compact region in M, and g_0 be a $C^\infty(D)$ metric and h a metric on M which satisfies $h \leq g_0 \leq (1+\delta)h$. Let $g(t), t \in [0,T]$ be a $C^\infty(D\times[0,T])$ solution to the h flow with Dirichlet boundary conditions $g|_{\partial D}(\cdot,t) = g_0(\cdot)$, $g(0) = g_0$. For every $\sigma > 0$, there exists an $S = S(n, k_0, \sigma) > 0$ such that*

$$g_{ij} \leq (1+\delta)(1+\sigma)h_{ij},$$

for all $t \in [0,S]\cap[0,T]$.

Proof : Choose $m(n)$, and $\alpha(m)$ so that

$$m \geq 64n^2 + 1 + 2c(n,h), \tag{2.3}$$

and large enough so that

$$(2n)^{\frac{1}{m}} \leq (1+\sigma) \tag{2.4}$$

and choose $\alpha = \alpha(m) > 0$ so small that

$$(1-\alpha)^{m-1} \geq \frac{3}{4}. \tag{2.5}$$

By the previous theorem, there exists an $S = S(n, k_0, \alpha) > 0$ such that

$$g(\cdot, t) \geq h(1-\alpha) \text{ for all } t \in [0,S] \cap [0,T]. \tag{2.6}$$

Equation (2.3) implies that

$$\begin{aligned} 4 - (m-1)(1-\alpha)^{m-1}\frac{1}{2(2n)^{1+\frac{1}{m}}} &\leq 4 - (1-\alpha)^{m-1}\frac{64n^2}{8n^2} \\ &= 4 - 8(1-\alpha)^{m-1}, \end{aligned}$$

which combined with (2.5) gives

$$4 - (m-1)(1-\alpha)^{m-1}\frac{1}{2(2n)^{1+\frac{1}{m}}} \leq -2. \tag{2.7}$$

Similar to Shi [Sh], we define

$$G = h^{i_1 j_1} g_{j_1 i_2} h^{i_2 j_2} g_{j_2 i_3} \dots h^{i_m j_m} g_{j_m i_1}, \tag{2.8}$$

and for $(1+\delta)^m - [\frac{1}{2n}](G) > 0$, we define $F = \frac{1}{(1+\delta)^m - [\frac{1}{2n}](G)}$. See [Sh], Lemma 2.3, equation (79). In our preferred co-ordinate system,

$$F(x,t) = \frac{1}{(1+\delta)^m - [\frac{1}{2n}](\lambda_1^m(x,t) + \lambda_2^m(x,t) + \dots \lambda_n^m(x,t))}. \tag{2.9}$$

From (2.8) and the fact that h is $1+\delta$ fair to g_0 we see that

$$(1+\delta)^m - [\frac{1}{2n}](G(x,0)) \geq (1+\delta)^m - [\frac{n}{2n}](1+\delta)^m \geq \frac{1}{2}(1+\delta)^m, \tag{2.10}$$

and hence $F(x,0) = \frac{1}{(1+\delta)^m - [\frac{1}{2n}](G(x,0))} < \infty$, is well defined at time zero. Since D is compact, and $g(x,t)$ is a priori smooth, there is some maximal

$T' \in [0,T] \cap [0,S]$, such that $F(\cdot,t)$ is well defined (not infinity) for all $t \in [0,T')$, and if $\sup_{D\times[0,T')} F < \infty$, then $[0,T'] = [0,T] \cap [0,S]$. Since the function F is well defined on $[0,T')$, we see that $(1+\delta)^m - [\frac{1}{2n}](G(x,t)) > 0$ for all $t \in [0,T')$, that is $\sum_i^n \lambda_i^m \leq (2n)(1+\delta)^m$, which implies

$$\lambda_i \leq (2n)^{\frac{1}{m}}(1+\delta) \text{ for all } i \in \{1,\dots,n\}, \text{ for all } t \in [0,T'). \qquad (2.11)$$

Then we calculate as in Shi (but remove the error stemming from (82), that is he should have there $-2\frac{\lambda_i}{\lambda_\alpha}\tilde{R}_{i\alpha i\alpha}$ not $-2\frac{1}{\lambda_i\lambda_\alpha}\tilde{R}_{i\alpha i\alpha}$. Once corrected one calculates as he does and no problem occurs) up to equation (89) on page 241, that

$$\frac{\partial}{\partial t}F \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta F + F^2\frac{m}{(1-\alpha)}\Big(4-(m-1)(1-\alpha)^{m-1}\frac{1}{(1+\delta)(2n)^{1+\frac{1}{m}}}\Big)|\tilde{\nabla} g|^2$$
$$+\frac{mn^2k_0}{(1-\alpha)}F^2 \quad \forall t \in [0,T'),$$

where here we have used (2.3),(2.6) and (2.11) to arrive at this evolution equation in the same way Shi does. Substituting $(1+\delta) < 2$ and then (2.5) into the above we get

$$\frac{\partial}{\partial t}F \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta F + \frac{mn^2k_0}{(1-\alpha)}F^2, \forall t \in [0,T').$$

From the parabolic maximum principle, and equation(2.10) we obtain

$$F(\cdot,t) \leq a(t) \leq \frac{a_0}{s(t)}, \qquad (2.12)$$

where $a(t) = \sup_{(x,t)\in D\times[0,T')} F(x,t)$, $a_0 = a(0)$, $s(t) = 1 - a_0b_0t$, and $b_0 = \frac{mn^2k_0}{(1-\alpha)}$. Substituting (2.9) into (2.12) we see that

$$(1+\delta)^m - \frac{1}{2n}G \geq \frac{s(t)}{a_0} \geq s(t)\frac{(1+\delta)^m}{2}\forall t \in [0,T'),$$

in view of (2.10). Rewriting this equation we get

$$G \leq 2n(1+\delta)^m(1-\frac{s(t)}{2}) = 2n(1+\delta)^m(\frac{1}{2}+a_0b_0t), \forall t \in [0,T'), \qquad (2.13)$$

in view of the definition of $s(t)$. Without loss of generality, we assume that $S \leq \frac{1}{4a_0b_0}$, which implies that $a_0b_0t \leq \frac{1}{4}$ for all $t \in [0,T')$, which when substituted into (2.13) implies that

$$\lambda_1^m(x,t) + \lambda_2^m(x,t) + \ldots + \lambda_n^m \leq \frac{3n}{2}(1+\delta)^m, \forall t \in [0,T') \qquad (2.14)$$

Equation (2.14) then implies that

$$(1+\delta)^m - [\frac{1}{2n}](G) \geq \frac{1}{4}(1+\delta)^m > 0 \forall t \in [0,T'),$$

and hence $T' = \min(S,T)$. From equation (2.14) we see, for fixed $i \in \{1,\dots n\}$, that

$$\lambda_i(x,t) \leq (\frac{3n}{2})^{\frac{1}{m}}(1+\delta) \text{ for all } t \in [0,S] \cap [0,T].$$

Substituting (2.4) into this inequality gives us $\lambda_i(x,t) \leq (1+\delta)(1+\sigma)$ as required. ♢

Theorem 2.3. *Let D be a compact region in M, and g_0 be a $C^\infty(D)$ metric and h a metric on M which satisfies $\frac{1}{\delta_1}h \leq g_0 \leq \delta_2 h$, where $\delta_i \geq 1$ for $i \in \{1,2\}$. Let $g(t), t \in [0,T]$ be a $C^\infty(D \times [0,T])$ smooth solution to h flow . For every $\sigma > 0$, there exists an $S = S(n,k_0,\delta_1,\delta_2,\sigma) > 0$ such that*

$$(1-\sigma)\frac{1}{\delta_1}h_{ij} \leq g_{ij} \leq \delta_2(1+\sigma)h_{ij},$$

for all $t \in [0,S] \cap [0,T]$.

Proof : The proof relies on some simple scaling arguments. First note that if $g(\cdot,t)$ is a solution to $hflow$, then so is $cg(\cdot,\frac{1}{c}t)$, with initial data $cg(\cdot,0)$. Let $\tilde{g}(\cdot,t) = \delta_1 g(\cdot,\frac{1}{\delta_1}t)$. Then $\tilde{g}_0(\cdot) = \tilde{g}(\cdot,0) = \delta_1 g_0(\cdot)$ satisfies $h \leq \tilde{g}(\cdot,0) \leq \delta_1\delta_2 h$. From the previous two lemmas, we may find an $S = S(\sigma,\delta_1\delta_2,n,k_0)$ so that $(1-\sigma)h_{ij} \leq \tilde{g}_{ij} \leq \delta_1\delta_2(1+\sigma)h_{ij}$, for all $t \in [0,S]\cap[0,T]$. Multiplying the above equation by $\frac{1}{\delta_1}$, we obtain the result. ♢

Lemma 2.4. *Let $g(t), t \in [0,S]$ be a $C^\infty(D \times [0,S])$ solution to the h flow, for some h which is $1+\varepsilon(n)$ fair to $g(t)$, for all $t \in [0,S]$ ($\varepsilon(n)$ to be specified in the proof below). Then*

$$\sup_{x\in D} |{}^h\nabla g(x,t)|^2 \leq c(n,h,\delta,D,\sup_D |{}^h\nabla g_0|^2) \textit{ for all } t \in [0,S].$$

Proof : Let

$$\phi(x,t) = g_{j_1 i_1}(x,t)h^{i_1 j_2} g_{j_2 i_2}(x,t) h^{i_2 j_3} \dots g_{j_m i_m}(x,t)h^{i_m j_1}. \tag{2.15}$$

We may always choose co-ordinates at a point so that $h_{ij}(p,t) = \delta_{ij}$, $g_{ij}(p,t) = \lambda_i\delta_{ij}$, and then $\phi(p,t) = (\lambda_1)^m + \dots + (\lambda_n)^m$. Notice that since $(1-\varepsilon)h(x) \leq g(x,t) \leq (1+\varepsilon)h$, we get

$$1-\varepsilon \leq \lambda_i \leq (1+\varepsilon) \tag{2.16}$$

in our preferred co-ordinate system. We will calculate the evolution equation for the function $\psi = (\phi(x,t) + a(n))|\overset{h}{\nabla} g(x,t)|^2$. Calculating as in Shi, and using the fact that $\hat{h}$ is a priori $(1+\varepsilon)$ fair to $\hat{g}(x,t)$, we see, as in Shi ([Sh], §4, page 250, estimate (33)) that the function ψ satisfies

$$\frac{\partial}{\partial t}\psi \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\psi - \frac{1}{2}\psi^2 + c_0(n, k_0, k_1),$$

as long as $\varepsilon(n) > 0$ is chosen small enough (as is the case in [Sh], §4, (33)) . For now we only need the fact that this implies that

$$\frac{\partial}{\partial t}(\psi - c_0 t) \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\psi - c_0 t),$$

although the term $-\frac{1}{2}\psi^2$ shall be important later. From the maximum principle we obtain that

$$\sup_{D\times[0,S]}(\psi - c_0 t) \leq \sup_{\partial D\times[0,S]\cup D\times 0}\psi. \tag{2.17}$$

Applying lemma 3.1, VI, §3 [LSU] to the evolution equation (1.5) for g, we get $\sup_{\partial D\times[0,S]}|\overset{h}{\nabla} g| \leq c(n, \delta, \partial D)$, in view of the apriori parabolicity (Theorem 2.3). Upon substituting this inequality into (2.17) we obtain the result. ◇

3. Solution to the Dirichlet problem.

Theorem 3.1. *Let g_0 be a $C^\infty(D)$ metric and h a metric which is $1+\varepsilon(n)$ fair to g_0 on D, where $D \subset M$ is a compact domain in M ($\varepsilon(n)$ as in lemma 2.4). There exists an $S = S(n, k_0, \delta) > 0$ and a family of metrics $g(t), t \in [0,S]$ which solves h flow, h is $1+2\varepsilon(n)$ fair to $g(t)$ for all $t \in [0,S]$, and $g|_{\partial D}(\cdot,t) = g_0(\cdot)$, $g(0) = g_0$.*

Proof : We consider the family of evolution problems

$$\begin{aligned} {}^sL({}^sg(x,t)) &= 0, \text{ for } (x,t) \in D\times[0,T], \\ {}^sg(\cdot,0) &= (1-s)g_0(\cdot) + sh(\cdot), \\ {}^sg(x,t) &= (1-s)g_0(x) + sh(\cdot)(x) \text{ for all } x \in \partial D, \text{ for all } t \in [0,T], \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} {}^sL(v)_{kl} &= \tfrac{\partial}{\partial t}v_{kl} - \hat{v}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j v_{kl} + \hat{v}^{cd}\hat{v}_{kp}h^{pq}\tilde{R}_{lcqd} + \hat{v}^{cd}\hat{v}_{lp}h^{pq}\tilde{R}_{kcqd} \\ &\quad -\tfrac{1}{2}\hat{v}^{cd}\hat{v}^{pq}(\tilde{\nabla}_a\hat{v}_{pc}\cdot\tilde{\nabla}_b v_{qd} + 2\tilde{\nabla}_c\hat{v}_{ap}\cdot\tilde{\nabla}_q v_{bd} \\ &\quad -2\tilde{\nabla}_c\hat{v}_{ap}\cdot\tilde{\nabla}_d v_{bq} - 4\tilde{\nabla}_a\hat{v}_{pc}\cdot\tilde{\nabla}_d v_{bq}), \end{aligned}$$

where $\tilde{R}_{abcd} = \text{Riem}(h)_{abcd}$ and $\tilde{\nabla} = {}^h\nabla$, $\hat{v}(\cdot,t) = (1-s)v(\cdot,t) + sh(\cdot)$ and $s \in [0,1]$ (strictly speaking the notation $\hat{v}$ should be ${}^s\hat{v}$ since the operator $\hat{}$ depends on s). Then if ${}^sg(x,t), t \in [0,T]$ is a classical solution to (3.2), one may verify that $\hat{v}(x,t) = \widehat{{}^sg}(x,t), t \in [0,T]$ solves

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{v}_{kl} &= \hat{v}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\hat{v}_{kl} - (1-s)\hat{v}^{cd}\hat{v}_{kp}h^{pq}\tilde{R}_{lcqd} - (1-s)\hat{v}^{cd}\hat{v}_{lp}h^{pq}\tilde{R}_{kcqd} \\
&- \frac{1}{2}\hat{v}^{cd}\hat{v}^{pq}(\tilde{\nabla}_a\hat{v}_{pc}\cdot\tilde{\nabla}_b\hat{v}_{qd} + \ldots), \\
v(\cdot,0) &= (1-s)g_0 + sh, \\
v(x,t) &= (1-s)g_0(x) + sh(x), \text{ for all } x \in \partial D, \text{ for all } t \in [0,T].
\end{aligned}$$

This is essentially equation (1.5), and we may use the same techniques we used there to obtain lemma 2.3 and lemma 2.4 for all classical solutions ${}^sg(\cdot,t), t \in [0,S]$ of $L_s({}^sg) = 0$ (independently of s) for all t in $[0,S]$, where $S = S(k_0, n, 1+\varepsilon)$ is as in Theorem 2.3. Then h is a priori $1+2\varepsilon$ fair to ${}^sg(t)$ for all $t \in [0,S]$ because of Theorem 2.3, and the fact that h is $1+\varepsilon$ fair to ${}^sg(0)$. Also using lemma 2.4, we get that $\sup_D |{}^h\nabla{}^sg(t)| \leq c(D, g_0, h, 1+\varepsilon)$. Hence the equation $L_s({}^sg) = 0$ may be written as

$$-\frac{\partial}{\partial t}\widehat{{}^sg}_{kl} + \widehat{{}^sg}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\widehat{{}^sg}_{kl} = f_{kl},$$

where f_{kl} is bounded uniformly by a constant depending only on $k_0, n, h, g_0|_D, \delta$ and D. Using the $W^{2,1}_q$ estimates for parabolic equations (see [LSU], chapter. IV, §9, thm. 9.1) we obtain $g \in W^{2,1}_q(D \times (0,T))$ with $\|g\|_{W^{2,1}_q(D\times(0,T))} \leq c(k_0, n, h, g_0|_D, \delta, D, q)$, for all integral q, and hence using the Sobolev imbedding Theorem (see for example Lemma 3.3, Section 3, Chapter II [LSU]) we get ${}^h\nabla g$ is Hölder continuous in space and time with a norm bounded by

$$\|{}^h\nabla g\|_{C^{\alpha,\frac{\alpha}{2}}((D\times(0,T))} \leq c(k_0, n, h, g_0|_D, \delta, D, \alpha).$$

The standard existence Theorem using the Leray Schauder fixed point Theorem (See for example [LSU] §6, ch. V, the argument given at the beginning of the chapter and in Theorem 6.1, with the family of quasi-linear problems given by L_s above) then applies to obtain existence of solutions to (1.5) for some short time $[0,T]$ where $T > 0$.

Notice that in this argument, all derivatives (time like and spatial) of the solutions ${}^sg(\cdot,t), t \in [0,T)$, and hence of the solution $g(x,t), t \in [0,T)$ to (1.5), are bounded by constants depending on g_0,h,$1+\varepsilon$, n and D , as long as $T \leq S(n, k_0, \delta, \delta)$, S as in thm. 2.3. Hence we obtain a solution to $hflow$ $g(x,t), (x,t) \in D \times [0,T)$ for every $T \leq S$. ♢

4. A priori interior estimates for the gradient and higher order mixed derivatives of g.

To obtain interior estimates for the first derivative of $g(x,t)$ we may modify the argument used in Shi.

Lemma 4.1. *Let $g(t), t \in [0,S]$ be a $C^\infty(D \times [0,S])$ solution to the h flow, for some h which is $1+\varepsilon(n)$ fair to $g(t)$, for all $t \in [0,S]$ ($\varepsilon(n)$ as in Lemma 2.4). Then*

$$\sup_{B(h)(x_0,r)} |{}^h\nabla g(x,t)|^2 \leq c(n,h,\frac{1}{r})\frac{1}{t}, \text{ for all } t \in [0,S],$$

where $B(h)(x_0,r)$ denotes a ball of radius r with centre x_0 calculated with respect to the metric h.

Proof : In lemma 2.4 we saw that the function $\psi = (\phi(x,t) + a(n))(|{}^h\nabla g(x,t)|^2$, satisfies

$$\frac{\partial}{\partial t}\psi \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\psi - \frac{1}{2}\psi^2 + c_0(n,\varepsilon(n),k_0,k_1) \text{ for all } (x,t) \in M \times [0,T].$$

We have been careful to include the dependence of the constant c_0 here, and note that it does not depend upon g_0 or D. Using this inequality we see that the function $f(x,t) = \psi(x,t)t$ satisfies

$$\frac{\partial}{\partial t}f \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta f - \frac{1}{2t}f^2 + c_0 t + \frac{f}{t} \text{ for all } (x,t) \in M \times [0,T]. \tag{4.2}$$

Let x_0 be fixed in M, and $\Omega = B(h)(x_0,r)$ a ball of radius r in M, where here we have used the notation $B(h)(x_0,r)$ to make clear that the ball $B(h)(x_0,r)$ is calculated in terms of the metric h. That is

$$\Omega = B(h)(x_0,r) = \{x \in M : dist(h,x,x_0) \leq r\}.$$

For fixed x_0, we use the background metric h and the Hessian comparison principle to construct a time independent cut off function η satisfying

$$\begin{aligned}
\eta(x) &= 1 \forall x \in B(h)(x_0,r), && (d1)\\
\eta(x) &= 0 \forall x \in M - B(h)(x_0,2r), && (d2)\\
0 &\leq \eta(x) \leq 1 \forall x \in M, && (d3)\\
{}^h|\tilde{\nabla}\eta|^2 &\leq c_1(\tfrac{1}{r})\eta, && (d4)\\
\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\eta &\geq -c_2(\sup_M \tilde{\text{Riem}}, \tfrac{1}{r})h_{\alpha\beta} = c_2(k_0,\tfrac{1}{r}). && (d5)
\end{aligned}$$

(see [Sh] Theorem 1.1). Note that the constants c_1 and c_2 decrease (get better) as r increases. Note also that the function is C^∞ almost everywhere, and Lipschitz everywhere. If we mollify the function η then we obtain a C^∞ function satisfying the same properties, but for slightly different balls ($B(h)(x_0, r-\varepsilon)$ and $B(h)(x_0, 2r+\varepsilon)$) and slightly different constants ($c_1+\varepsilon$, $c_2+\varepsilon$).

Using (d3) in equation (4.2) we get

$$\begin{aligned}\tfrac{\partial}{\partial t}(f\eta) \leq\ & g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(f\eta) - \tfrac{1}{16t}f^2\eta - 2g^{\alpha\beta}\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta\eta - fg^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\eta \\ & + c(n,r,h) + \tfrac{f\eta}{t}, \text{ for all } x\in\Omega, t\in[0,S].\end{aligned} \tag{4.3}$$

In this proof a large number of constants depending on n, h, r appear. To simplify the proof we use a small c to denote a constant $c = c(n,h,r)$. We often rewrite algebraic expressions involving c and other constants simply as c. For example $\frac{n}{2}c^2 + c^4$ would be turned into c.

Let us assume that (x_0, t_0) is an interior point of $B(h)(x_0, 2r)\times[0,T]$ where the supremum of $(f\eta)(x,t)$, for $(x,t)\in B(h)(x_0,2r)\times[0,T]$ is obtained. Since $(f\eta)(x_0,t_0)$ is a maximum of the function $f\eta(\cdot, t_0)$, we get

$$\begin{aligned}-2g^{\alpha\beta}\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta\eta &= -2g^{\alpha\beta}\tfrac{1}{\eta}\tilde{\nabla}_\alpha(f\eta)\tilde{\nabla}_\beta\eta + 2g^{\alpha\beta}\tfrac{f}{\eta}|\tilde{\nabla}\eta|^2 \\ &= 2g^{\alpha\beta}\tfrac{f}{\eta}|\tilde{\nabla}\eta|^2,\end{aligned}$$

at the point (x_0,t_0), which implies

$$-2g^{\alpha\beta}\tilde{\nabla}_\alpha f\tilde{\nabla}_\beta\eta \leq cf, \tag{4.4}$$

at (x_0,t_0), in view of (d4). Substituting inequality (4.4) and (d5) into (4.3), we obtain $\frac{\partial}{\partial t}(f\eta)(x_0,t_0) \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(f\eta) - \frac{1}{16t_0}f^2\eta + cf + c + \frac{f\eta}{t_0}$, at (x_0,t_0), which implies that

$$0 \leq \frac{\partial}{\partial t}(f\eta)(x_0,t_0) \leq -\frac{1}{16t_0}f^2\eta + cf + c + \frac{f\eta}{t_0}, \tag{4.5}$$

in view of the fact that $\frac{\partial}{\partial t}(f\eta)(x_0,t_0)\geq 0$, and $g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(f\eta)(x_0,t_0)\leq 0$, at the maximal point (x_0,t_0). Multiplying equation (4.5) by $\eta(x_0)$ we get

$$\frac{1}{16t_0}(f\eta)^2(x_0,t_0) - c(f\eta)(x_0,t_0) - \frac{(f\eta)(x_0,t_0)}{t_0} \leq c,$$

which implies

$$(f\eta)(x_0,t_0)\Big(\frac{1}{16t_0}(f\eta)(x_0,t_0) - c - \frac{1}{t_0}\Big) \leq c,$$

that is $(f\eta)(x_0,t_0) \leq c(n,h)$, in view of the fact that $t_0 \leq S = S(n,k_0)$. This implies that $\sup_{B(h)(x_0,r)} f(x,t) \leq \max(\sup_{B(h)(x_0,2r)} f(x,0), c) = c$, since $f(x,0) = 0$. Using $1+\varepsilon(n)$ fairness and the definition of f we obtain the result. $\diamondsuit$

We have now obtained the important a priori parabolic estimates and the a priori interior gradient estimate. To obtain further interior estimates we may apply the above techniques and those of Shi.

Lemma 4.2. *Let $g(t), t \in [0,S]$ be a $C^\infty(D) \times [0,S]$ solution to the h flow, for some h which is $1+\varepsilon(n)$ fair to $g(t)$, for all $t \in [0,S]$, $\varepsilon(n)$ as in lemma 2.4 Then*

$$\sup_{B(h)(x_0,r)} |{}^h\nabla^i g|^2 \leq \frac{c(n,i,\frac{1}{r},k_0,\ldots,k_{i+1})}{t^{p(i,n)}} \text{ for all } t \in [0,S], i \in \{1,2,\ldots\},$$

where $p(i,n)$ is an integer and $B(h)(x_0,r)$ denotes a ball of radius r contained in D.

Proof : Whenever we write $|T|$ for some tensor T in the following calculation, we shall mean ${}^h|$: the modulus of the tensor taken with respect to the metric h. We calculate similar to Shi ([Sh], Lemma 4.1, equation (4),(5),...) using the evolution equation (1.5) for h flow, that

$$\begin{aligned}
\frac{\partial}{\partial t}{}^h|{}^h\nabla^m g|^2 &= g^{ij}\,{}^h\nabla_i{}^h\nabla_j|{}^h\nabla^m g|^2 - 2g^{ij}\,{}^h\nabla_i({}^h\nabla^m g)\,{}^h\nabla_j({}^h\nabla^m g)\\
&+2\sum_{i+j+k=m,i,j,k,l\leq m} {}^h\nabla^i g^{-1} * {}^h\nabla^j g * {}^h\nabla^k \mathrm{Riem}(h) * {}^h\nabla^m g\\
&+2\sum_{i+j+k+l=m+2,i,j,k,l\leq m+1} {}^h\nabla^i g^{-1} * {}^h\nabla^j g^{-1} * {}^h\nabla^k g * {}^h\nabla^l g * {}^h\nabla^m g,\\
&\text{for all } x \in \Omega, \text{ for all } t \in [0,S],
\end{aligned}$$

where here $T * S$, (T and S are tensors), refers to some trace with respect to the metric h which results in a tensor of the appropriate type (in the above formula the tensor product should result in a function). Using the fact that h is C^∞ and $1+\varepsilon$ fair to $g(x,t)$, we get

$$\begin{aligned}
\tfrac{\partial}{\partial t}{}^h|{}^h\nabla^m g|^2 &\leq g^{ij}\,{}^h\nabla_i{}^h\nabla_j|{}^h\nabla^m g|^2 - 2g^{ij}\,{}^h\nabla_i({}^h\nabla^m g)\,{}^h\nabla_j({}^h\nabla^m g)\\
&+2c(n,h)\textstyle\sum_{i+j+k=m,i,j,k,l\leq m} |{}^h\nabla^i g||{}^h\nabla^j g||{}^h\nabla^m g|\\
&+2\textstyle\sum_{i+j+k+l=m+2,i,j,k,l\leq m+1}(1+\varepsilon)^4|{}^h\nabla^i g||{}^h\nabla^j g||{}^h\nabla^k g||{}^h\nabla^l g||{}^h\nabla^m g|,\\
&\text{for all } x \in \Omega, \text{ for all } t \in [0,S].
\end{aligned} \tag{4.6}$$

We will prove interior gradient bounds by induction in m. Assume that we know already that

$${}^h|{}^h\nabla^i g|^2 \leq \frac{c(n,h,m)}{t^{p(i,n)}} \text{ for all } x \in \Omega, t \in [0,T], i \in \{1,2,\ldots,m-1\},$$

where $p(i,n)$ is an integer. We show that this implies a similar bound for $|{}^h\nabla^m g|^2$. We will write $c(n,h,m)$ simply as c, to simplify readability of the proof (as in the proof of Theorem 4.1). Combining the evolution equation (4.6) with our induction hypothesis, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}{}^h|{}^h\nabla^m g|^2 \;\leq\;& g^{ij}\,{}^h\nabla_i {}^h\nabla_j |{}^h\nabla^m g|^2 - 2g^{ij}\,{}^h\nabla_i({}^h\nabla^m g)\,{}^h\nabla_j({}^h\nabla^m g)\\ &+\frac{c}{t^q}|{}^h\nabla^m g| + c|{}^h\nabla^m g|^2\\ &+c\sum_{i+j+k+l=m+2,\,m\leq i,j,k,l\leq m+1}|{}^h\nabla^i g||{}^h\nabla^j g||{}^h\nabla^k g||{}^h\nabla^l g||{}^h\nabla^m g|,\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial}{\partial t}|{}^h\nabla^m g|^2 \;\leq\;& g^{ij}\,{}^h\nabla_i {}^h\nabla_j |{}^h\nabla^m g|^2 - 2g^{ij}\,{}^h\nabla_i({}^h\nabla^m g)\,{}^h\nabla_j({}^h\nabla^m g)\\ &+\frac{c}{t^q}|{}^h\nabla^m g| + c|{}^h\nabla^m g|^2 + c|{}^h\nabla^m g|^2(|\nabla^2 g| + |\nabla g|^2)\\ &c|{}^h\nabla^{m+1} g||{}^h\nabla^m g||\nabla g|,\end{aligned}$$

where $q = q(n,p,m)$ is some integer. In what follows we shall freely replace powers of q by q. For example $2q^2$ will be replaced by q. Since $m \geq 2$, we may use our induction hypothesis on the gradient terms of order one and two, and the Cauchy-Schwarz inequality to obtain,

$$\begin{aligned}\tfrac{\partial}{\partial t}|{}^h\nabla^m g|^2 \;&\leq g^{ij}\,{}^h\nabla_i {}^h\nabla_j |{}^h\nabla^m g|^2 - 2g^{ij}\,{}^h\nabla_i({}^h\nabla^m g)\,{}^h\nabla_j({}^h\nabla^m g)\\ &+\tfrac{c}{t^q}|{}^h\nabla^m g|^3 + \tfrac{(1+\varepsilon)}{2}|{}^h\nabla^{m+1} g|^2 + \tfrac{c}{t^q}.\end{aligned} \tag{4.7}$$

Finally, substituting

$$2g^{ij}\,{}^h\nabla_i({}^h\nabla^m g)\,{}^h\nabla_j({}^h\nabla^m g) \geq \frac{1}{(1+\varepsilon)}{}^h|{}^h\nabla^{m+1} g|^2$$

into (4.7), we get

$$\begin{aligned}\frac{\partial}{\partial t}|{}^h\nabla^m g|^2 \;\leq\;& g^{ij}\,{}^h\nabla_i {}^h\nabla_j |{}^h\nabla^m g|^2 - \frac{1}{(1+\varepsilon)}{}^h|{}^h\nabla^{m+1} g|^2\\ &+\frac{c}{t^q}|{}^h\nabla^m g|^3 + \frac{c}{t^q} \text{ for all } x \in \Omega, \text{ for all } t \in [0,S].\end{aligned}$$

Similarly

$$\frac{\partial}{\partial t}|{}^h\nabla^{m-1} g|^2 \leq g^{ij}\,{}^h\nabla_i {}^h\nabla_j |{}^h\nabla^{m-1} g|^2 - \frac{1}{(1+\varepsilon)}{}^h|{}^h\nabla^m g|^2 + \frac{c}{t^q},$$

in view of the induction hypothesis. Following Shi ([Sh], Lemma 4.2 equation (80)) we define

$$\psi(x,t) = (a + {}^{h}|{}^{h}\nabla^{m-1} g|^2)\,{}^{h}|{}^{h}\nabla^{m} g|^2,$$

where a is a constant to be chosen later. In view of the previous two evolution equations we get

$$\begin{aligned}
\tfrac{\partial}{\partial t}\psi(x,t) \quad &\leq \quad g^{ij}\,{}^{h}\nabla_i {}^{h}\nabla_j \psi + \left(-\tfrac{1}{(1+\varepsilon)}\,{}^{h}|{}^{h}\nabla^{m} g|^4 + \tfrac{c}{t^q}\,{}^{h}|{}^{h}\nabla^{m} g|^2\right) \\
&\quad + (a + {}^{h}|{}^{h}\nabla^{m-1} g|^2)\left(-\tfrac{1}{(1+\varepsilon)}\,{}^{h}|{}^{h}\nabla^{m+1} g|^2 + \tfrac{c}{t^q}|{}^{h}\nabla^{m} g|^3 + \tfrac{c}{t^q}\right) \\
&\quad - 2g^{ij}\,{}^{h}\nabla_i\,{}^{h}|{}^{h}\nabla^{m-1} g|^2\,{}^{h}\nabla_j\,{}^{h}|{}^{h}\nabla^{m} g|^2 \\
&\leq \quad -\tfrac{1}{2(1+\varepsilon)}\,{}^{h}|{}^{h}\nabla^{m} g|^4 - \tfrac{a}{2(1+\varepsilon)}\,{}^{h}|{}^{h}\nabla^{m+1} g|^2 + \tfrac{c(1+a^4)}{t^q} \\
&\quad - 2g^{ij}\,{}^{h}\nabla_i\,{}^{h}|{}^{h}\nabla^{m-1} g|^2\,{}^{h}\nabla_j\,{}^{h}|{}^{h}\nabla^{m} g|^2,
\end{aligned} \tag{4.8}$$

where here it is clear that we have used our inductive assumption (and not that of Shi) and the Cauchy-Schwartz inequality. The last term satisfies

$$\begin{aligned}
-2g^{ij}\,{}^{h}\nabla_i\,{}^{h}|{}^{h}\nabla^{m-1} g|^2\,{}^{h}\nabla_j\,{}^{h}|{}^{h}\nabla^{m} g|^2 \quad &\leq \quad 2(1+\varepsilon)\,{}^{h}|{}^{h}\nabla^{m} g|^2\,{}^{h}|{}^{h}\nabla^{m-1} g|\,{}^{h}|{}^{h}\nabla^{m+1} g| \\
&\leq \quad \frac{1}{4(1+\varepsilon)}\,{}^{h}|{}^{h}\nabla^{m} g|^4 + \frac{c}{t^q}\,{}^{h}|{}^{h}\nabla^{m+1} g|^2,
\end{aligned}$$

where we have used our inductive assumption on the term ${}^{h}|{}^{h}\nabla^{m-1} g|$ and the Cauchy-Schwartz inequality. Substituting this inequality into (4.8) we get

$$\frac{\partial}{\partial t}\psi \leq g^{ij}\,{}^{h}\nabla_i {}^{h}\nabla_j \psi + \left(\frac{c}{t^q} - \frac{a}{2(1+\varepsilon)}\right){}^{h}|{}^{h}\nabla^{m+1} g|^2 - \frac{1}{4(1+\varepsilon)}\,{}^{h}|{}^{h}\nabla^{m} g|^4 + \frac{c(1+a^4)}{t^q}, \tag{4.9}$$

where here it is clear that we have used our inductive assumption (and not that of Shi), and $(1+\varepsilon)$ fairness to obtain upper and lower bounds for the metric $g(x,t)$ in terms of h (as in Shi). We now modify ψ to our purposes. We consider the new function w defined by

$$w = t^{q+1}(2c(1+\varepsilon)t^{-q} + |{}^{h}\nabla^{m-1} g|^2)|{}^{h}\nabla^{m} g|^2,$$

where q and c are the constants appearing in equation (4.9) (the constant $q = q(n,p,m)$ is now fixed!). That is we have chosen a to be a constant depending on t (who's time derivative we must therefore calculate), and multiplied the whole function by a power of t. Note that this function is zero at time zero and hence must attain a maximum at some time bigger than zero. Without loss of generality we may assume ${}^{h}|{}^{h}\nabla^{i} g|^2 \leq a(t)$, for all

$i \in \{1, \ldots, m\}$. Also note that in (4.9) we then get that $(\frac{c}{t^q} - \frac{a}{2(1+\varepsilon)}) \leq 0$ and $\frac{c(1+a^4)}{t^q} \leq \frac{c}{t^{5q}}$. In view of these two inequalites and (4.9), we get

$$\begin{aligned}\frac{\partial}{\partial t}w &\leq g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j w - \frac{1}{4(1+\varepsilon)}t^{q+1}|\overset{h}{\nabla}^m g|^4 + \frac{(q+1)}{t}w - qc|\overset{h}{\nabla}^m g|^2 + \frac{c}{t^{4q}},\\ &= g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j w - \frac{1}{4(1+\varepsilon)}\frac{w^2}{t^{q+1}(a(t)+|\overset{h}{\nabla}^{m-1}g|^2)^2} + \frac{(q+1)}{t}w\\ &\quad -c|\overset{h}{\nabla}^m g|^2 + \frac{c}{t^{4q}},\end{aligned}$$

where the last equality follows from the definition of w. Hence

$$\begin{aligned}\frac{\partial}{\partial t}w &\leq g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j w - \frac{1}{48(1+\varepsilon)}\frac{w^2}{t^{q+1}a^2(t)} + \frac{(q+1)}{t}w + \frac{c}{t^{4q}}\\ &= g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j w - \frac{1}{c^2}t^{q-1}w^2 + \frac{(q+1)}{t}w + cw + \frac{c}{t^{4q}},\end{aligned} \tag{4.10}$$

in view of the fact that $a(t) = ct^{-q}$, and ${}^h|\overset{h}{\nabla}^{m-1}g|^2 \leq a(t)$. Now, as in the estimate of the first derivative of g, we multiply w by a cut off function η and calculate the evolution equation of $w\eta$. Using (4.10) and d1 – d5 as in the estimate of the first derivative of g, we get

$$\frac{\partial}{\partial t}(w\eta) \leq g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j(w\eta) - \frac{1}{c^2}t^{q-1}w^2\eta + \frac{(q+1)}{t}(\eta w) + cw + \frac{c}{t^{4q}}.$$

At an interior point (x_0, t_0) of $\Omega \times [0,T]$ which is a maximum of w we argue as in the proof of lemma 4.1 to get

$$(\eta w)(x_0,t_0)\Big(\frac{1}{c^2}t_0^{5q-1}(\eta w)(x_0,t_0) - (q+1)\Big) \leq c,$$

from which we obtain $(\eta w)(x_0,t_0) \leq \frac{c}{t_0^{5q}}$. Using the definition of w and the above inequality, we get

$$t^{q+1}(4ct^{-q} + |\overset{h}{\nabla}^{m-1}g|^2)|\overset{h}{\nabla}^m g|^2 \leq \frac{c}{t^{5q}}, \text{ for all } x \in B_r(x_0), t \in [0,T],$$

which implies the desired result. $\diamondsuit$

Theorem 4.3. *Let $g(t), t \in [0,S]$, h be as in Theorem 4.2. Then*

$$\sup_{x\in M}|\overset{h}{\nabla}^i g(x,t)|^2 \leq \frac{c(n,i,R_i)}{t^i} \textit{ for all } t \in [0,T], i \in \{1,2,\ldots\},$$

where $R_i^2 = \max(k_0, k_1, k_2, \ldots, k_i)$, k_j as in definition 1.1.

Proof : We derive this corollary from Theorem 4.2 and a scaling argument. Let $\hat{g}(\cdot,t) = \frac{1}{R}g(x,Rt)$ for some constant $R > 0$, and $\hat{h}(\cdot) = \frac{1}{R}h(\cdot)$. Then $\hat{h}$ is $1+\varepsilon$ fair to $\hat{g}(x,t)$ and $\hat{g}(\cdot,t), t \in [0, \frac{S}{R}]$ solves $\hat{h}flow$. Without loss of generality we assume that $S \leq 1$. For a given $t_0 \in [0,S]$, let $R = t_0 \leq 1$. Then the $\hat{k}_i \leq k_i$, where $\hat{k}_i = \sup_M \overset{\hat{h}}{|}\overset{\hat{h}}{\nabla}^i \mathrm{Riem}(\hat{h})|^2$. Hence by lemma 4.2 we get

$$\overset{\hat{h}}{|}\hat{\nabla}^i g|^2|_{(x,1)} \leq c(n,\hat{k}_0,\ldots,\hat{k}_i) \leq c(n,k_0,\ldots,k_i).$$

But

$$\begin{aligned}&\overset{\hat{h}}{|}\hat{\nabla}^i \hat{g}|^2|_{(x,1)} \\ &= (\hat{h}^{j_1k_1}\hat{h}^{j_2k_2}\ldots\hat{h}^{j_ik_i}\hat{h}^{mn}\hat{h}^{pq}\hat{\nabla}_{j_1}\hat{\nabla}_{j_2}\ldots\hat{\nabla}_{j_i}\hat{g}_{mp}\hat{\nabla}_{k_1}\hat{\nabla}_{k_2}\ldots\hat{\nabla}_{k_i}\hat{g}_{nq})(x,1) \\ &= R^i|\overset{h}{\nabla}^i g|^2(x,R),\end{aligned}$$

from which the result follows. ◊

5. Existence of entire solutions.

Lemma 5.1. *Let g_0 be a $C^\infty(M)$ metric and h a metric on M which is $1+\varepsilon(n)$ fair to g_0, $\varepsilon(n)$ as in lemma 2.4. There exists a $T = T(n,k_0)$ and a family of metrics $g(t), t \in [0,T]$ in $C^\infty(M \times [0,T])$ which solves h flow, h is $(1+2\varepsilon)$ fair to $g(\cdot,t)$ for $t \in [0,T]$, and*

$$|\overset{h}{\nabla}^i g|^2 \leq \frac{c_i(n,k_0,\ldots,k_i)}{t^i}, \text{ for all } t \in (0,T], i \in \{1,2,\ldots\}.$$

Proof : If M is compact, then we obtain the result using Theorems 3.1 and 4.2. Let $\{D_i\}, i \in \{1,2,\ldots,\infty\}$ be a family of compact sets which exhaust M, $D_i = B(g_0)(x_0,i)$, where $B(g_0)(x_0,i)$ is the ball of radius i for some fixed arbitrary x_0, with respect the metric given by g_0. Let $\overset{i}{g}(\cdot,t), t \in [0,T]$ be the solutions obtained to the Dirichlet problem on D_i with boundary data g_0. Using the interior estimates (Theorem 4.2) and Arzela-Ascolii Theorem, we may let $i \to \infty$ and take a diagonal subsequence to obtain a limit $g(\cdot,t), t \in (0,T]$ which solves h flow for $t > 0$ and satisfies the interior estimates. As the initial data is smooth, as is the solution for $t > 0$, we see that $g(\cdot,t), t \in [0,T]$ solves h flow. ◊

Theorem 5.2. *Let g_0 be a complete metric and h a complete metric on M which is $1+\varepsilon(n)$ fair to g_0, $\varepsilon(n)$ as in Lemma 2.4. There exists a $T = T(n,k_0)$ and a family of metrics $g(t), t \in (0,T]$ in $C^\infty(M \times (0,T])$*

which solves h flow for $t \in (0,T]$,h is $(1+2\varepsilon)$ fair to $g(t)$ for $t \in (0,T]$, and

$$\lim_{t\to 0}\sup_{x\in\Omega'} {}^{h}|g(\cdot,t) - g_0(\cdot)| = 0,$$

$$\sup_{x\in M} |{}^{h}\nabla^{i} g|^2 \le \frac{c_i(n,k_0,\ldots,k_i)}{t^i}, \text{ for all } t \in (0,T], i \in \{1,2,\ldots\},$$

where Ω' is any open set satisfying $\Omega' \subset\subset \Omega$, where Ω is any open set on which g_0 is continuous.

Remark. *In particular, if M is compact, and g_0 is continuous then $g(\cdot,t) \to g_0(\cdot)$ uniformly on M as $t \to 0$.*

Proof : Let $\{\overset{\alpha}{g}_0\}_{\alpha\in\mathbf{N}}$ be a sequence of smooth metrics which satisfy $\lim_{\alpha\to\infty}\{\overset{\alpha}{g}_0\} = g_0$, where the limit is uniform in the C^0 norm. It follows then that h is $(1+\frac{\varepsilon}{2})$ fair to $\overset{\alpha}{g}_0$ for all $\alpha \ge N$ for some $N \in \mathbf{N}$. We flow each metric $\overset{\alpha}{g}_0$ by h flow (using Lemma 5.1) to obtain a family of metrics $\overset{\alpha}{g}(\cdot,t), t\in[0,T]$, $T = T(n,k_0)$ independent of α which satisfy

$$|{}^{h}\nabla^{j}(\overset{\alpha}{g}(\cdot,t))|^2 \le \frac{c_j}{t^j}, \text{ for all } t \in (0,T],$$

independently of α, for all $\alpha \ge N$. We then obtain a limiting solution $g(x,t)$, $t \in (0,T)$ via $g(x,t) = \lim_{\alpha\to\infty} \overset{\alpha}{g}(x,t)$, which is defined for all $t \in (0,T)$. This limit is obtained using the Theorem of Arzela-Ascolii (is uniform on compact subsets of M), and it may be necessary to pass to a sub-sequence to obtain the limit. It remains to show that the metrics $g(\cdot,t)|_{\Omega'}$ uniformly approaches $g_0(\cdot)|_{\Omega'}$ as t approaches zero. As a first step we obtain estimates on the rate at which $g(\cdot,t) \to g_0(\cdot,t)$ as $t \to 0$ if $g_0(\cdot)$ is smooth.

Let $\varepsilon > 0$ be given. Arguing as in [Sh] Lemma 2.2, we see from (66) and (68) in the proof of Lemma 2.2, and using $(1+\varepsilon(n))$ fairness that g^{ij} satisfies

$$\frac{\partial}{\partial t} g^{ij} \le g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta g^{ij} + c(h,n)g^{ij} - S^{ij},$$

where S^{ij} is a positive tensor obtained from the square of $\tilde\nabla g$ (the last term in [Sh] Lemma 2.2 , equation (68)). Since S^{ij} is positive, we get

$$\frac{\partial}{\partial t}(g^{ij}-l^{ij}) \le g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta(g^{ij}-l^{ij})+c(h,n)(g^{ij}-l^{ij})+g^{\alpha\beta}\tilde\nabla_\alpha\tilde\nabla_\beta l^{ij}+c(h,n)l^{ij}, \tag{5.1}$$

for any time independent tensor l^{ij}. Fix x_0 in Ω', and fix a co-ordinate chart around x_0, $\psi : U \to M$, $x_0 \in U \subset\subset \Omega$. Define the $(0,2)$ tensor l by

$$l(V,W)(x) = V_i(x)W_j(x)h_{qp}(x_0)g_0^{qi}(x_0)h^{pj}(x),$$

where on the right hand side we have used our fixed co-ordinate chart to help us define this tensor. That this is a well defined tensor (for example linearity) follows from the definition. Notice that the right hand side in the above definition *is dependent* on the coordinate chart. That is we have used our fixed co-ordinate chart to help us define this tensor. Also notice that $l^{ij}(x_0) = g_0^{ij}(x_0)$. By definition of l , we get

$$\begin{aligned} {}^h|g_0^{ij}(x) - l^{ij}(x)| &\leq {}^h|g_0^{ij}(x) - g_0^{ij}(x_0)| + {}^h|g_0^{ij}(x_0) - h_{qp}(x_0)g_0^{qi}(x_0)h^{pj}(x)| \\ &\leq \tfrac{\varepsilon}{2}, \end{aligned} \tag{5.2}$$

for all $x \in B(h)(x_0, r) \subseteq U$ for some small $r = r(g_0, h, \varepsilon) > 0$, where the last inequality follows from the continuity of g_0^{ij} and the continuity of h^{ij}. This gives us that

$$(1 - 2\varepsilon)h \leq l \leq (1 + 2\varepsilon)h, \tag{5.3}$$

for all $x \in B(h)(x_0, r)$, in view of (5.2) and the fact that h is $(1 + \varepsilon)$ fair to g_0.

We also have that

$$\sup_{B(h)(x_0,r)} {}^h|{}^h\nabla {}^h\nabla l| \leq c(h, n, U), \tag{5.4}$$

as a consequence of the definition of l, the inequality (5.3), and the fact that $U \subseteq \Omega$ is some fixed compact set. Substituting (5.4) into (5.1) and using (5.3) we get

$$\frac{\partial}{\partial t}(g^{ij} - l^{ij}) \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(g^{ij} - l^{ij}) + c(h, n, U)(g^{ij} - l^{ij}) + c(h, n, U)h^{ij},$$

and hence

$$\frac{\partial}{\partial t}(e^{-ct}(g^{ij} - l^{ij}) - cth^{ij}) \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(e^{-ct}(g^{ij} - l^{ij}) - cth^{ij}),$$

for all $x \in B(h)(x_0, r)$. Define the tensor f to be $f^{ij} = e^{-ct}((g^{ij} - l^{ij}) - cth^{ij})$. We construct a cut off function η, as in the proof of Lemma 4.1, for the ball $B(h)(x_0, r)$, with $\eta \equiv 1$ on $B(h)(x_0, \frac{r}{2})$ and $\eta \equiv 0$ on $\partial B(h)(x_0, r)$. Using the properties of η, as in the proof of lemma 4.1, we see that

$$\frac{\partial}{\partial t}(\eta f^{ij}) \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\eta f^{ij}) + c_1(\eta f^{ij}) - 2\tilde{\nabla}_\alpha\eta\tilde{\nabla}_\beta(f^{ij}\eta) + c_1 f^{ij},$$

which combined with the fact that f is bounded gives

$$\frac{\partial}{\partial t}(\eta f^{ij} - c_1 h^{ij}t) \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(\eta f^{ij} - c_1 h^{ij}t),$$

where $c_1 = c_1(\frac{1}{r}, n, h, U)$. Using the maximum principle and the fact that $\eta f^{ij} - c_1 h^{ij} t = -c_1 h^{ij} t \le 0$ on $\partial B(h)(x_0, r)$, we get that $\eta(\cdot) f^{ij}(\cdot, t) - c_1 h^{ij}(\cdot) t \le \eta f^{ij}(\cdot, 0) \le \frac{\varepsilon}{2} h^{ij}$ for all $x \in B(h)(x_0, r)$, where the last inequality follows from (5.2), and the definition of f. This implies that $f^{ij}(\cdot, t) \le (c_1 t + \frac{\varepsilon}{2}) h^{ij}$ for all $x \in B(h)(x_0, \frac{r}{2})$ and for all $t \in [0, S]$ in view of the fact that η is equal to one on $B(h)(x_0, \frac{r}{2})$. Substituting $t \le \frac{\varepsilon}{2c_1}$ into the above inequality, we get $f^{ij}(\cdot, t) \le \varepsilon h^{ij}$ for $x \in B(h)(x_0, \frac{r}{2})$, for all $t \le \frac{\varepsilon}{2c_1}$. Substituting the definition of f into this inequality, we see that $(g^{ij} - l^{ij}) \le e^{ct}(\varepsilon + ct) h^{ij}$, which implies that $(g^{ij} - l^{ij}) \le 2\varepsilon h^{ij}$, for all x in $B(h)(x_0, \frac{r}{2})$, for all $t \le T(c, c_1, \varepsilon)$. Substituting the inequality (5.2) into the above inequality, we get that

$$g^{ij} - {g_0}^{ij} = (g^{ij} - l^{ij}) + (l^{ij} - {g_0}^{ij}) \le 3\varepsilon h^{ij}, \tag{5.5}$$

for all x in $B(h)(x_0, \frac{r}{2})$, for all $t \le T(c, c_1, \varepsilon)$.

Notice that this argument applies to each solution ${}^{\alpha}g(\cdot, t)$ defined at the beginning of the proof. That is, ${}^{\alpha}g^{ij} \le {}^{\alpha}{g_0}^{ij} + 3\varepsilon h^{ij}$ for all $x \in B(h)(x_0, \frac{r_\alpha}{2})$, for all $t \le T(n, U, h, \frac{1}{r_\alpha}, \varepsilon)$, where here we write r_α, as r_α may possibly depend on α. In the estimate (5.2) we see that $r_\alpha > 0$ is chosen so that

$$^{h}|{}^{\alpha}g_0^{ij}(x) - {}^{\alpha}l^{ij}(x)| \le \frac{\varepsilon}{2},$$

for all $x \in B(h)(x_0, r_\alpha)$. But then for $x \in B(h)(x_0, r_\alpha)$, $\beta > \alpha$ we get

$$\begin{aligned} ^{h}|{}^{\beta}g_0^{ij}(x) - {}^{\beta}l^{ij}(x)| &\le {}^{h}|{}^{\beta}g_0^{ij}(x) - {}^{\alpha}g_0^{ij}(x)| + {}^{h}|{}^{\alpha}g_0^{ij}(x) - {}^{\alpha}l^{ij}(x)| \\ &\quad + {}^{h}|{}^{\alpha}l^{ij}(x) - {}^{\beta}l^{ij}(x)| \\ &\le 3\varepsilon, \end{aligned}$$

if α, β are chosen large enough, due to the continuity of h, the definition of l and the fact that ${}^{\alpha}g_0 \to g_0$ in Ω as $\alpha \to \infty$. So we see that we may choose $r > 0$ independent of α. Hence we obtain (5.5) for the metric $g(\cdot, t) = \lim_{\alpha \to \infty} {}^{\alpha}g(\cdot, t)$.

Let ϕ be the function defined in (2.15). Arguing similarly to Shi, we see that ϕ satisfies

$$\frac{\partial}{\partial t}\phi \le g^{\alpha\beta}\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \phi + c_0(h, n) - \frac{m^2}{8} {}^{h}|\nabla g|^2,$$

as shown in [Sh] (§4, equation (19)), where we use $(1 + \varepsilon(n))$ fairness as Shi does. Arguing as above, but for ϕ instead of g^{ij}, we see that there exists a $S = S(n, h, \Omega', g_0, \varepsilon) > 0$ such that

$$\phi(\cdot, t) \le \phi(\cdot, 0) + 3\varepsilon, \tag{5.6}$$

for all $t \in [0, S]$, for all $x \in \Omega'$. Using the inequalities (5.5) and (5.6) we see that $\sup_{\Omega'} {}^{h}|g_0(\cdot) - g(\cdot, t)| \le c(n)\varepsilon^{\frac{1}{m(n)}}$, for all $t \in [0, S]$, for all $x \in \Omega'$. $\diamondsuit$

6. Applications to metrics with 'curvature bounded from above and below'.

Assume that our initial metric g_0 is a 'metric with bounded curvature' on a compact manifold M, in the sense of Aleksandrov ([Al], see [BN] for a good overview). Such metrics are locally $C^{1,\alpha}$, and using a Theorem of Nikolaev, we may approximate g_0 by a family of smooth Riemannian metrics whose sectional curvatures are bounded from above and below by constants which approximate the bounds for g_0. Furthermore the bounds from above and below for Ricci curvature and curvature operator of the approximating metrics are approximately the same as those for g_0. We state this more precisely.

Lemma 6.1. *Let g be a 'metric with bounded curvature' on a manifold M, with curvature $K(g)$*

$$C' \leq K(g) \leq C$$

in the sense of Aleksandrov. We may approximate g by smooth Riemannian metrics, $\{\overset{\alpha}{g}\}_{\alpha \in \mathbf{N}}$ so that

$$C' - \frac{1}{\alpha} \leq K(\overset{\alpha}{g}) \leq C + \frac{1}{\alpha},$$

and

$$\lim_{\alpha\to\infty} |\overset{\alpha}{g} - g|_{C^{1,\beta}(\Omega)} \to 0, \lim_{\alpha\to\infty} {}^{g_0}|\overset{\alpha}{g} - g|_{C^0(M)} \to 0 \tag{6.1}$$

for open $\Omega \subseteq M$ whose closure is compact. Furthermore if the curvature satisfies

$$\begin{aligned} B'g &\leq \text{Ricci}(g) \leq Bg, \\ (B'G &\leq \mathcal{R}(g) \leq BG,) \end{aligned}$$

then

$$(B' - \frac{1}{\alpha})\overset{\alpha}{g} \leq \text{Ricci}(\overset{\alpha}{g}) \leq (B + \frac{1}{\alpha})\overset{\alpha}{g}, \tag{6.2}$$

$$((B' - \frac{1}{\alpha})\overset{\alpha}{G} \leq \mathcal{R}(\overset{\alpha}{g}) \leq (B + \frac{1}{\alpha})\overset{\alpha}{G}). \tag{6.3}$$

Proof: The approximation is achieved by mollifying or regularising g. Here we use Sobolev averaging and a partition of unity (Nikolaev used De Rham regularisation to obtain the estimates for the sectional curvatures: see [Re]).

Let $\{U_s\}$ be a locally finite cover by co-ordinate neighborhoods of M, and $U_s \subseteq U_{s'}$ for some co-ordinate chart $U_{s'}$. For $x \in U_s \subseteq U_{s'}$ define

$$\overset{s,\alpha}{g}_{ij}(x) = \int_{|z|\leq 1} \rho(z) g_{ij}(x - \frac{1}{\alpha} z) dz,$$

where here $\frac{1}{\alpha}$ is small enough so that $x - \frac{1}{\alpha} z \in U_{s'}$, for all $z \in B_1(0)$ (which then means that $g_{ij}(x - \frac{1}{\alpha} z)$ is well defined for this fixed co-ordinate system, $(U_{s'}, \psi)$). From work of Berestovskij [Be] we know that (M, g) is actually a manifold (and not just a metric space) and that g is continuous. Nikolaev [Ni] then used these facts to prove that locally $g \in W^{2,p}$. It then follows (see Berestovskij, Nikolaev[BN]) that g has a second derivative except on a set of measure zero $\Sigma_1 \subseteq M$. Hence we have the formula

$$\frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} \overset{s,\alpha}{g}_{ij}(p) = \overset{s,\alpha}{} (\frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} g_{ij})(p), \text{ for all } p \in M \tag{6.4}$$

where here we have used $g \in W^{2,p}$ in order to make sense of the right hand side. The local formula for the Riemannian curvature tensor of a metric g is given by

$$\begin{aligned} \mathrm{R}(g)_{ijkl} \;=\; & \tfrac{\partial}{\partial x^j} \tfrac{\partial}{\partial x^k} g_{il} + \tfrac{\partial}{\partial x^i} \tfrac{\partial}{\partial x^l} g_{jk} - \tfrac{\partial}{\partial x^j} \tfrac{\partial}{\partial x^l} g_{ik} - \tfrac{\partial}{\partial x^i} \tfrac{\partial}{\partial x^k} g_{jl} \\ & + (g^{-1} * \partial g * \partial g)_{ijkl}, \end{aligned} \tag{6.5}$$

where the last term is a product of two first derivatives of g and the inverse of g. Since the first derivatives of g are continuous (as is g itself) we obtain, in view of (6.4) and (6.5)

$$\mathrm{R}(\overset{s,\alpha}{g})_{ijkl}(p) = \overset{s,\alpha}{} (\mathrm{R}(g)_{ijkl})(p) \pm \varepsilon_{ijkl}(p), \text{ for all } p \in M,$$

where ε_{ijkl} is a tensor, $|\varepsilon|$ goes to zero as $\alpha \to \infty$. Using the partition of unity $\{U_s, \eta_s\}$, we construct our approximating metric $\overset{\alpha}{g} = \eta_s \overset{s,\alpha}{g}$. From construction (6.1) is true, and

$$\mathrm{R}(\overset{\alpha}{g})_{ijkl} = \overset{\alpha}{} (\mathrm{R}(g)_{ijkl}) \pm c \varepsilon_{ijkl}, \tag{6.6}$$

where the constant $c = c(\eta_1, \ldots, \eta_N)$ comes from taking first and second derivatives of the unity functions η_s. The estimates (6.2) and (6.3) then follow simply by taking traces with respect to $\overset{\alpha}{g}$ of (6.6), in view of (6.1). $\diamondsuit$

If the dimension of X is three, and (X, g_0) is a space with curvature bounded from above and below with $\mathrm{Ricci}(g_0) \geq 0$, then we may use the *hflow* to flow g_0 and so obtain a family $g(t)$, $t \in (0, T)$ of smooth metrics all of which satisfy $\mathrm{Ricci}(g(t)) \geq 0$.

Theorem 6.2. *Let g_0 be a complete metric with bounded curvature on a manifold M^3 of dimension three, $-k_0^2 \leq K \leq k_0^2$, such that*

$$\text{Ricci}(g_0) \geq 0$$

in the Aleksandrov sense. Then there exists a metric h which is $1+\varepsilon(n)$ fair to g_0 ($\varepsilon(n)$ as in lemma 2.4), a $T(n,h,k_0) > 0$, and a family of smooth Riemannian metrics $g(x,t), t \in [0,T]$ such that $g(x,t), t \in (0,T]$ solves h flow, $g(\cdot,t) \to g_0(\cdot)$ uniformly on compact subsets of M as $t \searrow 0$, and

$$0 \leq \text{Ricci}(g(x,t)) \leq c^2(k_0,n,\delta,h) \textit{ for all } t \in (0,T].$$

Proof : Let $\overset{\alpha}{g}_0$, be the approximating metrics for g_0 (obtained from lemma 6.1), and let h be $\overset{N}{g}_0$ a metric which is $1+\varepsilon(3)$ fair to all $\overset{\alpha}{g}_0$ for α big enough.Also let $\overset{\alpha}{g}(x,t), t \in [0,T]$ denote the corresponding solutions to the $hflow$, and $g(x,t), t \in (0,T]$ the limit (as $\alpha \to \infty$) solution. Note that each $\overset{\alpha}{g}_0$ satisfies $\sup_M {}^h|{}^h\nabla(\overset{\alpha}{g}_0)| \leq c_1$, from (6.1), and hence we see, arguing as in the proof of lemma 4.1 (but without multiplying our test function by time t) that $\sup_{M\times[0,T)} {}^h|{}^h\nabla(\overset{\alpha}{g})| \leq c_1$. Calculating the evolution equation of the function $t(a + {}^h|{}^h\nabla(\overset{\alpha}{g})|^2){}^h|{}^h\nabla^2(\overset{\alpha}{g})|^2$ as in the proof of lemma 4.2, and arguing as in the proof of lemma 4.2, we get that $\sup_{M\times[0,T]} {}^h|{}^h\nabla^2(\overset{\alpha}{g})| \leq \frac{c_2}{\sqrt{t}}$, in view of the fact that $|{}^h\nabla(\overset{\alpha}{g})|^2$ is bounded.

This then implies that the tensor $V(\overset{\alpha}{g}) = \overset{\alpha}{g}_{ij}(\Gamma^{ij}_k(\overset{\alpha}{g}) - \Gamma^{ij}_k(h))$ satisfies

$$\begin{aligned} &\sup_{M\times[0,T]} {}^h|V(g)| \leq c_3, \\ &\sup_{M\times[0,T]} {}^h|\text{Riem}(g)| + {}^h|{}^h\nabla V(g)| \leq \tfrac{c_4}{\sqrt{t}}. \end{aligned} \tag{6.7}$$

We wish to calculate the evolution of the curvature tensor of the metrics $\overset{\alpha}{g}$ For a fixed point p, let $\phi : B_\varepsilon(p) \times [0,\varepsilon] \to M$ be a time dependent local diffeomorphism satisfying the equation

$$\begin{aligned} \tfrac{\partial}{\partial t}\phi^i(p,t) &= (\phi_t * V)(\phi_t(p),t) = \hat{V}(\phi_t(p),t), \\ V(p,t) &= ({}^{g(t)}\Gamma^i_{jk} - {}^h\Gamma^i_{jk})(p,t), \\ \phi(p,0) &= \text{Id}(p), \end{aligned}$$

and define $\hat{g}(t) = (\phi_t{}^* g)(t)$. As explained in the introduction, $\hat{g}(t)$ satisfies the Ricci flow equation (1.1). Also

$$\text{Ricci}_{ij}(g(t))(p) = \text{Ricci}_{ij}(\phi_{t*}\hat{g}(t))(p) = \text{Ricci}_{\alpha\beta}(\hat{g}(t))(\phi_t(p))\frac{\partial}{\partial x^i}\phi^\alpha \frac{\partial}{\partial x^j}\phi^\beta,$$

which gives us that

$$
\begin{aligned}
&\tfrac{\partial}{\partial t}\mathrm{Ricci}_{ij}(g(t))(p)\\
&= \left(\tfrac{\partial}{\partial t}\mathrm{Ricci}_{\alpha\beta}(\hat{g}(t))(\phi_t(p)) + \tfrac{\partial}{\partial x^\eta}\mathrm{Ricci}_{\alpha\beta}(\hat{g}(t))(\phi_t(p))\tfrac{\partial}{\partial t}\phi_t(p)^\eta\right)\tfrac{\partial}{\partial x^i}\phi^\alpha\tfrac{\partial}{\partial x^j}\phi^\beta\\
&\quad +\mathrm{Ricci}_{\alpha\beta}(\hat{g}(t))(\phi_t(p))\tfrac{\partial}{\partial x^i}\tfrac{\partial}{\partial t}\phi^\alpha\tfrac{\partial}{\partial x^j}\phi^\beta + \mathrm{Ricci}_{\alpha\beta}(\hat{g}(t))(\phi_t(p))\tfrac{\partial}{\partial x^i}\phi^\alpha\tfrac{\partial}{\partial x^j}\tfrac{\partial}{\partial t}\phi^\beta\\
&= \Delta\mathrm{Ricci}(g)_{ij} + \theta(\mathrm{Ricci}(g))_{ij} + (\nabla_s\mathrm{Ricci}_{ij})V^s\\
&\quad +\mathrm{Ricci}_{lj}{}^{g}\nabla_i V^l + \mathrm{Ricci}_{il}{}^{g}\nabla_j V^l,
\end{aligned}
\tag{6.8}
$$

where in order to obtain the last equality we have used the fact that $\hat{g}(t)$ satisfies the Ricci flow equation (1.1) (as explained in the introduction) and here θ is a quadratic term coming from the curvature evolution equation. In dimension three, the evolution of the Ricci curvature for a family of metrics $\hat{g}$ evolving by Ricci flow is given by

$$\frac{\partial}{\partial t}\mathrm{Ricci}(\hat{g}) = {}^{\hat{g}}\Delta\mathrm{Ricci}(\hat{g}) + \theta(\mathrm{Ricci}(\hat{g})),$$

where $\theta(\mathrm{Ricci})$ is a quadratic in the Ricci curvature (See [Ha 1]). More specifically, if we choose co-ordinates around x_0 for given t_0 so that $\mathrm{Ricci}_{ij}(x_0, t_0)$ is diagonal, with values $\mathrm{Ricci}_{11} = \lambda \le \mathrm{Ricci}_{22} = \mu \le \mathrm{Ricci}_{33} = \nu$, then

$$\theta(\mathrm{Ricci})_{11} = (\mu-\nu)^2 + \lambda(\mu+\nu-2\lambda) \ge RR_{11} - 3g^{kl}R_{1k}R_{1l}, \tag{6.9}$$

and similarly

$$\theta(\mathrm{Ricci})_{ij} \ge R_{ij}R - 3g^{kl}R_{ik}R_{jl}.$$

Clearly θ satisfies the conditions of Theorem 7.3 and so, in view of (6.7) and the initial conditions and (6.9), we may apply the corollary 7.4 to the tensor $N = \mathrm{Ricci}({}^{\alpha}g(t))$ whose evolution equation is given by (6.8), to obtain

$$\mathrm{Ricci}(\overset{\alpha}{g}(x,t)) \ge -\frac{4}{\alpha}, \text{ for all } t \in [0, T''],$$

where $T'' = T''(3, k_0)$. Similarly we may apply Theorem 7.3 to the function $N = {}^{h}|\mathrm{Riem}(\overset{\alpha}{g})|^2$ to obtain $\sup_{M\times[0,T'']} {}^{h}|\mathrm{Riem}(\overset{\alpha}{g})|^2 < c(3, k_0, h)$. Letting α go to infinity gives us the result. $\diamondsuit$

Hence, if M is compact, we may apply the result of Hamilton ([Ha 2]) to obtain M is diffeomorphic to a quotient of one of the spaces S^3 or $S^2 \times \mathrm{R}^1$, or R^3 by a group of fixed point free isometries in the standard metric.

When the dimension of M is two we obtain a similar result by examining scalar curvature and arguing as in the theorem above. Note that in dimension two, the scalar curvature evolves according to the equation $\frac{\partial}{\partial t}R = \Delta R + R^2$.

Theorem 6.3. *Let g_0 be a complete metric on a manifold M^2,*

$$\begin{aligned} -k_0 &\leq K(g_0) \leq 0 \\ (0 &\leq K(g_0) \leq k_0) \end{aligned}$$

in the sense of Aleksandrov. Then there exists a metric h which is $1+\varepsilon(n)$ fair to g_0 ($\varepsilon(n)$ as in Lemma 2.4), a $T(n,k_0) > 0$, and a family of smooth Riemannian metrics $g(x,t), t \in [0,T]$ such that $g(x,t), t \in (0,T]$ solves h flow, $g(\cdot,t) \to g_0(\cdot)$ uniformly on compact subsets of M as $t \searrow 0$, and

$$\begin{aligned} -c^2(k_0,h) &\leq \mathrm{R}(g(x,t)) \leq 0 \textit{ for all } t \in (0,T] \\ (0 &\leq \mathrm{R}(g(x,t)) \leq c^2(k_0,h) \textit{ for all } t \in (0,T]). \end{aligned}$$

We can actually slightly weaken the hypothesis of 'curvature bounded from above' for Theorem 6.2 to a uniform Lipschitz condition on the initial sequence of metrics.

Definition 6.4. *Let M be a three dimensional manifold, and g be a locally Lipschitz complete metric on M. We say that* $\mathrm{Ricci}(g) \geq 0$, *if there exists a family $\{{}^\alpha g\}_{\alpha \in \{1,2,\ldots\}}$ of smooth metrics on M which satisfy* $\mathrm{Ricci}({}^\alpha g) \geq -\frac{1}{\alpha}$, *and $\lim_{\alpha\to\infty} \sup_M {}^g|{}^\alpha g - g| = 0$, and ${}^g|\Gamma({}^\alpha g) - \Gamma(g)| \leq k$ for all $\alpha \in \{1,2,\ldots\}$, where k is some constant which does not depend on α, and $\Gamma(g)$ refers to the Christoffel symbols of g.*

Theorem 6.5. *Let M^3 be a three dimensional manifold and g_0 be a complete locally Lipschitz metric on M which satisfies* $\mathrm{Ricci}(g_0) \geq 0$, *in the sense of definition 6.4. Then there is a solution $g(x,t), t \in (0,T]$ to h flow of g_0 for some smooth metric h and some $T = T(k_0)$, and it satisfies* $\mathrm{Ricci}(g(x,t)) \geq 0$ *for all $t \in (0,T]$ in the usual smooth sense.*

Proof : The proof is the same as for the case of bounded curvature from above (theorem 6.2), except that we use the family ${}^\alpha g_0$ coming from the definition 6.4, and not the family ${}^\alpha g_0$ constructed in the proof of theorem 6.2 (which come from lemma 6.1). ♢

We now examine the evolution equation of the curvature operator $\mathcal{R}$. In [Ha 3], Hamilton uses time dependent isomorphisms $u(t) : (TM, g_0) \to (TM, g(t))$ to examine the evolution of the curvature operator. In particular if $(M, g_{ij}(t)$ is a solution to the Ricci flow, then the pull back of the curvature operator is

$$\mathcal{R}(t)(\phi,\psi) = \mathrm{R}(t)_{abcd}\phi^{ab}\psi^{cd},$$

where $\mathrm{R}(t)_{abcd} = \mathrm{Riem}(g(t))_{ijkl}u^i_a u^j_b u^k_c u^l_a$, and the pull back of the metric is $g_{ab} = u^i_a(t)u^j_b(t)g_{ij}(t)$, and the isomorphisms $u(t)$ are chosen so that g_{ab} has zero time derivative, and hence g_{ab} is independent of t. That is

$$\frac{\partial}{\partial t} u^i_a = g^{ij} R_{jk} u^k_a.$$

The evolution of $\mathcal{R}$ is then derived in [Ha 3] to be

$$\frac{\partial}{\partial t}\mathcal{R} = \Delta\mathcal{R} + \mathcal{R}^2 + \mathcal{R}\#\mathcal{R},$$

where $\mathcal{R}^2$ is the square of the curvature operator, $\#$ is the operator given by $T\#N_{\alpha\beta} = c_\alpha^{\gamma\eta} c_\beta^{\delta\theta} T_{\gamma\delta} N_{\eta\theta}$, and $c_{\alpha\gamma\eta}$ are the structure constants given by $c^{\alpha\beta\eta} =< \phi^\alpha, [\phi^\beta, \phi^\eta] >$.

Theorem 6.6. *Let g_0 be a metric with bounded curvature on a manifold M, $-k_0^2 \le K \le k_0^2$, such that $\mathcal{R}(g_0) \ge 0$ in the Aleksandrov sense. Then there exists a metric h which is $1+\varepsilon(n)$ fair to g_0 ($\varepsilon(n)$ as in Lemma 2.4), a $T(n,k_0) > 0$, and a family of smooth Riemannian metrics $g(x,t), t \in [0,T]$ such that $g(x,t), t \in (0,T]$ solves h flow, $g(\cdot,t) \to g_0(\cdot)$ uniformly on compact subsets of M as $t \searrow 0$, and*

$$0 \le \mathcal{R}(g(x,t)) \le c^2(k_0, n, h) \text{ for all } t \in (0,T].$$

The same result is achieved if we replace the curvature operator (in the above hypotheses) by the scalar curvature.

Proof : We argue as in Theorem 6.2. Notice that we obtain initially that $\sup_{M\times[0,T]} |\mathrm{Riem}(\overset{\alpha}{g})| \le c(k,n)$ which implies that $\sup_{M\times[0,T]} \lambda_i(x,t) \le c'(k,n)$, for all $i \in 1,2,\ldots,\frac{n(n-1)}{2}$ where λ_i are the eigenvalues of the curvature operator $\mathcal{R}$. Then the evolution equation for $\mathcal{R}$ fulfills the conditions of corollary 7.5. In particular the tensor a appearing in Theorem 7.1 (which is needed for corollary 7.5) will be a matrix coming from the eigen values of $\mathcal{R}$ and then $\sup_{M\times[0,T]} {}^g|a| \le c(k,n)$. We may apply corollary 7.5 to the family of solutions $\overset{\alpha}{g}(x,t), t \in [0,T]$, and then take the limit as $\alpha \to \infty$ to obtain the result. $\diamondsuit$

For completeness we mention the following results which are proved using the same techniques as above. Let M^4 be a four manifold and $\mathcal{I}$ denote the isotropic curvature on this manifold (see [Ha 5]).

Theorem 6.7. *Let g_0 be a metric with bounded curvature on a compact real four manifold M^4 , $-k_0^2 \le K \le k_0^2$, such that $\mathcal{I}(g_0) \ge 0$ in the weak Aleksandrov sense. Then there exists a metric h which is $1+\varepsilon(4)$ fair to g_0, a $T(k_0) > 0$, and a family of smooth Riemannian metrics $g(x,t), t \in (0,T]$ such that $g(x,t), t \in (0,T]$ solves h flow, $g(\cdot,t) \to g_0(\cdot)$ uniformly on compact subsets of M as $t \searrow 0$, and*

$$0 \le \mathcal{I}(g(x,t)) \le c^2(k_0, h) \text{ for all } t \in (0,T].$$

Proof : In four dimensions one can decompose the real two forms Λ^2 into the direct sum of Λ^2_+ and Λ^2_-. Then the curvature operator defined on $\Lambda^2 \otimes \Lambda^2$ decomposes as a block matrix

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}.$$

The manifold has non-negative isotropic curvature if and only if $a_1 + a_2 \geq 0$ and $c_1 + c_2 \geq 0$, where a_1 and a_2 are the two smallest eigenvalues of A and c_1, c_2 are the two smallest eigenvalues of C. The ordinary differential equation for the evolution of a (c is the same) under Ricci flow is (see [Ha 5], proof of Theorem 1.2)

$$\frac{\partial}{\partial t}(a_1 + a_2) \geq a_1^2 + a_2^2 + 2(a_1 + a_2)a_3 + b_1^2 + b_2^2 \text{ a.e. } t \in [0,T], \text{ a.e. } x \in M,$$

where b_1 and b_2 are the two smallest eigenvalues of B (we ignore the lapcaian term for the moment). We consider the function $f(x,t) = a_1(x,t) + a_2(x,t)$ and note that it satisfies the ODE

$$\frac{\partial}{\partial t} f \geq 2a_3 f \text{ a.e. },$$

where $\sup_{M \times [0,T]} |a_3| \leq k < \infty$. We may argue then as in Theorem 6.2 to show that if $f(x,0) \geq 0$ in the Aleksandrov sense then g_0 can be evolved by $hflow$ for some h to obtain $g(x,t), t \in [0,T]$, and $f(x,t) \geq 0$ for all $t \in [0,T]$, where $f(x,t) = a_1(x,t) + a_2(x,t)$ and $a_1(x,t), a_2(x,t)$ are the two smallest eigen values of $A(x,t)$, where $A(x,t)$,$B(x,t)$,$C(x,t)$ are the curvature operator matrices (as above) for the metric $g(x,t)$. Similarly we obtain $c_1 + c_2 \geq 0$. $\diamondsuit$

We now show that the theorems of Hamilton for manifolds with non-negative Ricci curvature in three dimensions, and non-negative curvature operator in n-dimensions can be epsilon improved. To do this we argue by contradiction and apply Cheeger's finiteness Theorem([Ch]), and Gromov's compactness Theorem (see [Pe] for an exposition), The argument here was inspired by the argument given in Berger [Ber 1], where it is shown that there as a $\delta < \frac{1}{4}$ such that any compact, even dimensional manifold which is δ pinched is either homeomorphic to an n-dimensional sphere or is isometric to one of the symmetric spaces of rank one (the complex projective space CP^n, the quaternion projective space HP^n, or the Cayley projective space CaP^2). This is an epsilon improvement on the $\frac{1}{4}$ pinched rigidity Theorem (which is the same result for $\frac{1}{4}$ pinched manifolds). See Berger [Ber 2], Klingenberg [Kl] and Rauch [Ra] for the Sphere Theorem, rigidity Theorems and their generalisations.

Theorem 6.8. *Let $\mathcal{M}(n, k_0, d, v)$ be the set of (M^n, g) such that M^n is an n-dimensional compact manifold and g is a metric with curvature $K(M, g)$ bounded from above and below which satisfies*

$$-k_0 \leq K(M,g) \leq k_0, vol(M,g) \geq v, diam(M,g) \leq d.$$

There exists an $\varepsilon_1(3, k_0, d, v) > 0$, $\varepsilon_2(n, k_0, d, v) > 0$, and $\varepsilon_3(4, k_0, d, v) > 0$ with the following properties. If (M^3, g) is an element of $\mathcal{M}(3, k_0, d, v)$ and satisfies Ricci$(g) \geq -\varepsilon_1 g$, *then there exists a smooth Riemannian metric g' on M^3 where (M^3, g') has non-negative Ricci-curvature. If (M, g) is an element of $\mathcal{M}(n, k_0, d, v)$ and satisfies $\mathcal{R}(g) \geq -\varepsilon_2 G(g)$, then there exists a smooth Riemannian metric g' on M where (M, g') has non-negative curvature operator. If (M^4, g) is an element of $\mathcal{M}(4, k_0, d, v)$ and satisfies $\mathcal{I}(g) \geq -\varepsilon_3$, then there exists a smooth Riemannian metric g' on M^4 where (M^4, g') has non-negative Isotropic curvature. If (M, g) is an element of $\mathcal{M}(n, k_0, d, v)$ and satisfies $R(g) \geq -\varepsilon_4$, (scalar curvature), then there exists a smooth Riemannian metric g' on M where (M, g') has non-negative scalar curvature.*

Proof : All of these results are proved in the same way using Gromov's compactness result and Cheeger's finiteness Theorem for manifolds in $\mathcal{M}(n, k_0, d, v)$. We prove the Ricci curvature result here. Fix k_0, d and v. Assume, to the contrary that there is no such $\varepsilon_1 > 0$. Then we have for $i \in \{1, 2, \ldots\}$, manifolds M_i with metrics g_i such that $(M_i, g_i) \in \mathcal{M}(3, k_0, d, v)$, and Ricci$(g_i) \geq -\frac{1}{i}$, but there is *no* smooth ${g_i}'$ on M_i such that ${g_i}'$ has non-negative Ricci curvature. By Cheeger's finiteness Theorem, after taking a sub-sequence if necessary, we may assume that $M_i = M$. By Gromov's compactness Theorem, $g_i \to g$ in $C^{1,\alpha}$ for some $g \in \mathcal{M}(3, k_0, d, v)$ which satisfies Ricci$(g) \geq 0$ on M in the sense of Aleksandrov. We may flow this metric g using *hflow* (Theorem 6.2) to obtain a metric g' on M which is smooth and has non-negative Ricci curvature: a contradiction. $\diamondsuit$

7. Non-compact tensor maximum principles.

For scalar parabolic equations on non-compact manifolds there exist already versions of the maximum principle. For example Ecker and Huisken [EH] prove a maximum principle for a scalar function on a non-compact manifold which is evolving by a very general heat flow like equation (with a back ground metric which may depend on time), as long as the function satisfies a priori various spatial growth conditions and the metric satisfies a priori various spatial and temporal growth conditions. It is well known that for non-compact manifolds the maximum principle may be violated if at some fixed time the function has very large (bigger than exponential) growth in space. Here we prove a maximum principle for tensors which evolve parabolically (that is we consider a system of equations) on non-compact manifolds.

For the proof below we introduce the notation Σ^2 to be the set of two by two symmetric matrices.

Theorem 7.1 Non-compact Tensor Maximum Principle. *Let (M^n, g) be a smooth complete Riemannian manifold (non-compact or compact), and $N(t), t \in [0,T]$ be a family of symmetric two tensors on M evolving according to the evolution equation*

$$\begin{aligned} \frac{\partial}{\partial t}N_{ij}(\cdot,t) &= {}^{g(t)}\Delta N(\cdot,t)_{ij} + \nabla_s N_{ij}(\cdot,t))W^s + fN_{ij} + F_i{}^k N_{kj} + \phi(N_{ij}), \\ N_{ij}(\cdot,0) &= N_0(\cdot)_{ij}, \end{aligned} \tag{7.1}$$

where N_0 is a covariant two tensor satisfying $N_0 \geq 0$, W is a covariant three tensor, and $\phi : \Sigma^2 \to \Sigma^2$ satisfies $\phi(P)_{ij} \geq a_i^{kp} a_j^{lq} P_{kl} P_{pq} - g^{kl} P_{ik} P_{lj}$, where a_i is some smooth symmetric contravariant two tensor for each $i \in \{1, \ldots, n\}$, and f, w, F are smooth tensors (functions) on M whose type is indicated in equation (7.1). Assume also that

$$\sup_{M\times[0,T]} \text{Ricci}(g(t)) \geq -k, \tag{a}$$

$$\sup_{M\times[0,T]} {}^{g}|a| + {}^{g}|\frac{\partial}{\partial t}g| + {}^{g}|N| + |f| + {}^{g}|F| + |W| \leq k, \tag{b}$$

where k is a constant. Then the solution $N(t), t \in [0,T]$, to the equation (7.1) satisfies $N(t) \geq 0$.

Remark. *It is often the case that if in condition (b) we replace $\sup_{M\times[0,T]} |N| \leq k$ with $\sup_M |N_0| \leq k$ then the smoothing properties of flows will ensure that this bound exists for all $t \in [0,T]$.*

Remark. *There is an earlier weaker version of the non-compact maximum principle for tensors in [Ha 6].*

Proof : We prove the result initially for the simpler evolution equation $\frac{\partial}{\partial t}N \geq^g \Delta N$, as the more general case is merely a minor adjustment of this argument.

Step 1. All metrics are equivalent. Condition (b) implies that

$$-kg \geq \frac{\partial}{\partial t}g \geq -kg, \tag{7.2}$$

which implies that all metrics $g(t), t \in [0,T]$ are equivalent: $\frac{1}{c(k,n,T)} g_0 \le g(t) \le c(k,n,T) g_0$. Similarly if $\gamma : [a,b] \to M$ is a smooth curve in M and $l(t)(\gamma)$ is the length of γ with respect to $g(t)$, then

$$\begin{aligned} \tfrac{\partial}{\partial t} l(t)(\gamma) &= \tfrac{\partial}{\partial t} \int_\gamma \sqrt{g(\tfrac{\partial}{\partial s}\gamma, \tfrac{\partial}{\partial s}\gamma)}ds \ge -kl(t)(\gamma) \text{ for all } t \in [0,T], \\ \tfrac{\partial}{\partial t} l(t)(\gamma) &\le kl(t)(\gamma) \text{ for all } t \in [0,T]. \end{aligned} \tag{7.3}$$

We define $\rho(x,t) = dist(g(t))(x,x_0)$, $\rho_0(x) = dist(g_0)(x,x_0)$ where x_0 is some fixed arbitrary point in M. Then $\rho^2(p,t)$ is Lipschitz continuous in t for $p \notin Cut(g(t))(x_0)$, and we get

$$\begin{aligned} & e^{-2kt}\rho_0^2 \le \rho^2 \le \rho_0^2 e^{2kt} \text{ for all } t \in [0,T] \\ & \tfrac{\partial}{\partial t}(\rho^2) \ge -kn\rho^2 \text{ for a.e. } t \in [0,T] \end{aligned} \tag{7.4}$$

in view of (7.3). Without loss of generality we may assume $T = 1$. As many constants appear in this proof, we shall often use a small c to denote a constant depending on k, n. For example it is understood that $5c(k,n) + c^2(k,n)$ may be replaced by $c(k,n)$ without any harm. This implies that $\frac{1}{c^2(k,n)}\rho_0(\cdot) \le \rho(\cdot,t) \le c^2(k,n)\rho_0(\cdot)$.

Step 2. Compactification of the problem. Let

$$\tilde{N}ij(\cdot,t) = N_{ij}(\cdot,t) + \varepsilon E_{ij}(\cdot,t), \tag{7.5}$$

where E is defined by $E_{ij} = e^{b(x,t)}g_{ij}$, where $b(x,t) = (1+\beta t)(1+\rho^2(x,t))$, and where $\beta = \beta(k,n)$ is a constant to be determined later. In view of the fact that $E \ge 0$ we get $\tilde{N}_0 = N_0 + \varepsilon E_0 > 0$. Since $\sup_M |N| \le k$, we have

$$-cg \le N \le cg, \text{ for all } t \in [0,T]. \tag{7.6}$$

Choose $R = R(\varepsilon,k,n)$ so large that $e^{\rho^2} \ge \frac{2c}{\varepsilon}$ for all $x \in M - B(g_0)_R(x_0)$, for all $t \in [0,T]$. Substituting this inequality and (7.6) into the definition of $\tilde{N}$, we get

$$\begin{aligned} \tilde{N}_{ij}(\cdot,t) &= N_{ij}(\cdot,t) + \varepsilon e^{b(x,t)} g_{ij}(\cdot,t) \\ &\ge -cg_{ij} + \varepsilon\frac{2c}{\varepsilon}\rho^2(x,t)g_{ij} \\ &> 0 \text{ for all } t \in [0,T], \text{ for all } x \in M - B_R(g_0)(x_0). \end{aligned}$$

Step 3. Evolution of $\tilde{N}$. From the definition of $\tilde{N}$ we get

$$\begin{aligned} \tfrac{\partial}{\partial t}\tilde{N} &= \tfrac{\partial}{\partial t}N + \varepsilon\beta(1+\rho^2)e^{b(x,t)}g + \varepsilon 2\tfrac{\partial}{\partial t}\rho\rho(1+\beta t)e^{b(x,t)}g + \varepsilon e^{b(x,t)}\tfrac{\partial}{\partial t}g \\ &\ge \tfrac{\partial}{\partial t}N + \varepsilon e^{b(x,t)}(-c+\beta)(1+\rho^2)g, \text{ for a.e. } t \in [0, \tfrac{1}{\beta^2}] \end{aligned} \tag{7.7}$$

in view of (7.2) and (7.4) and the fact that $1+\beta t \leq 2$ for all $t \in [0, \frac{1}{\beta^2}]$. Also from the definition of $\tilde{N}$ we get

$$\begin{aligned} \Delta\tilde{N} &= \Delta N + \varepsilon(\Delta e^{b(x,t)})g \\ &= \Delta N + \varepsilon((1+\beta t)\Delta(1+\rho^2) + (1+\beta t)^2|\nabla\rho^2|^2)g \\ &= \Delta N + \varepsilon e^{b(x,t)}((1+\beta t)2\rho\Delta\rho + 2(1+\beta t)|\nabla\rho|^2 \\ &\quad +(1+\beta t)^2 4\rho^2|\nabla\rho|^2)g. \end{aligned}$$

We use the following facts from Geometry: (i) ${}^g|{}^g\nabla\rho|^2 = 1$ for all $x \in M - Cut(t)(x_0)$, (ii) ${}^g\Delta\rho \leq (n-1)k\frac{(1+\rho^2)}{\rho}$, for all $x \in M - Cut(t)(x_0)$, where (i) is true for any smooth complete Riemannian manifold (M, g), and (ii) is true under the extra assumption that Ricci$(g) \geq -kg$. Substituting (i) and (ii) into the calculation of the Laplacian of $\tilde{N}$ we get

$$\Delta\tilde{N} \leq \Delta N + \varepsilon c(k,n)(1+\rho^2)e^{b(x,t)}g \quad \text{for all } x \in M - Cut(t)(x_0), \text{ for all } t \in [0, \tfrac{1}{\beta^2}]. \tag{7.8}$$

Subtracting (7.8) from (7.7) we get

$$(\frac{\partial}{\partial t} - \Delta)\tilde{N} \geq (\frac{\partial}{\partial t} - \Delta)N + \varepsilon(1+\rho^2)e^{b(x,t)}(-2c+\beta)g.$$

Choose $\beta > 2c$. Assume that at some first time $t_0 > 0$, $\tilde{N}(t_0) \not> 0$. Then due to the compactification of the problem (see step 1), there exists some $p_0 \in B_R(x_0)$ and a vector v_{p_0} such that $N(t_0)(v_{p_0}, v_{p_0}) = 0$. We see that if $p_0 \notin M - Cut(t)(x_0)$ then we may argue as in the proof of the compact maximum principle for tensors ([Ha 1], Theorem 9.1) to obtain a contradiction. If $p_0 \in M - Cut(t)(x_0)$ then we must use a trick of Calabi ([Ca]). Let γ : $[0, s]$ be a geodesic with (respect to the metric $g(t_0)$) going from x_0 to $p_0 \in Cut(t_0)(x_0)$, and let $q = \gamma(r)$ for some very small $r \in (0, s)$. Then q is not a point in the cut locus of x_0 with respect to the metric $g(t)$ for every $t \in [t_0 - \varepsilon', t_0 + \varepsilon']$ for $\varepsilon' > 0$ small. Then we define a new function ${}^q\rho(x,t) = dist(g(t))(x_0, q) + dist(g(t))(q, x)$. Notice that ${}^q\rho(x,t) \geq \rho(x,t)$, in view of the triangle inequality and hence, defining ${}^q\tilde{N} = N + \varepsilon e^{(1+\beta t)(1+{}^q\rho^2)}g$, we get ${}^q\tilde{N} \geq \tilde{N} > 0$ for all $t \in [0, t_0)$, and also ${}^q\tilde{N}(p_0, t_0)(v_{p_0}, v_{p_0}) = 0$ due to the definition of ${}^q\tilde{N}$. Using the same argument we used for $\tilde{N}$, we also get

$$(\frac{\partial}{\partial t} - \Delta){}^q\tilde{N}(\cdot, t) > 0, \text{ for all } t \in [t_0 - \varepsilon', t_0 + \varepsilon'],$$

in a small neighbourhood of $p_0 \in M$ in view of the fact that ${}^q\rho(x,t) - \rho(x,t) \leq 2r$ for t near t_0, and x in a small neighbourhood of $p_0 \in M$, and the fact that

$r \in (0,s)$ was chosen small. But the tensor ${}^q\tilde{N}(\cdot,\cdot)$ is smooth in space and time in a small neighbourhood of $(x_0,t_0) \in M \times [0,T]$, and so we may argue as in the proof of the compact maximum principle for tensors to obtain a contradiction. Hence $\tilde{N}(\cdot,t) > 0$ for all $t \in [0,\frac{1}{b^2}]$, for all $x \in M$. Letting $\varepsilon \to 0$ gives us that $N(\cdot,t) \geq 0$ for all $t \in [0,\frac{1}{b^2}]$, for all $x \in M$. Iterating this argument we obtain $N(\cdot,t) \geq 0$ for all $t \in [0,T]$.

Step. 4 The general case. For the general case we argue as above to obtain

$$\begin{aligned}(\frac{\partial}{\partial t} - {}^{g(t)}\Delta)\tilde{N}_{ij} \geq\ & \varepsilon(\beta - c(k,n))(1+\rho^2)e^{b(x,t)}g_{ij} + \nabla_s N_{ij}(\cdot,t)\cdot W^s \\ & + fN_{ij} + F_i^k N_{jk} + \phi(N)_{ij},\end{aligned}$$

which we then rewrite as

$$\begin{aligned}&(\tfrac{\partial}{\partial t} - {}^{g(t)}\Delta)\tilde{N}_{ij} \\ &= \varepsilon(\beta - c(k,n))(1+\rho^2)e^{b(x,t)}g_{ij} + \nabla_k(N+\varepsilon E)_{ij}W^k + f(N+\varepsilon E)_{ij} \\ &\quad + F_i^k(N+\varepsilon E)_{jk} - g^{kl}(N_{ik}+\varepsilon E_{ik})N_{jl} + a_i^{kl}a_j^{pq}N_{kp}N_{lq} \\ &\quad - \varepsilon\nabla_k E_{ij}W^k - \varepsilon f E_{ij} - \varepsilon E_{ik}N_{jl}g^{kl} - \varepsilon F_i^k E_{jk}, \\ &\geq \varepsilon(\beta - c)(1+\rho^2)e^{b(x,t)}g_{ij} + \nabla_k\tilde{N}_{ij}W^k + f\tilde{N}_{ij} \\ &\quad + F_i^k\tilde{N}_{jk} + a_i^{kl}a_j^{pq}N_{kp}N_{lq}, \text{ for all } t \in [0,\tfrac{1}{\beta^2}],\end{aligned} \tag{7.9}$$

in view of (b) and the definition of $\tilde{N}$ and E. Let v_{p_0} be an arbitrary non-zero vector of length one. Then in orthonormal co-ordinates at p_0 for which g and N are diagonal, we have,

$$(a_i^{kl}a_j^{pq}N_{kp}N_{lq})(v^i,v^j) = a^{kl}(v)a^{pq}(v)N_{kp}N_{lq} = (a^{kl}(v))^2 N_{kk}N_{ll}.$$

For fixed k,l we see that either (case 1) $(a(v)kl)^2 N_{kk}N_{ll} \geq 0$ or (wlog) (case 2) $N_{kk} < 0$, $N_{ll} \geq 0$. In case 2, we get

$$\begin{aligned}(a^{kl}(v))^2 N_{kk}N_{ll} &= (a^{kl}(v))^2(N_{kk}+\varepsilon E_{kk})N_{ll} - \varepsilon(a^{kl}(v))^2 E_{kk}N_{ll} \\ &= (a^{kl}(v))^2\tilde{N}_{kk}N_{ll} - \varepsilon(a^{kl}(v))^2E_{kk}N_{ll} \\ &\geq -\varepsilon(a^{kl}(v))^2E_{kk}N_{ll} \\ &\geq -\varepsilon c(k,n)(1+\rho^2)e^{b(x,t)}g_{ij} \text{ for all } t \in [0,t_0],\end{aligned}$$

in view of (b) and the fact that $\tilde{N} \geq 0$ for all $t \in [0,t_0]$. Taking the sum over all k and l, and substituting this inequality into (7.9) we get

$$(\frac{\partial}{\partial t} - {}^{g(t)}\Delta)\tilde{N}_{ij} \geq \varepsilon(\beta - c)(1+\rho^2)e^{b(x,t)}g_{ij} + \nabla_k\tilde{N}_{ij}W^k + f\tilde{N}_{ij} + F_i^k\tilde{N}_{jk},$$

for all $t \in [0,t_0]$. The result follows using the argument at the end of step 3. ◇

Corollary 7.2. *Let (M^n, g) be a non-compact smooth Riemannian manifold, and $N(t), t \in [0,T]$ be a family of symmetric two tensors on M evolving according to the evolution equation (7.1) in the Theorem above. Assume that N_0 satisfies $N_0 \geq -\varepsilon {g_0}_{ij}$. Assume that all the conditions of the theorem above are satisfied, (except for $N_0 \geq 0$). Then*

$$N_{ij}(t) \geq -2\varepsilon e^{(1+\beta t)} g_{ij}(t), \ \textit{for all } t \in [0,T], \tag{7.10}$$

where $\beta = \beta(n,k)$.

Proof : The argument is essentially contained in the proof above. Fix an arbitrary $x_0 \in M$, and define $\tilde{N} = N + 2\varepsilon E$ as above. Then $\tilde{N}_0 > 0$ since $N_0 + \varepsilon {g_0}_{ij} > 0$. we argue as before to obtain $\tilde{N} > 0$ for all $t \in [0,T]$. In particular at $x = x_0$ where $\rho(x) = 0$ we get (7.10). As x_0 was arbitrary, the proof is finished. ♢

Theorem 7.3. *Let $N(t), t \in [0,T]$ be as in thm. 7.1, with all the conditions of the theorem being satisfied. Assume that in place of condition (a) and (b) we have*

$$\sup_{M\times[0,T]} \text{Ricci}(g(t)) \geq -\frac{k}{\sqrt{t}}, \tag{a'}$$

$$\sup_{M\times[0,T]} \frac{{}^g|a|}{\sqrt{t}} + {}^g|\frac{\partial}{\partial t} g| + {}^g|N| + |f| + |F| + |W| \leq \frac{k}{\sqrt{t}}. \tag{b'}$$

Then the solution $N(t), t \in [0,T]$ satisfies $N(t) \geq 0$.

Proof : The proof is the same as the proof of Theorem 7.1, with some small changes. Wherever in the proof we use (a) or (b) to estimate we must now use (a') and (b'). To compensate, we define a modified E,

$$E_{ij} = e^{b(x,t)} g_{ij}, \text{ where}$$
$$b(x,t) = (1+\beta\sqrt{t})(1+\rho^2(x,t)),$$

where β is as in the theorem above, and set $\tilde{N} = N + \varepsilon E$. Then all the estimates carry through. Note that although at time zero, $\tilde{N}(\cdot,t)$ is not smooth in t, this causes no problems, as $\frac{\partial}{\partial t}\tilde{N}(\cdot,t) \geq 0$ for all $t \in (0, T(\varepsilon)]$ implies $\frac{\partial}{\partial t}\tilde{N}(\cdot,t^2) \geq 0$ for all $t \in (0, T^2(\varepsilon)]$, and $\tilde{N}(\cdot,t^2)$ is smooth. Hence $\tilde{N}(\cdot,t^2) > 0$ for all $t \in [0, T^2(\varepsilon)]$, which implies $\tilde{N}(\cdot,t) \geq 0$ for all $t \in [0, T(\varepsilon)]$. We then argue as before. ♢

Corollary 7.4. *Let $N(t), t \in [0,T]$ be as in Corollary 7.2. Assume that in place of (a),(b) we have (a') and (b'). Then*

$$N(t)_{ij} \geq -2\varepsilon e^{(1+\beta\sqrt{t})} g(t)_{ij}, \ \textit{for all } t \in [0,T],$$

where $\beta = \beta(n,k)$.

Proof : The corollary follows by making the modifications to the proof of corollary 7.2 mentioned in the proof of Theorem 7.3. ◇

Corollary 7.5. *Let* $(M, g(t))$ *be as in Theorem 7.1, and* $\mathcal{R}(t) : \Lambda^2(M) \otimes \Lambda^2(M) \to \mathbf{R}$*, satisfy*

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathcal{R}(\cdot,t) &= {}^{G(t)}\Delta\mathcal{R}(\cdot,t) + \langle\nabla\mathcal{R}(\cdot,t)), W\rangle + f\mathcal{R} + \phi(\mathcal{R}), \\ \mathcal{R}(\cdot,0) &= \mathcal{R}_0(\cdot), \end{aligned}$$

where f, W, ϕ *are as above, and* $\sup_M {}^{G}|\mathcal{R}| \le \frac{k}{\sqrt{t}}$*, where* $G(t)$ *is the operator defined as in (1.8). Then* $\mathcal{R}(\cdot,0)_{\alpha\beta} \ge -\varepsilon G_{0\alpha\beta}$*, implies that*

$$\mathcal{R}(\cdot,t)_{\alpha\beta} \ge -\varepsilon e^{(1+\beta\sqrt{t})} G(t)_{\alpha\beta}, \text{ for all } t \in [0,T],$$

where $\beta = c(n,k)$. ◇

Proof : The proof is as in Corollary 7.2 with some minor changes. In the proof we replace E by $E = e^{b(x,t)}G(x,t)$, where $b(x,t)$ is as in Corollary 7.2. Note that as the metric $G(t)$ is compatible with $g(t)$, ${}^{G(t)}\Delta\rho^2 = {}^{g(t)}\Delta\rho^2$, and so on. ◇

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