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Optimal Curvature Estimates for Homogeneous Ricci Flows

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We prove uniform curvature estimates for homogeneous Ricci flows: For a solution defined on $[0, t]$ the norm of the curvature tensor at time t is bounded by the maximum of $C(n)/t$ and $C(n)(\mathrm{scal}(g(t)) - \mathrm{scal}(g(0)))$. This is used to show that solutions with finite extinction time are Type I, immortal solutions are Type III and ancient solutions are Type I, with constants depending only on the dimension n. A further consequence is that a non-collapsed homogeneous ancient solution on a compact homogeneous space emerges from a unique Einstein metric on that space. The above curvature estimates follow from a gap theorem for Ricci-flatness on homogeneous spaces. This theorem is proved by contradiction, using a local $W^{2,p}$ convergence result which holds without symmetry assumptions.

1 Introduction

The proof of Thurston's geometrization conjecture by Perelman [57–59] using Hamilton's Ricci flow [34] can certainly be considered a major break through. There are however interesting related problems which remain open. For instance, Lott asked in [50], whether the three-dimensional Ricci flow detects the homogeneous pieces in the geometric decomposition proposed by Thurston. In the same article this was proved to be true for immortal solutions, assuming a Type III behavior of the curvature tensor

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and a natural bound on the diameter of the underlying closed oriented manifold. More recently, Bamler showed in a series of articles that for the Ricci flow with surgery there exist only finitely many surgery times, and that the Type III behavior holds after the last surgery time. In many cases, convergence to a geometric piece could be established. We refer to [9] and the articles quoted therein.

Recall that a Ricci flow solution is called *homogeneous*, if it is homogeneous at every time. In dimension 3 homogeneous Ricci flows are well understood: see [20, 31, 36, 39, 49]. For results in higher dimensions we refer to [1, 2, 6, 17, 19, 37, 46, 47, 56], among others. Notice that except for [47], assumptions on the algebraic structure or on the dimension were made.

Our first main result is

Theorem 1. Let $(M^n, g(t))_{t\in[a,b]}$ be a homogeneous Ricci flow solution. Then the norm of the Riemannian curvature tensor at the final time b can be estimated by

$$\|\mathrm{Rm}(g(b))\|_{g(b)} \leq C(n)\cdot \max\left\{\tfrac{1}{b-a}\ ,\ \mathrm{scal}(g(b)) - \mathrm{scal}(g(a))\right\}. \qquad \square$$

Symbols like $c(n), C(n)$, etc. refer to positive constants which depend only on the dimension. A first immediate consequence of the above estimate is that for homogeneous Einstein spaces of a fixed dimension, the Einstein constant controls the norm of the curvature tensor. Notice that this is not true already in the case of cohomogeneity one Einstein spaces with positive Einstein constant [12].

Corollary 2. Let $(M^n, g(t))_{t\in I}$ be a homogeneous Ricci flow solution. Then the following holds: If the solution has finite extinction time T, i.e. $I = [0, T)$, and $\mathrm{scal}(g(0)) = 1$, then there exists $\delta(n) \in (0, 1)$ such that for all $t \in [\delta(n)\cdot T, T)$

$$\|\mathrm{Rm}(g(t))\|_{g(t)}\cdot (T-t) \quad \in \quad [\tfrac{1}{8}, C(n)].$$

If the solution is immortal, i.e. $I = [0,\infty)$, and $\mathrm{scal}(g(0)) = -1$, then for all $t \in I$

$$\|\mathrm{Rm}(g(t))\|_{g(t)}\cdot t \quad \in \quad [0, C(n)].$$

If the solution is ancient, i.e. $I = (-\infty, -1]$, and $\mathrm{scal}(g(-1)) = 1$, then for all $t \in I$

$$\|\mathrm{Rm}(g(t))\|_{g(t)}\cdot |t| \quad \in \quad [c(n), C(n)]. \qquad \square$$

The above assumptions on the scalar curvature can always be achieved: see Remark 4.2. Note also, that in [14] the first two upper curvature bounds were shown, however with constants depending on the initial metric.

Regarding homogeneous solutions with finite extinction time, recall that by [10] a homogeneous space always admits such solutions, if its universal cover is not diffeomorphic to Euclidean space, and that starting from dimension 7 there exists infinitely many homotopy types of simply-connected homogeneous spaces [7]. Notice also, that the homogeneity assumption cannot be dropped in the above corollary, see [26, 33, 38]. Moreover, we show in Lemma 4.3 that there cannot exist a uniform upper bound for $\| \mathrm{Rm}(g(t)) \|_{g(t)} \cdot (T - t)$ for small times: On S^3 there exists a sequence of homogeneous Ricci flows, such that at the initial time the norm of the curvature tensor is one, the scalar curvature is positive, but the extinction times are unbounded.

Using the above corollary, it then follows from [52] and [28] that a homogeneous Ricci flow solution with finite extinction time subconverges, after appropriate scaling, to a non flat homogeneous gradient shrinking soliton. By [61], such a forward limit soliton is a finite quotient of a product of a compact homogeneous Einstein space and a non-compact flat factor, where the latter might be absent.

Next, recall that by [10] and [45] *any* homogeneous Ricci flow on a homogeneous space whose universal cover is diffeomorphic to Euclidean space is immortal and that starting in dimension 3 there are uncountably many homogeneous spaces whose underlying manifold is Euclidean space [11]. For immortal solutions the lower bound 0 in the above estimate is again optimal: already in dimension three, there are examples where the curvature tensor converges exponentially fast to zero [36]. Concerning forward limit non-gradient solitons of immortal homogeneous solutions we refer to [15].

We turn to homogeneous ancient solutions. Notice first, that the homogeneity assumption cannot be dropped in the above corollary in view of [8, 58]. Recall also that homogeneous ancient solutions are called non-collapsed, if the corresponding curvature normalized metrics have a uniform lower injectivity radius bound. In this case, by the above corollary and by [21, 52] these solutions admit a non-flat homogeneous asymptotic soliton as one goes backwards in time.

All known non-compact ancient homogeneous Ricci flow solutions are the Riemannian product of a compact ancient solution and a flat factor. In the compact case our estimates yield the following

Theorem 3. The asymptotic soliton of a non-collapsed, homogeneous ancient solution on a compact homogeneous space is compact and unique. □

In fact, we show that non-collapsed ancient solutions on a compact homogeneous space must emanate from a homogeneous Einstein metric on the same space. Examples of such solutions have been described in [8, 19], both with compact and non-compact

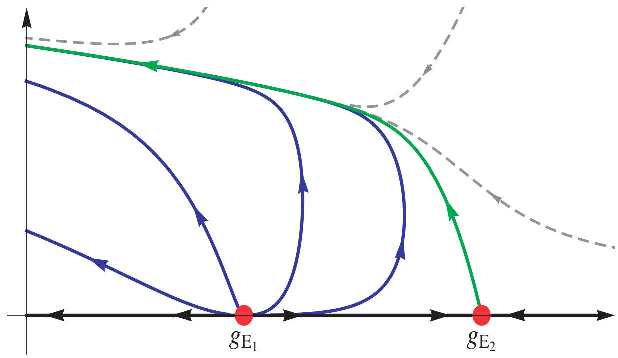

Fig. 1. A family of ancient solutions on M^{12}.

forward limit soliton. Let us also mention that on a compact homogeneous space which is not a homogeneous torus bundle, ancient homogeneous solutions are non-collapsed: see Remark 5.3.

Since along the volume-normalized Ricci flow the scalar curvature is non-decreasing, the Einstein metric from which an ancient solution emanates cannot be a local maximum of the total scalar curvature functional restricted to the space of homogeneous metrics. Conversely, if an Einstein metric is not a local maximum in this sense, there exists an ancient solution emanating from it: see Lemma 5.4.

Next, we turn now to collapsed homogeneous ancient solutions on compact homogeneous spaces. Since they are collapsed, the asymptotic soliton can only exist in the sense of Riemannian groupoids, as introduced by Lott [49]: see Section 6. A nice example is given by the Berger metrics on S^{2n+1}. They have the non-compact asymptotic soliton $\mathbb{CP}^n \times \mathbb{R}$, and the round sphere as a compact forward limit soliton.

The following compact homogeneous space is the first example admitting a collapsed ancient solution with non-compact forward limit soliton. Moreover, this example also shows that the geometry of the asymptotic soliton does not depend continuously on ancient solutions.

Example. There exists a compact homogeneous space M^{12} which admits a one-parameter family of homogeneous ancient solutions with the same asymptotic soliton $(E^{11}, g_{E_1}) \times \mathbb{R}$ and the same forward limit soliton $S^3 \times \mathbb{R}^9$. Moreover, in the closure of these solutions there is a single ancient solution emanating from $(E^{11}, g_{E_2}) \times \mathbb{R}$. □

Here, g_{E_1}, g_{E_2} are non-isometric Einstein metrics on the compact homogeneous space E^{11}. In appropriate coordinates these solutions are depicted in Figure 1. For further details and higher dimensional examples see Section 6.

Our second main result, which is crucial for the proof of Theorem 1, is

Theorem 4 (Gap Theorem). There exists $\epsilon(n) \in (0,1)$ such that for any homogeneous space (M^n, g) the Weyl curvature can be estimated by

$$\|\mathrm{W}(g)\|_g \leq \big(1-\epsilon(n)\big) \cdot \|\mathrm{Rm}(g)\|_g. \qquad \square$$

It follows that a homogeneous Ricci flat space is flat, a result which was proved by Alekseevski and Kimel'fel'd [3] in 1975. But it also shows that a non-flat homogeneous space cannot be "too" Ricci flat. The optimal gap size $\epsilon(n)$ is unknown, but converges to 0 as $n \to \infty$: see Section 7. It is worthwhile mentioning that the Gap Theorem is equivalent to the statement

$$\|\,\mathrm{Rm}(g)\|_g \leq C(n) \cdot \|\,\mathrm{Ric}(g)\|_g.$$

The Gap Theorem is proved by contradiction. We show that a contradiction sequence subconverges locally in $C^{1,\alpha}$-topology to a smooth local limit space, when assuming norm-normalized curvature tensors. Such a local limit is a smooth Ricci flat metric. Since it is also locally homogeneous, it must be flat [65]. On the other hand, as already remarked by Anderson [4], subconvergence can even be assumed in $W^{2,p}$-topology for some $p > n/2$, which yields a positive lower bound for the norm of the curvature tensor.

Since the corresponding curvature estimates might be of independent interest we state them here. We would like to mention, that in the following theorem there are no symmetry or completeness assumptions.

Theorem 5. Given $0 < v \leq V$ and $p \in (n/2, \infty)$, there exists a constant $\varepsilon = \varepsilon(v, V, n, p) > 0$ such that the following holds. Let $(D_i^n, g_i, x_i)_{i\in\mathbb{N}}$ be a sequence of smooth manifolds, such that $B_1^{g_i}(x_i)$ is compactly contained in D_i^n for all $i \in \mathbb{N}$. Assume that $vr^n \leq \mathrm{vol}(B_r^{g_i}(x)) \leq Vr^n$ for all $r \leq 1$, for all $B_r^{g_i}(x) \subseteq B_1^{g_i}(x_i)$, and

$$\lim_{i\to\infty} \int_{B_1^{g_i}(x_i)} \|\,\mathrm{Ric}(g_i)\|^p d\mu_{g_i} = 0 \quad \text{and} \quad \int_{B_1^{g_i}(x_i)} \|\,\mathrm{Rm}(g_i)\|^{n/2} d\mu_{g_i} \leq \varepsilon.$$

Then, for all $s \in (0,1)$, $(B_s^{g_i}(x_i), g_i, x_i)_{i\in\mathbb{N}}$ subconverges in the pointed $W^{2,p}$-topology to a C^∞-smooth limit manifold $(B_s^{g_\infty}(x), g_\infty, x)$, and we have

$$\lim_{i\to\infty} \int_{B_s^{g_i}(x_i)} \|\,\mathrm{Rm}(g_i)\|^p d\mu_{g_i} = \int_{B_s^{g_\infty}(x)} \|\,\mathrm{Rm}(g_\infty)\|^p d\mu_{g_\infty}. \qquad \square$$

At the moment no algebraic proof of the Gap Theorem is known, not even for the fact that Ricci flat homogeneous spaces are flat. Hence we propose the following

Problem. Provide an algebraic proof for the Gap Theorem. □

The article is organized as follows: In Section 2 we prove the Gap Theorem using Theorem 5, whose proof is provided in Section 3. In Section 4 we show how Theorem 1 can be deduced from Theorem 4 and we give the proof of Corollary 2. In Section 5 we prove Theorem 3, and in Section 6 and we will provide examples of homogeneous ancient solutions. Finally, in Section 7 examples of left-invariant metrics on solvable Lie groups are given, which show that the constant $\epsilon(n)$ in the Gap Theorem must converge to zero for $n \to \infty$.

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2 Locally and Globally Homogeneous Spaces

In this section, we define locally homogeneous spaces and show that they are real analytic Riemannian manifolds. Moreover, we provide injectivity radius estimates and a result which shows how to extend local isometries on simply connected domains. All of this is used to prove Theorem 2.6: there, we show that a contradiction sequence to Theorem 4, lifted to the tangent spaces, has to subconverge in the $C^{1,\alpha}$-topology to a Ricci-flat space which is locally homogeneous and thus flat. Then at the end of the section we prove Theorem 4.

A Riemannian manifold (M^n, g) is called globally homogeneous if for all points $p, q \in M^n$ there exists an isometry $f_{p,q}$ of (M^n, g) mapping p to q. In other words, the isometry group acts transitively on M^n. It is a classic result that any globally homogeneous Riemannian manifold is complete.

A Riemannian manifold (M^n, g) is called *locally homogeneous* if for all $p, q \in M^n$ there exists $\varepsilon = \varepsilon_{p,q} > 0$, depending possibly on p and q, such that $B_\varepsilon(p)$ and $B_\varepsilon(q)$ are isometric with the induced metric. Here, $B_\varepsilon(p)$ denotes the open ε-ball around p in (M^n, g). Notice that a locally homogeneous manifold is not necessarily complete. Moreover, recall that there exists (incomplete) locally homogeneous manifolds which are not locally isometric to any globally homogeneous manifold, see [43].

Lemma 2.1. A locally homogeneous space (M^n, g) is a real analytic Riemannian manifold. □

Proof. By [67] local homogeneity of (M^n, g) is equivalent to the existence of an Ambrose-Singer-connection ∇^*, in short AS-connection. An AS-connection is a metric connection, which has parallel torsion and parallel curvature. Now by Theorem 7.7 from Chapter VI in [40] it follows that both M^n and the AS-connection ∇^* are real analytic. It remains to show that the Riemannian metric g is real analytic as well.

Let $x : U \to \mathbb{R}^n$ be a chart from the analytic atlas of M^n, let $V := x(U)$ and let $(e_1, ..., e_n)$ denote the standard basis on $\mathbb{R}^n$. Pulling back the metric $g|_U$ by x^{-1} to V we obtain an metric $\hat{g} = (\hat{g}_{jk})_{1\le j,k\le n}$ on V with $\hat{g}(e_j, e_k) = \hat{g}_{jk}$. Pulling back the connection ∇^* to V we get an analytic connection $\hat{\nabla}^*$ on V. Hence the Christoffel symbols $\hat{\Gamma}^*_{ij,k} : V \to \mathbb{R}$ of $\hat{\nabla}^*$ are real analytic, $1 \le i, j, k \le n$.

Recall that $\hat{\nabla}^*$ is a metric connection, that is for $1 \le i, j, k \le n$ we have

$$e_i\hat{g}(e_j, e_k) = \hat{g}(\hat{\nabla}^*_{e_i} e_j, e_k) + \hat{g}(e_j, \hat{\nabla}^*_{e_i} e_k).$$

Let $v_0 \in V$, $v \in \mathbb{R}^n$ with $\|v\|_{\text{std}} = 1$ and $c_v : (-\varepsilon, \varepsilon) \to V$; $t \mapsto v_0 + t \cdot v$. We set $\hat{g}_{jk}(t) := \hat{g}_{jk}(c_v(t))$ for $1 \le j, k \le n$. Then the vector $\hat{G}(t) = (\hat{g}_{11}(t), ..., \hat{g}_{nn}(t))$ satisfies a linear ordinary differential equation $\hat{G}'(t) = \hat{A}(t) \cdot \hat{G}(t)$, where $\hat{A}(t)$ is real analytic. Now by Theorem 10.1 in [66] the solution $\hat{G}(t)$ is real analytic and defined on the entire interval $(-\varepsilon, \varepsilon)$, that is

$$\hat{g}_{jk}(t) = \sum_{l=0}^{\infty} \frac{\hat{g}^{(l)}_{jk}(0)}{l!} \cdot t^l,$$

for all $1 \le j, k \le n$ and all $t \in (-\varepsilon, \varepsilon)$. A computation shows that $\hat{g}_{jk}(0) = \hat{g}_{jk}(v_0)$, $\hat{g}^{(1)}_{jk}(0) = \langle (\nabla \hat{g}_{jk})_{v_0}, v\rangle$ and

$$\hat{g}^{(2)}_{jk}(0) = \tfrac{d}{dt}|_{t=0} \big\langle (\nabla \hat{g}_{jk}(c_v(t))), v\big\rangle = \big(\mathrm{Hess}(\hat{g}_{jk})\big)_{v_0}(v, v).$$

Inductively we get corresponding formulae for the higher derivatives. This shows that at the points $c_v(t)$ the functions $\hat{g}_{jk}$ can be written as a power series. Since this works for any v with $\|v\|_{\text{std}} = 1$ we deduce that the metric coefficients $\hat{g}_{jk}$ are real analytic. This shows the claim. ■

Next, we consider globally homogeneous spaces (M^n, g) with sectional curvature bound $|K_g| \le 1$ at one and hence any point of M^n. We consider also for a point $p \in M^n$

the Riemannian exponential map $\exp_p : T_pM^n \to M^n$. Since $|K_g| \leq 1$, by the Rauch comparison theorems

$$\exp_p|_{\hat{B}_\pi(0_p)} : \hat{B}_\pi(0_p) \to B_\pi(p)$$

is an immersion. Hence we can pull back the metric $g|_{B_\pi(p)}$ to a metric $\hat{g}$ on $\hat{B}_\pi(0_p) \subset T_pM^n$. The metric $\hat{g}$ is locally homogeneous, clearly incomplete, but still real analytic by Lemma 2.1. Here, we used that the exponential map is analytic: see Proposition 10.5 in [35].

Definition 2.2. We call $(\hat{B}_\pi(0_p), \hat{g})$ a *geometric model* for the globally homogeneous space (M^n, g). □

We mention here that any homogeneous space has an associated *infinitesimal model* which encodes the algebraic data of the space: see [67].

Using once again that $|K_{\hat{g}}| \leq 1$, we see that $\mathrm{inj}_{\hat{g}}(x) \geq i(r) > 0$ for any $x \in \hat{B}_\pi(0_p)$ in view of [23] or [22]. Here $r = d_{\hat{g}}(0_p, x) = \|x\|$ and $i : [0, \pi) \to \mathbb{R}_+$ is an explicit continuous function with $i(r) \leq \pi - r$. Notice that i does not depend on the particular choice of the local model $(\hat{B}_\pi(0_p), \hat{g})$.

For the convenience of the reader we will provide a proof of a much stronger estimate.

Lemma 2.3. If $(\hat{B}_\pi(0_p), \hat{g})$ is a geometric model and $x \in \hat{B}_\pi(0_p)$ with $r = d_{\hat{g}}(0_p, x)$, then $i(r) = \pi - r$. □

Proof. Let $x \in \hat{B}_\pi(0_p)$ be given and suppose that $\varepsilon := \mathrm{inj}_{\hat{g}}(x) < \pi - r$. Using the triangle inequality we see that the closure of $B^{\hat{g}}_\varepsilon(x)$ is a subset of $\hat{B}_\pi(0_p)$. Since $|K_{\hat{g}}| \leq 1$, by Klingenberg's Lemma (cf. [60], p. 182) there exists a geodesic loop c centered at x of length 2ε possibly not closing up smoothly. Since the angle between a Killing field and a geodesic does not change, the loop must close up smoothly. Here we have used that on a simply connected, locally homogeneous space (M^n, g) for each point p there exist Killing vector fields on M^n spanning T_pM^n (see Theorem 1 in [53]).

Now the injectivity radius is a continuous function at x (see Chapter VIII, Theorem 7.3 in [41]). As a consequence, for small positive t there exist closed geodesics c_t centered at $(1-t)\cdot x$ of length $2\varepsilon + \delta(t)$, with $\lim_{t\to 0}\delta(t) = 0$. We claim that in fact $\delta \equiv 0$. To this end, we pick a sequence $(t_i)_{i\in\mathbb{N}}$ converging to zero, such that the closed geodesics c_{t_i} converge to a limit geodesic $\tilde{c}$ of length 2ε. Since the metric $\hat{g}$ is real analytic, on compact sets the length functional has only finitely many critical values by [63]. Consequently for large i the length of c_{t_i} must be 2ε. We can now conclude that also at the center point 0_p the injectivity radius is less than or equal to $\varepsilon < \pi$. This is a contradiction. ■

In the next lemma, we show that local isometries can be extended to balls of uniform size.

Lemma 2.4. Let $(\hat{B}_\pi(0_p), \hat{g})$ be a geometric model, $x, y \in \hat{B}_\pi(0_p)$, and let

$$\delta := \delta(x, y) := \min\{i(\|x\|), i(\|y\|)\}.$$

Then the open distance balls $B^{\hat{g}}_\delta(x)$ and $B^{\hat{g}}_\delta(y)$ are isometric. □

Proof. Since $(\hat{B}_\pi(0_p), \hat{g})$ is locally homogeneous, there exists $\varepsilon > 0$ and an isometry $h : B^{\hat{g}}_\varepsilon(x) \to B^{\hat{g}}_\varepsilon(y)$. We need to show that we can extend this isometry to an isometry $H : B^{\hat{g}}_\delta(x) \to B^{\hat{g}}_\delta(y)$.

Since by Lemma 2.1 the locally homogeneous space $(\hat{B}_\pi(0_p), \hat{g})$ is real analytic, it follows as in the proof of Proposition 10.5 in [35] that both the exponential maps $\exp_x : \hat{B}_\delta(0_x) \to B^{\hat{g}}_\delta(x)$ and $\exp_y : \hat{B}_\delta(0_y) \to B^{\hat{g}}_\delta(y)$ are real analytic diffeomorphisms by Lemma 2.3. We set

$$H : B^{\hat{g}}_\delta(x) \to B^{\hat{g}}_\delta(y) \ ; \ \ z \mapsto \left(\exp_y \circ (\mathrm{D}h)_x \circ \left(\exp_x\right)^{-1}\right)(z).$$

The map H coincides with h on $B^{\hat{g}}_\varepsilon(x)$. Moreover H is analytic. As a consequence, the analytic tensors $\hat{g}|_{B^{\hat{g}}_\delta(x)}$ and $H^*(\hat{g}|_{B^{\hat{g}}_\delta(y)})$ are equal on $B^{\hat{g}}_\varepsilon(x)$. By analyticity they coincide on all of $B^{\hat{g}}_\delta(x)$, hence H is the desired extension. Clearly, H is a diffeomorphism. ■

Remark 2.5. The previous two lemmata imply that there exist positive constants $v_n \leq V_n$ such that

$$v_n r^n \leq \mathrm{vol}(B^{\hat{g}}_r(x)) \leq V_n r^n \tag{1}$$

for all $x \in \hat{B}_\pi(0_p)$ and all $0 < r < \pi - |x|$. □

For a Riemannian metric g we denote by $\mathrm{Ric}(g)$ its Ricci tensor.

Theorem 2.6. Let $(\hat{B}_\pi(0_{p_i}), \hat{g}_i)_{i\in\mathbb{N}}$ be a sequence of locally homogeneous geometric models. Suppose that $\mathrm{Ric}(\hat{g}_i) \to 0$ for $i \to \infty$. Then, there exists a subsequence converging in $C^{1,\alpha}$-topology to a smooth flat limit space $(X, \hat{g})$. □

Proof. Since for geometric models we have $\mathrm{inj}_{\hat{g}_i}(0_{p_i}) = \pi$ and $|K_{\hat{g}_i}| \leq 1$, by the Cheeger-Gromov-compactness theorem we may assume that $(\hat{B}_\pi(0), \hat{g}_i, 0)_{i\in\mathbb{N}}$ converges to a $C^{1,\alpha}$-smooth manifold $(X, \hat{g}, x_0)$ in the pointed $C^{1,\alpha}$ topology. Using $\mathrm{Ric}(\hat{g}_i) \to 0$ as $i \to \infty$ this can be improved: $(X, \hat{g})$ is smooth and satisfies $\mathrm{Ric}(\hat{g}) = 0$ (see [4] and [60], Theorem 5.5). The type of convergence is described for example in Theorem 3.3.

We claim now that this limit space $(X, \hat{g})$ is locally homogeneous. To this end, let $y \neq x_0$ be a point in $B^{\hat{g}}_{\pi-3\delta}(x_0)$, and assume after pulling-back by the corresponding diffeomorphisms that convergence takes place in $B^{\hat{g}}_{\pi-\delta}(x_0)$. Then by Lemma 2.4, for all $i \in \mathbb{N}$ there exists a $\hat{g}_i$-isometry f_i between $B^{\hat{g}_i}_\delta(x_0)$ and $B^{\hat{g}_i}_\delta(y)$. Choose a countable dense subset $S = \{s^k\}_{k\in\mathbb{N}}$ of $B^{\hat{g}}_\delta(x_0) \subseteq X$. The sequence $\big(f_i(s^k)\big)_{i\in\mathbb{N}}$ has a convergent subsequence with limit $f_\infty(s^k) \in B^{\hat{g}}_\delta(y)$. Using a Cantor diagonal procedure, one can choose a subsequence $(f_{i_l})_{l\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$, such that for any $k \in \mathbb{N}$ the sequence $(f_{i_l}(s^k))_{l\in\mathbb{N}}$ converges as l goes to infinity. This defines a one-Lipschitz map $f_\infty : S \to X$, that is, $d_{\hat{g}}(f_\infty(s^j), f_\infty(s^k)) = d_{\hat{g}}(s^j, s^k)$. Such a map can be extended to a distance preserving map from $B^{\hat{g}}_\delta(x_0)$ to $B^{\hat{g}}_\delta(y)$. Clearly, this map is injective. Now, since $(X, \hat{g})$ is smooth, distance preserving maps are smooth as well (see for instance [35, Theorem 11.1]). Hence $f_\infty : B^{\hat{g}}_\delta(x_0) \to B^{\hat{g}}_\delta(y)$ is also surjective, and this shows that these two balls are isometric. Consequently, the limit space $(X, \hat{g})$ is locally homogeneous.

Finally, by [65] locally homogeneous Ricci flat Riemannian manifolds are flat. ■

Proof of the Gap Theorem. As noticed in the introduction, it is sufficient to prove the equivalent estimate $\|\mathrm{Rm}(g)\|_g \leq C(n) \cdot \|\mathrm{Ric}(g)\|_g$. Let V_n, v_n be the volume bounds mentioned in Remark 2.5, $\varepsilon(n) = \varepsilon(n, v_n, V_n) > 0$ and $L(n) = L(n, v_n, V_n)$ be as in Theorem 3.3. W.l.o.g. $\tilde{\varepsilon}(n) := \frac{\varepsilon(n)}{V_n 3^n} \leq 1$.

The proof goes by contradiction. Suppose that there exists a sequence (M^n_i, g_i) of n-dimensional homogeneous spaces with $\|\mathrm{Rm}(g_i)\| = \tilde{\varepsilon}(n)^{2/n} \leq 1$ and $\|\mathrm{Ric}(g_i)\| \to 0$ for $i \to \infty$. Notice that $\|\mathrm{Rm}(g_i)\| \leq 1$ implies $|K_{g_i}| \leq 1$. Hence, by the above discussion every such space has a geometric model $(\hat{B}_\pi(0_{p_i}), \hat{g}_i)$. By Theorem 2.6 we may assume that the sequence $(\hat{B}_\pi(0_{p_i}), \hat{g}_i, 0_{p_i})_{i\in\mathbb{N}}$ converges to a flat limit space $(X, \hat{g}, x_0)$.

On the other hand, this contradicts the estimates in Theorem 3.3 as we will show now. The balls $B^{\hat{g}_i}_3(0)$ are compactly contained in $\hat{B}_\pi(0_{p_i})$ and satisfy the volume estimates of Remark 2.5. Clearly, $\|\mathrm{Ric}(\hat{g}_i)\| \to 0$ for $i \to \infty$. Moreover, by our normalization we have that

$$\int_{B_3^{\hat{g}_i}(0)} \| \operatorname{Rm}(\hat{g}_i)\|^{n/2} d\mu_{\hat{g}_i} = \tilde{\varepsilon}(n) \cdot \operatorname{vol}(B_3^{\hat{g}_i}(0)) \in \left[\varepsilon(n)\tfrac{v_n}{V_n}, \varepsilon(n)\right] \tag{2}$$

in view of the volume estimates. Thus, we can apply Theorem 3.3 and conclude that the convergence to $(X, \hat{g}, x_0)$ is in fact in the $W^{2,p}$-topology, for any $p > n/2$. Using this, together with the volume estimates (1) and the fact that the limit is flat we obtain

$$\begin{aligned} v_n^{1/p} \cdot \tilde{\varepsilon}(n)^{n/2} &= v_n^{1/p} \cdot \| \operatorname{Rm}(\hat{g}_i)\| \\ &\leq \operatorname{vol}(B_1^{\hat{g}_i}(0))^{1/p} \cdot \| \operatorname{Rm}(\hat{g}_i)\| \\ &= \left(\int_{B_1^{\hat{g}_i}(0)} \| \operatorname{Rm}(\hat{g}_i)\|^p \right)^{1/p} \underset{i\to\infty}{\longrightarrow} 0, \end{aligned}$$

a contradiction. ■

3 A Weak Convergence Result

In this section, we will state a convergence result and curvature estimates for Riemannian manifolds (D^n, g) without boundary (possibly incomplete) satisfying certain integral curvature and volume bounds. These results were applied to geometric models of homogeneous spaces in the proof of the Gap Theorem. Notice though that in this section no symmetry assumptions are made on (D^n, g) whatsoever.

We start by defining the $W^{1,p}$ harmonic radius of (D^n, g). Given a chart

$$\psi : V \to \psi(V) = Y \subset \mathbb{R}^n,$$

$V \subset D^n$, we denote by

$$(g_{jk}^{\psi})_{1\leq j,k\leq n} : Y \to \mathbb{R}$$

the coordinate functions of g in the chart ψ. Then we set

$$\|Dg\|_{L^p}(\psi) := \left(\int_{\psi(V)} \sum_{j,k,l=1}^{n} |\partial_l g_{jk}^{\psi}|^p dy \right)^{\frac{1}{p}}, \tag{3}$$

where dy refers to Lebesgue measure on $\mathbb{R}^n$ and $\partial_l g_{jk}^{\psi}$ refers to the standard Euclidean partial derivative in the l-th direction of the function g_{jk}^{ψ}.

Clearly the above norm depends on the choice of the chart. For instance if $\psi(V)$ is the Euclidean standard ball $B_1^{\mathrm{std}}(0)$ of radius one and if $\psi_r := d_r \circ \psi$, $d_r(y) = r \cdot y$ is the dilation by factor $r > 0$, then

$$\|Dg\|_{L^p}(\psi_r) = r^{\frac{n}{p}-3} \cdot \|Dg\|_{L^p}(\psi).$$

We assume now that distance ball $B_1^g(y)$ is compactly embedded in D^n. Furthermore, we assume that there exists constants $0 < v < V$ such that for all $B_r^g(x) \subseteq B_1^g(y)$ and all $0 \le r \le 1$ we have

$$vr^n \le \mathrm{vol}(B_r^g(x)) \le Vr^n. \tag{4}$$

Definition 3.1 (Harmonic radius). Let (D^n, g) be as above and let $0 < p < \infty$. Then the p-harmonic radius $r^g_{\mathrm{har},p}(x)$ at a point $x \in B_1^g(y)$ is the supremum of all $r > 0$ with the following property: There exists a C^∞-smooth chart

$$\psi_r = (\psi_r^1, \dots, \psi_r^n) : V \to B_r^{\mathrm{std}}(0)$$

around x with $\psi_r(x) = 0$, $V \subseteq B_1^g(y)$ with the following properties:

(i) $\frac{1}{2}\delta_{jk} < g_{jk}^{\psi_r} < 2\delta_{jk}$, and $g_{ij}(0) = \delta_{ij}$.
(ii) $r^{1-\frac{n}{p}} \cdot \|Dg\|_{L^p}(\psi_r) < 2$.
(iii) The map ψ_r is harmonic, that is $\Delta_g \psi_r^m = 0$ for all $m \in \{1, \dots, n\}$. □

Nowhere in the definition do we require that $V = \psi_r^{-1}(B_r(0))$ be a geodesic ball. All three conditions are invariant under the simultaneous scaling $(g, r) \mapsto (\lambda^2 \cdot g, \lambda \cdot r)$ for some $\lambda > 0$, where then the ball $B_1^g(y)$ has to be replaced by $B_{1\lambda}^{\lambda^2 \cdot g}(y)$ in the above definition. As a consequence, the harmonic radius scales as a radius.

The proof given here follows essentially the proof given in Appendix B of [64], which, as explained there, essentially follows the proof of Main Lemma 2.2 of [4] (see Remark 2.1 there), using some notions coming from [5].

Theorem 3.2. Let $0 < v \le V$ and $p \in (n/2, \infty)$ be fixed constants. Then there exist $\varepsilon = \varepsilon(v, V, n, p) > 0$ and $L = L(v, V, n, p) > 0$ such that the following holds. Let (D^n, g) be a smooth Riemannian manifold without boundary, $y \in D^n$, such that $B_1^g(y)$ is compactly contained in D^n. Assume that the volume estimates of (4) are satisfied, and

$$\int_{B_1^g(y)} \|\mathrm{Ric}(g)\|^p d\mu_g \le 1, \tag{5}$$

and

$$\int_{B_1^g(y)} \| \operatorname{Rm}(g)\|^{n/2} d\mu_g \leq \epsilon. \tag{6}$$

Then for all $x \in B_s^g(y)$, $s < 1$ we have

$$r^g_{\mathrm{har},2p}(x) \geq L(1-s). \tag{7}$$

□

Proof. We are going to prove that $r^g_{\mathrm{har},2p}(x) \geq L\,\mathrm{dist}_g(x, \partial B_1^g(y))$ for all $x \in B_s^g(y)$. Assume that the result is false. We start with the more difficult case $p \in (n/2, n)$. Then there exists a sequence $(D_i^n, g_i, y_i)_{i\in\mathbb{N}}$ of pointed Riemannian manifolds without boundary, such that $B_1^{g_i}(y_i)$ is compactly contained in (D_i^n, g_i) with the following properties: We have

$$\int_{B_1^{g_i}(y_i)} \| \operatorname{Rm}(g_i)\|^{n/2} d\mu_{g_i} \leq \tfrac{1}{i} \tag{8}$$

and there exist $\tilde{x}_i \in B_1^{g_i}(y_i)$ such that

$$h_i(\tilde{x}_i) = \frac{r^{g_i}_{\mathrm{har},2p}(\tilde{x}_i)}{\mathrm{dist}_{g_i}(\tilde{x}_i, \partial(B_1^{g_i}(y_i)))} \to 0 \quad \text{as} \quad i \to \infty.$$

Since $r^{g_i}_{\mathrm{har},2p}$ is lower semi-continuous on $\overline{B_1^{g_i}(y_i)}$ $(2p > n)$, there exist points $x_i \in B_1^{g_i}(y_i)$ with $h_i(x_i) \leq h_i(x)$ for all $x \in B_1^{g_i}(y_i)$. Clearly $\lim_{i\to\infty} h_i(x_i) = 0$. This in turn implies $\lim_{i\to\infty} r^{g_i}_{\mathrm{har},2p}(x_i) \to 0$. We deduce that

$$\mu_i := \frac{1}{r^{g_i}_{\mathrm{har},2p}(x_i)} \to \infty \quad \text{as} \quad i \to \infty.$$

Next, we consider the rescaled metrics $\tilde{g}_i := \mu_i^2 \cdot g_i$. Notice that the functions h_i, defined above, are invariant under such a scaling, if we replace the ball of radius one by the balls $B^{\tilde{g}_i}_{\mu_i}(y_i)$ in the definition of harmonic radius.

We also have $r^{\tilde{g}_i}_{\mathrm{har},2p}(x_i) = 1$,

$$\int_{B^{\tilde{g}_i}_{\mu_i}(y_i)} \| \operatorname{Ric}(\tilde{g}_i)\|^{p} d\mu_{\tilde{g}_i} \to 0 \quad \text{as} \quad i \to \infty, \tag{9}$$

and

$$\int_{B^{\tilde{g}_i}_{\mu_i}(y_i)} \| \operatorname{Rm}(\tilde{g}_i)\|^{n/2} d\mu_{\tilde{g}_i} \to 0 \quad \text{as} \quad i \to \infty, \tag{10}$$

since $p > \frac{n}{2}$ and the inequality (8) holds; notice that the integral in (10) is invariant under scaling of the metric. Hence by the above mentioned scale invariance of the functions h_i and the choice of points x_i we have for all $x \in B^{\tilde{g}_i}_{\mu_i}(y_i)$

$$\frac{r^{\tilde{g}_i}_{\mathrm{har},2p}(x)}{\mathrm{dist}_{\tilde{g}_i}(x,\partial(B^{\tilde{g}_i}_{\mu_i}(y_i)))} \geq \frac{r^{\tilde{g}_i}_{\mathrm{har},2p}(x_i)}{\mathrm{dist}_{\tilde{g}_i}(x_i,\partial(B^{\tilde{g}_i}_{\mu_i}(y_i)))} = \frac{1}{\mathrm{dist}_{\tilde{g}_i}(x_i,\partial(B^{\tilde{g}_i}_{\mu_i}(y_i)))} \to 0 \quad \text{as} \quad i \to \infty.$$

Next, as in [64] a simple triangle inequality estimate shows that for any $\rho > 0$ and any $x \in B^{\tilde{g}_i}_{\rho}(x_i)$ for $i \geq N(\rho)$ large enough we have

$$r^{\tilde{g}_i}_{\mathrm{har},2p}(x) \geq \tfrac{1}{2}.$$

For ease of reading, we remove the tildes from $\tilde{g}_i$ and write again g_i.

We set $r_0 := \frac{1}{100}$. Using the volume estimates and (i) we find harmonic coordinate charts $\psi^s_i : U^s_i \to B^{\mathrm{std}}_{r_0}(0)$ with $\psi^s_i(x^s_i) = 0$, $s = 1, \ldots, N = N(v, V, \rho, n)$, such that the sets $(U^s_i)^N_{s=1}$ cover $B^{g_i}_{\rho}(x_i)$, and their intersection number is bounded from above by $Z(v, V, n)$. We write

$$\varphi^s_i : B^{\mathrm{std}}_{r_0}(0) \to U^s_i;\ y \mapsto (\psi^s_i)^{-1}(y)$$

and call these maps charts also. We proceed as in [64] (cf. [60]): There exists a limit space (X, d_X, x_∞) of the sequence $(D^n_i, d(g_i), x_i)$ in pointed Gromov-Hausdorff-topology by Theorem 7.4.15 in [18] in view of the volume estimates. Arguing exactly as in [60] after the proof of Fact 4 and at the beginning of Fact 5, we see first that X is a C^0 manifold, with coordinate charts $\varphi_r : B^{\mathrm{std}}_{r_0}(0) \to V_r$, and their construction implies the following: if $\varphi_t(B^{\mathrm{std}}_{2\epsilon}(v)) \subseteq V_r \cap V_t$, then the maps $(\alpha_i)^t_r := (\varphi^r_i)^{-1} \circ \varphi^t_i : B^{\mathrm{std}}_{\epsilon}(v) \to \mathbb{R}^n$ subconverge with respect to the C^0 norm to the maps $\alpha^t_r := (\varphi^r)^{-1} \circ \varphi^t : B^{\mathrm{std}}_{\epsilon}(v) \to \mathbb{R}^n$ as $i \to \infty$.

Next, we show that the limit space X is a $C^{2,\beta}$-manifold. We have to prove that the transition functions

$$(\alpha_i)^s_{\tilde{s}} := (\varphi^{\tilde{s}}_i)^{-1} \circ \varphi^s_i$$

have a convergent subsequence in $C^{2,\beta}$. We sketch the argument.

We can assume, choosing i large enough, that $(\alpha_i)^s_{\tilde{s}}$ is defined on a small ball $B^{\mathrm{std}}_{2\delta}(z) \subseteq B^{\mathrm{std}}_{r_0}(0)$, independent of i. The indices s and $\tilde{s}$ are fixed for the moment. By assumption (i) and (ii) we have $W^{1,2p}(B^{\mathrm{std}}_{r_0}(0))$-bounds for $g^{\psi^s_i}_{jk}$, with $2p > n$. By Morrey's Embedding theorem (see Theorem 7.17 in [30]) we obtain $C^{\alpha}(B^{\mathrm{std}}_{r_0}(0))$-bounds for $g^{\psi^s_i}_{jk}$ for some $0 < \alpha < 1$ and by the Arzela-Ascoli-Theorem we obtain, after taking a subsequence,

$g^{\psi_i^s} \to h^s$ in $C^\alpha(B^{\text{std}}_{r_0}(0))$ as $i \to \infty$ for some Riemannian metric $h^s \in C^\alpha(B^{\text{std}}_{r_0}(0))$: we use the fact that $\frac{1}{2}\delta \le g^{\psi_i^s} \le 2\delta$ freely, sometimes without explicit mention, where δ denotes the standard metric on $\mathbb{R}^n$. It is well known, see remark B.2 in [64], that the transition functions for harmonic coordinates satisfy

$$\sum_{j,k=1}^{n} g^{jk}_{\psi_i^s} \cdot \partial_j \partial_k ((\alpha_i)^s_{\tilde{s}})^m = 0$$

for all $m = 1, \ldots, n$ on $B^{\text{std}}_{2\delta}(z)$.

Now by Schauder theory for the above elliptic differential equation one obtains bounds for $((\alpha_i)^s_{\tilde{s}})^m$ in $C^{2,\alpha}(B^{\text{std}}_{\delta}(z))$. By the Arzela-Ascoli-Theorem we obtain a subsequence, which converges to the limit transition function $(\alpha)^s_{\tilde{s}} = (\varphi^{\tilde{s}})^{-1} \circ \varphi^s$ in $C^{2,\beta}(B^{\text{std}}_{\delta}(z))$-topology for $\beta < \alpha$. Hence X is a $C^{2,\beta}$ smooth manifold. Also, pushing the metric h^s back to X using φ^s , we obtain a well defined C^α metric h on X.

The bulk of the rest of the argument is devoted to showing that the limit manifold is a smooth flat Riemannian manifold (X, h) having Euclidean volume growth and hence is isometric to standard Euclidean space. This is used to obtain a contradiction to the fact that the harmonic radii of the approximating sequence is bounded from above.

We sketch the proof given in [64]. We fix an s and the maps $\varphi^s : B^{\text{std}}_{r_0}(0) \to V_s$, $\varphi^s_i : B^{\text{std}}_{r_0}(0) \to V_s$ associated to this s. By (i) and (ii) in the definition of harmonic radius we have $W^{1,2p}(B^{\text{std}}_{r_0}(0))$-bounds for the metric coefficients $g^s_i = g^{\psi_i^s}$ where $2p \in (n, 2n)$. Of course this gives bounds in $W^{1,q}(B^{\text{std}}_{r_0}(0))$ for all $q < n$, since $2p > n$. Also, g^s_i subconverges to h^s in $L^q(B^{\text{std}}_{r_0}(0))$ for all $q \in (0, \infty)$ as $i \to \infty$, since g^s_i subconverges to h^s in $C^\alpha(B^{\text{std}}_{r_0}(0))$. It is well known, that in harmonic coordinates one has

$$g^{ab}\partial_a\partial_b g_{jk} = (g^{-1} * g^{-1} * Dg * Dg)_{jk} - 2\,\text{Ric}(g)_{jk}\,, \tag{11}$$

see the reference in [64]. Remembering that s is fixed, we now use the notation $g_i = g^s_i$. We know by (9) that $\text{Ric}(g_i)$ converges to zero in $L^p(B^{\text{std}}_{r_0}(0))$ as $i \to \infty$, and by Hölder's inequality the other terms are bounded in $L^p(B^{\text{std}}_{r_0}(0))$ in view of the fact that ∂g is bounded in L^{2p}, and g, g^{-1} are bounded by 2δ. Now by the L^p-theory we obtain bounds for g_i in $W^{2,p}(K)$ ([30], Theorem 9.11) for any compact $K \subseteq B^{\text{std}}_{r_0}(0)$, and hence by the Rellich–Kondrachov-Embedding theorem, g_i converges to h strongly in $W^{1,q}(K)$, for any $q < p^*$ where $p^* = \frac{np}{n-p} > 2p$. Note that $q = 2p$ is a valid choice here.

Next we are going to show that $(g_i)_{i\in\mathbb{N}}$ is a Cauchy-sequence in $W^{2,p}(K)$, which gives us $g_i \to h^s$ in $W^{2,p}(K)$. For simplicity we write $g = g_i$ and $\tilde{g} = g_{\tilde{i}}$, $h = h^s$ in the next paragraph: s is still fixed. We set

$$L_{jk} := (h^{ab} - g^{ab})\partial_a\partial_b g_{jk} - (h^{ab} - \tilde{g}^{ab})\partial_a\partial_b \tilde{g}_{jk} - 2\,\mathrm{Ric}(g)_{jk} + 2\,\mathrm{Ric}(\tilde{g})_{jk} \\ + (g^{-1} * g^{-1} * Dg * Dg)_{jk} - (\tilde{g}^{-1} * \tilde{g}^{-1} * D\tilde{g} * D\tilde{g})_{jk}. \tag{12}$$

Then we have

$$L_{jk} = h^{ab}\partial_a\partial_b(g - \tilde{g})_{jk}. \tag{13}$$

We are going to show that L_{jk} becomes as small in $L^p(K)$ as we like for $i, \tilde{i}$ large enough. The first term on the right of (12) may be estimated as follows

$$\int_K |(h^{ab} - g^{ab})\partial_a\partial_b g_{jk}|^p \leq C(n) \cdot \sup_K |h - g| \int_K |\partial^2 g|^p.$$

The integrals are taken with respect to the standard Lebesgue measure. Since g_i converges to h in $C^\alpha(K)$ and $g = g_i$ is bounded in $W^{2,p}(K)$, as we showed above, we see that this term is as small as we like in $L^p(K)$, as long as i is large enough. The second term can be estimated in a similar fashion. The third and the fourth term converge by (9) in $L^p(K)$ to zero. The last two terms are dealt with as follows: $Dg_i \to Dh$ and $Dg_{\tilde{i}} \to Dh$ as $i, \tilde{i} \to \infty$, in L^{2p}, as we explained above, and hence $(g^{-1} * g^{-1} * Dg * Dg)_{jk}$ and $(\tilde{g}^{-1} * \tilde{g}^{-1} * D\tilde{g} * D\tilde{g})_{jk}$ converge to $(h^{-1} * h^{-1} * Dh * Dh)_{jk}$ in $L^p(K)$, which implies that the sum of the last two terms of (12) converges to zero in $L^p(K)$.

We deduce from (13) and (12) that

$$h^{ab}\partial_a\partial_b(g_i - g_{\tilde{i}})_{jk} = f(i, \tilde{i})_{jk}$$

with

$$\int_K |f(i, \tilde{i})|^p dy \leq \varepsilon(i, \tilde{i})$$

and $\varepsilon(i, \tilde{i}) \leq \varepsilon$ for any $\varepsilon > 0$, if $i, \tilde{i} \geq N(\varepsilon)$ are large enough. Using again L^p-theory for elliptic operators we deduce by Theorem 9.11 in [30] that

$$\|g_i - g_{\tilde{i}}\|_{W^{2,p}(\tilde{K})} \leq C_K \cdot \left(\|f(i, \tilde{i})\|_{L^p(K)} + \|g_i - g_{\tilde{i}}\|_{L^p(K)}\right).$$

on any smooth compact subset $\tilde{K} \subset\subset K \subseteq B^{\mathrm{std}}_{r_0}(0)$.

This clearly implies that $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $W^{2,p}(K)$ for compact subsets $K \subseteq B^{\mathrm{std}}_{r_0}(0)$, and consequently $g_i \to h$ for $i \to \infty$ in $W^{2,p}(K) \cap W^{1,q}(K)$, for any $q \in [2p, p^*)$, in particular we can choose $q = 2p$ here.

We come now back to the identity (11): we can take the limit in $L^p(K)$ and deduce

$$h^{ab}\partial_a\partial_b(h)_{jk} = -(h^{-1} * h^{-1} * Dh * Dh)_{jk},$$

where the right hand of this equation is in $L^r(K)$ with $r = \frac{n}{2} + \sigma$ for some $\frac{n}{2} > \sigma > 0$, and hence $h \in W^{2,r}(K)$ by the L^r theory. The Sobolev Embedding Theorem tells us that $h \in W^{1,\frac{rn}{n-r}}(K)$, and we note that $\frac{rn}{n-r} = \frac{(n/2+\sigma)n}{(n/2-\sigma)} = n + 2\sigma \cdot (\frac{n}{n/2-\sigma}) \geq n + 4\sigma$, and hence the right hand side of the above equation is bounded in $L^{\frac{n}{2}+2\sigma}$.

Iterating this argument, we get $h \in W^{2,r}$ for all $r \in [1,\infty)$: we first choose $\sigma > 0$ very small, and $N \in \mathbb{N}$ large so that $\frac{n}{2} + (N-1)\sigma$ is as close as we like, but less than n. Then the Sobolev Embedding Theorem in the N'th iteration gives us that the right hand side is in $L^r(K)$. Now using the L^2 and the L^r theory, we conclude, as in [64], that h is $C^\infty(K)$.

At this stage we know that the limit manifold X is a $C^{2,\beta}$-manifold, but that the metric h is C^∞-smooth in our constructed harmonic coordinates. Using a similar but simpler argument to the one used above to show that h is C^∞, we see that the transition functions on X, and hence X, is C^∞: see the argument given in [64] for example.

Next, we show that the limit space (X,h) is flat. Let s be fixed and the map $\varphi_i^s : B_{r_0}^{\mathrm{std}}(0) \to U_i^s \subseteq X_i$ and $\varphi^s : B_{r_0}^{\mathrm{std}}(0) \to U^s$ in X be as above. Since the metrics $g_i = g_i^s$ converge to h^s in $W^{2,p}(K) \cap W^{1,2p}(K) \cap C^{0,\alpha}(K)$ in these coordinates, for fixed $s \in \{1,\dots,N\}$ we have writing $h = h_s$, abusing notation slightly,

$$\begin{aligned}
&\int_K \|\operatorname{Rm}(h)\|^p \, d\mu_h = \\
&\quad = \int_K \|h^{-1} * h^{-1} * D^2 h + h^{-1} * h^{-1} * Dh * Dh\|^p \, d\mu_h \\
&\quad = \lim_{i\to\infty} \int_K \|(g_i^s)^{-1} * (g_i^s)^{-1} * D^2 g_i^s + (g_i^s)^{-1} * (g_i^s)^{-1} * Dg_i^s * Dg_i^s\|^p \, d\mu_{g_i^s} \\
&\quad \leq \lim_{i\to\infty} \int_{B_{r_0}^{\mathrm{std}}(0)} \|\operatorname{Rm}(g_i)\|^p \, d\mu_{g_i} \overset{(10)}{=} 0.
\end{aligned}$$

This shows that the limit space (X,h) is flat. (X,h) has Euclidean volume growth since the $(B_\rho^{g_i}(x_i), g_i)$'s also do. This implies that (X,h) is isometric to $(\mathbb{R}^n,\delta)$.

As in the proof of the Main Lemma 2.2 in [4] one obtains now a contradiction: see [64] for more details.

The case $p \geq n$ is similar but easier. Arguing as above, we get (locally) $g_i \in W^{2,p}(K)$ and hence $\partial g \to \partial h$ in $L^s(K)$ for all $s \in [1,\infty)$, in view of the Rellich-Kondrachov embedding Theorem, and hence, arguing as above, $g_i \to h$ in $W^{2,p}(K) \cap W^{1,p}(K)$ for all smooth compact sets $K \subseteq B^{\mathrm{std}}_{r_0}(0)$. The rest of the argument is the same. ■

Theorem 3.3. Let $0 < v \leq V$ and $p \in (\frac{n}{2}, \infty)$ be fixed and $\varepsilon = \varepsilon(v, V, n, p) > 0$ and $L = L(v, V, n, p)$ be the constants from Theorem 3.2 above. Let $(D^n_i, g_i, y_i)_{i \in \mathbb{N}}$ be a sequence of pointed smooth Riemannian manifolds without boundary such that $B^{g_i}_1(y_i)$ is compactly contained in D^n_i for all $i \in \mathbb{N}$, $y_i \in D^n_i$. Assume that

$$vr^n \leq \mathrm{vol}(B^{g_i}_r(x)) \leq Vr^n.$$

for all $r \leq 1$, for all $B^{g_i}_r(x) \subseteq B^{g_i}_1(y_i)$, and

$$\int_{B^{g_i}_1(y_i)} \| \mathrm{Rm}(g_i) \|^{n/2} d\mu_{g_i} \leq \varepsilon,$$

and

$$\int_{B^{g_i}_1(y_i)} \| \mathrm{Ric}(g_i) \|^{p} d\mu_{g_i} \to 0 \quad \text{as} \quad i \to \infty$$

for some $p \in (\frac{n}{2}, \infty)$. Then for all $z \in B^{g_i}_s(y_i)$, $s < 1$, the $2p$-harmonic radius is bigger than $L(1-s)$. Furthermore, we find a smooth Ricci flat limit space (X, g, x_0) in the following sense. For all $s < 1$ $B^g_s(x_0)$ is compactly contained in X and there exist smooth diffeomorphisms $F_i : B^g_s(x_0) \to (F_i(B^{g_i}_s(y_i)) \subseteq B^{g_i}_1(y_i)$ with $F_i(x_0) = y_i$, such that $(F_i)^*(g_i) \to g$ in $W^{2,p}(B^g_s(x_0)) \cap W^{1,2p}(B^g_s(x_0))$, after taking a subsequence. Also the map $(F_i)^{-1} : (B^{g_i}_\sigma(m_i), d_{g_i}) \to (B^g_{2\sigma}(m), d_g)$, for any m_i with $F_i^{-1}(m_i) = m$ is an $\epsilon(i)$ Gromov–Hausdorff approximation, $\epsilon(i) \to 0$ as $i \to \infty$, if $B^g_{8\sigma}(m) \subseteq B^g_1(s)(x_0)$: distance converges with respect to this map and its inverse, as $i \to \infty$. In particular, we have

$$\lim_{i \to \infty} \int_{B^{g_i}_s(x)} \| \mathrm{Rm}(g_i) \|^p d\mu_{g_i} = \int_{B^g_s(x)} \| \mathrm{Rm}(g) \|^p d\mu_g ,$$

for all $s < 1$. □

Proof. By assumption we may apply Theorem 3.2 to all the metrics g_i, showing that the $2p$-harmonic radius on $B_1(y_i)$ is bounded uniformly from below by $L(1-s)$, where L is without loss of generality less than $1/100$. In these coordinates we have to establish the above claimed subconvergence.

Now after finding the contradiction subsequence g_i in the proof of Theorem 3.2 we only used the estimate (9) to construct the limit metric, which is then smooth. The estimate (6) was only used to prove the flatness of the limit metric. Hence we may proceed precisely as above, replacing the g_i's of the proof of Theorem 3.2 by the g_i's given in the statement of the Theorem here, and we deduce convergence of the sequence $(g_i)_{i\in\mathbb{N}}$ locally (in the harmonic coordinates from above) in the $W^{2,p}\cap W^{1,2p}$-topology. The diffeomorphisms F_i are constructed using the transition functions, which we know converge locally in $C^{2,\alpha}$, as explained in the proof of Theorem 3.2 above. This, combined with the fact that $(g_i)_{i\in\mathbb{N}}$ converges to h in $W^{2,p}\cap W^{1,2p}$ in the harmonic coordinates from above, implies that $(F_i)^*(g_i) \to g$ in $W^{2,p}(B^g_{1-2\delta}(x_0)) \cap W^{1,2p}(B^g_{1-2\delta}(x_0))$ as $i\to\infty$: see for example Appendix B in [64] for details on the construction of the diffeomorphisms F_i appearing in the statement of this Theorem, and why the previous statement is true. ■

4 Curvature Estimates

In this section, we discuss the applications of the Gap Theorem to the study of homogeneous Ricci flows. The proofs of Theorem 1 and Corollary 2 are presented. Moreover, we show that for homogeneous Ricci flows the Ricci curvature satisfies a *time doubling property*, similar to that of the full curvature tensor.

A Ricci flow solution is called homogeneous, if it is homogeneous at any time. Notice, that it is sufficient to assume homogeneity at the initial time: Homogeneity implies bounded curvature, hence for such initial metrics there exists a unique solution with bounded curvature. As is well-known it then follows that isometries of the initial metrics will also be isometries for all evolved metrics. Moreover, by [42] the isometry group does not change over time.

By the Gap Theorem there exists $C(n) > 0$, such that every n-dimensional homogeneous manifold (M^n, g) satisfies

$$\| \mathrm{Rm}(g)\| \leq C(n)\cdot \| \mathrm{Ric}(g)\|.$$

Theorem 1 follows now directly from

Theorem 4.1. If $(M^n, g(t))_{t\in[a,b]}$ is a homogeneous Ricci flow solution then

$$\| \mathrm{Rm}(g(b))\| \leq \max\left\{\tfrac{1}{8(b-a)},\ 16\cdot C(n)^2\cdot\big(\mathrm{scal}(g(b)) - \mathrm{scal}(g(a))\big)\right\}. \tag{14}$$

□

Proof. Let $K := \| \operatorname{Rm}(g(b))\|$. If $\frac{1}{8K} \geq b-a$ then $K \leq (8(b-a))^{-1}$ and the claim follows. We are left with the case $\frac{1}{8K} \leq b-a$. The above bound implies that

$$\begin{aligned}\int_a^b \| \operatorname{Rm}(g(t))\|^2 dt &\leq \tfrac{C(n)^2}{2} \int_a^b 2 \, \| \operatorname{Ric}(g(t))\|^2 dt \\ &= \tfrac{C(n)^2}{2} \int_a^b \operatorname{scal}(g(t))' dt \\ &= \tfrac{C(n)^2}{2} \big(\operatorname{scal}(g(b)) - \operatorname{scal}(g(a)) \big).\end{aligned}$$

Now if $t \in [b - \frac{1}{8K}, b]$, then by the doubling time estimate we have $\| \operatorname{Rm}(g(t))\| \geq \frac{1}{2} \cdot \| \operatorname{Rm}(g(b))\|$. Using that $a \leq b - \frac{1}{8K}$, we deduce that

$$\int_a^b \| \operatorname{Rm}(g(t))\|^2 dt \;\geq\; \int_{b-\frac{1}{8K}}^b \| \operatorname{Rm}(g(t))\|^2 dt \;\geq\; \tfrac{K^2}{4} \cdot \tfrac{1}{8K} = \tfrac{K}{32},$$

and the theorem follows. ■

Recall that along a homogeneous Ricci flow $(M^n, g(t))_{t\in[a,b]}$ the scalar curvature $s(t) := \operatorname{scal}(g(t))$ is a constant function on the manifold, hence

$$s(t)' = 2 \cdot \|\operatorname{Ric}(g(t))\|^2 \;\geq\; \tfrac{2}{n} \cdot s(t)^2.$$

If $s(t)$ does not vanish for $t \in [a, b]$ then by integrating we get

$$-\tfrac{1}{s(b)} + \tfrac{1}{s(a)} \geq \tfrac{2}{n} \cdot (b-a),$$

Assuming that $s(a) > 0$, which implies $s(b) > 0$, one gets that

$$s(a) \leq \tfrac{n}{2} \cdot (b-a)^{-1}, \tag{15}$$

and on the other hand if $s(b) < 0$ then also $s(a) < 0$ and one obtains

$$|s(b)| \leq \tfrac{n}{2} \cdot (b-a)^{-1}. \tag{16}$$

If in addition there exists a constant $C_1(n) > 0$ such that

$$s(t)' \;\leq\; C_1(n) \cdot s(t)^2 \tag{17}$$

then one also gets the reversed inequality

$$-\tfrac{1}{s(b)} + \tfrac{1}{s(a)} \leq C_1(n) \cdot (b-a). \tag{18}$$

Proof of Corollary 2. Let $(M^n, g(t)_{t\in[0,T)})$ be a homogeneous Ricci flow solution with finite extinction time $T < \infty$ and $\mathrm{scal}(g(0)) = 1$. Taking $a = 0, b = t$ in (14) yields

$$\| \mathrm{Rm}(g(t))\| \leq \max\left\{\tfrac{1}{8t}\,,\ 16 \cdot C(n)^2 \cdot \big(\mathrm{scal}(g(t)) - 1\big)\right\}.$$

Observe that the first expression on the right hand side is decreasing, whereas the second is increasing and diverges to $+\infty$ as $t \to T$ since the curvature blows up at T. Therefore there exists a unique $t_0 \in (0, T)$ such that

$$\tfrac{1}{8\,t} \leq 16 \cdot C(n)^2 \cdot (\mathrm{scal}(g(t)) - 1)$$

for all $t \in [t_0, T)$ with equality at t_0. Hence, on $[t_0, T)$ we have the estimate

$$\| \mathrm{Rm}(g(t))\| \leq C_2(n) \cdot \mathrm{scal}(g(t)), \tag{19}$$

The upper bound stated in the first statement of Corollary 2 now follows by taking $a = t > t_0, b \to T$ in (15). The lower bound is well-known and follows immediately from the doubling time estimate: see Lemma 6.1 in [25].

It remains to prove an upper bound for t_0 in terms of T. First notice that since $\mathrm{scal}(g(0)) = 1$, from (15) for $a = 0$, $b \to T$ we have that $t_0 < T \leq n/2$, which in turn gives a uniform upper bound

$$\mathrm{scal}(g(t_0)) \cdot t_0 \leq C_3(n)$$

by the very definition of t_0. On the other hand, the estimate (19) gives an upper bound for scal' as in (17). Thus from (18) for $a = t_0$, $b \to T$ we get

$$\tfrac{t_0}{C_3(n)} \leq \tfrac{1}{\mathrm{scal}(g(t_0))} \leq C_1(n) \cdot (T - t_0)\,,$$

and this implies that $t_0 \leq \delta(n) \cdot T$ for some uniform $\delta(n) \in (0, 1)$.

If the solution is immortal, i.e. defined for all $t \in [0, \infty)$, we take $a = t, b = 2t$ in (14) and obtain

$$\| \mathrm{Rm}(g(t))\| \leq \max\left\{\tfrac{1}{8t}, -16 \cdot C(n)^2 \cdot \mathrm{scal}(g(t))\right\}.$$

Taking $a = 0, b = t$ in (16) yields the desired estimate.

If the solution is ancient, i.e. defined for $t \in (-\infty, -1]$, we take $a \to -\infty, b = t < -1$ in (14). This gives us

$$\| \mathrm{Rm}(g(t))\| \leq C_2(n) \cdot \mathrm{scal}(g(t))\,, \tag{20}$$

since $\operatorname{scal}(g(a)) \to 0$ as $a \to -\infty$ by (15). The upper bound now follows by taking $b = -1, a = t \leq -1$ in (15). To prove the lower bound, notice that from (20) we immediately get an upper bound for the evolution of scalar curvature as in (17). Thus we can apply (18) for $b = -1$, $a = t < -1$, and this finishes the proof. ∎

Remark 4.2. The assumptions on the scalar curvature in Corollary 2 are not restrictive for non-flat solutions. Indeed, for the finite extinction time case, it follows from [45], and also from Theorem 4.1, that the scalar curvature blows up at the extinction time. Regarding immortal solutions, it is well-known that the scalar curvature cannot be positive, since this would imply finite extinction time. And if it vanishes, the solution is Ricci flat and hence flat. One can argue analogously for ancient solutions. In each case one can then scale the initial metric so that the assumptions are satisfied. □

The following example shows, that even along homogeneous Ricci flow solutions with positive scalar curvature the norm of the curvature tensor can decrease tremendously.

Lemma 4.3. On S^3 there exist a sequence $((g_l(t))_{t\in[0,T(l))})_{l\in\mathbb{N}}$ of homogeneous Ricci flow solutions with $\|\operatorname{Rm}(g_l(0))\| = 1$, $\operatorname{scal}(g_l(0)) > 0$ and $T(l) \to \infty$ for $l \to \infty$. □

Proof. On $S^3 = \mathrm{SU}(2)$ left-invariant metrics are in one-to-one correspondence with scalar product on the Lie algebra $\mathfrak{su}(2) = T_e\mathrm{SU}(2)$. Let $Q(X, Y) = \frac{1}{2}\operatorname{tr}(X \cdot Y^*)$ and consider for $x_1, x_2, x_3 > 0$ the left invariant metrics $g = g(x_1, x_2, x_3)$, given by

$$g = x_1 \cdot Q|_{\mathfrak{m}_1} \perp x_2 \cdot Q|_{\mathfrak{m}_2} \perp x_3 \cdot Q|_{\mathfrak{m}_3}.$$

Here $\mathfrak{m}_1$ is spanned by the diagonal matrix with entries $\pm i$, $\mathfrak{m}_2$ is spanned by the real skew-symmetric matrices and $\mathfrak{m}_3$ is spanned by the symmetric matrix having i at the off-diagonal entries. To compute the diagonal entries r_1, r_2, r_3 of the Ricci endomorphism $\operatorname{Ric}(g)$, see (21), of the metrics g with respect to the decomposition $\mathfrak{su}(2) = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ we use the formula (22). This yields

$$\begin{aligned} r_1 &= \tfrac{2}{x_1 x_2 x_3} \cdot (x_1^2 - (x_2 - x_3)^2) \\ r_2 &= \tfrac{2}{x_1 x_2 x_3} \cdot (x_2^2 - (x_1 - x_3)^2) \\ r_3 &= \tfrac{2}{x_1 x_2 x_3} \cdot (x_3^2 - (x_1 - x_2)^2). \end{aligned}$$

Here we have used that $b_1 = b_2 = b_3 = 8$ (see [68], p. 583), that $[123] = 4$ and that of course $d_1 = d_2 = d_3 = 1$. As is well-known the off-diagonal entries of the Ricci endomorphism vanish: see Chapter 1, Section 5 in [24].

The Ricci flow equation for these metrics is given by $x_i' = -2x_i \cdot r_i$, $i = 1, 2, 3$ and the volume normalized Ricci flow by $x_i' = -2x_i \cdot r_i^0$, $r_i^0 = r_i - \frac{1}{3}(r_1 + r_2 + r_3)$. Recall that after a reparametrization in space and time the Ricci flow and the volume normalized Ricci flow are equivalent. Solutions to the normalized Ricci flow will be denoted by $(\bar{g}(\bar{t}))_{\bar{t}\in[0,\bar{T})}$.

It is now convenient to introduce new coordinates $\alpha = \frac{x_2}{x_1}$ and $\beta = \frac{x_3}{x_1}$. Notice that $\frac{\alpha'}{\alpha} = \frac{x_2'}{x_2} - \frac{x_1'}{x_1} = 2(r_1 - r_2)$ and $\frac{\beta'}{\beta} = 2(r_1 - r_3)$. The volume constraint $x_1x_2x_3 = 1$ reads now $(\alpha\beta)^{\frac{1}{3}} = \frac{1}{x_1}$. Therefore, the volume normalized Ricci flow, fixing volume one, is equivalent to

$$\alpha' = \frac{8}{(\alpha\beta)^{\frac{2}{3}}} \cdot \alpha \cdot (1 - \alpha) \cdot (1 + \alpha - \beta)$$

$$\beta' = \frac{8}{(\alpha\beta)^{\frac{2}{3}}} \cdot \beta \cdot (1 - \beta) \cdot (1 + \beta - \alpha).$$

The sets $\{\alpha \equiv 1\}$, $\{\beta \equiv 1\}$ and $\{\alpha \equiv \beta\}$ are invariant under this ordinary differential equation and $(1, 1)$ is the unique zero in the domain $\{\alpha, \beta > 0\}$.

For $l \in \mathbb{N}$, $l \geq 100$, we choose initial values $\alpha_0^l := \frac{l}{4} + \frac{\sqrt{l-1}}{2}$ and $\beta_0^l := \frac{l}{4} - \frac{\sqrt{l-1}}{2}$. Clearly $\alpha_0^l, \beta_0^l > 1$ and $1 + \alpha_0^l - \beta_0^l > 0$, whereas $1 + \beta_0^l - \alpha_0^l < 0$. That is, for the solution $(\alpha(t), \beta(t))$ with initial values (α_0^l, β_0^l) at time $\bar{t} = 0$ we have $\alpha' < 0$ and $\beta' > 0$ until this solution reaches the line $\{1 + \beta - \alpha \equiv 0\}$ at a time $\bar{t}_0^l > 0$. Notice that $\beta(\bar{t}_0^l) > \beta_0^l$ of course and that the time $\bar{t}_0^l$ is unique.

The initial values or chosen such that $\mathrm{scal}(\bar{g}_l(0)) = 0$ and $\|\mathrm{Ric}(\bar{g}_l(0))\|^2 \geq c_1 \cdot l^{\frac{1}{3}}$ for $c_1 > 0$ independent of l. Another computation shows that at time $\bar{t}_0^l$ we have $\mathrm{scal}((\bar{g}_l(\bar{t}_0^l))^2 = \|\mathrm{Ric}(\bar{g}_l(\bar{t}_0^l))\|^2 \leq c_2 \cdot l^{-\frac{2}{3}}$ for a constant c_2, independent of l.

Next, let $(g(t))_{t\in[0,T)}$ denote a solution to the unnormalized Ricci flow and set $V(t) := \sqrt{x_1(t)x_2(t)x_3(t)}$. Notice that up to a constant this equals to the volume of $(S^3, g(t))$. As is well-known, but also follows from the above equations, we have $V'(t) = -V(t) \cdot s(t)$, where $s(t) := \mathrm{scal}(g(t))$ and consequently $V'(t) = -V^{\frac{1}{3}}(t) \cdot \bar{s}(t)$, with $\bar{s}(t) = -V^{\frac{2}{3}}(t) \cdot s(t)$. The function $(x_1x_2x_3)^{\frac{1}{3}} \cdot \mathrm{scal}(g(x_1, x_2, x_3))$ is scale invariant.

Let $(g_l(t))_{t\in[0,T(l))}$ denote the solution to the unnormalized Ricci flow with initial value $g_l(0)$ satisfying $V(0) = 1$, $\alpha(0) = \alpha_0^l$ and $\beta(0) = \beta_0^l$. By the above we know that $\|\mathrm{Ric}(g_l(0))\| \geq \sqrt{c_1} \cdot l^{\frac{1}{6}}$. Next, let us denote by $t_0^l \in (0, T(l))$ the unique time with $1 + \beta(t_0^l) - \alpha(t_0^l) = 0$. By the above we have $\bar{s}(t) \leq \bar{s}(t_0^l) \leq \epsilon(l) := \sqrt{c_2} \cdot l^{-\frac{1}{3}}$ for all $t \in [0, t_0^l]$, since the scalar curvature is still increasing along the volume normalized Ricci flow. Using $V'(t) = -V^{\frac{1}{3}}(t) \cdot \bar{s}(t)$ and $V(0) = 1$ we deduce $V(t) \geq (1 - \epsilon(l) \cdot t)^{\frac{3}{2}}$ for $t \leq \max\{t_0^l, \frac{1}{\epsilon(l)}\}$.

Suppose now $T(l) \leq C$, for all $l \geq 100$ and a constant $C > 0$. Clearly this implies $t_0^l < C$ and hence there exists $l_0 > 0$ such that $V(t_0^l) \geq 0.5$ for all $l \geq l_0$. Since $\| \operatorname{Ric}(\bar{g}_l(\bar{t}_0^l))\| = V^{\frac{1}{3}}(t_0^l) \cdot \| \operatorname{Ric}(g(t_0^l))\| \leq \epsilon(l)$, we conclude on the other hand side by the doubling time property of the Ricci flow that the extinction time of these solutions cannot be uniformly bounded. Contradiction.

Finally notice that if $(g(t))_{t\in[0,T)}$ is a Ricci flow solution then for any $\lambda > 0$ also $(\lambda \cdot g(\frac{t}{\lambda}))_{t\in[0,\lambda\cdot T)}$ is a solution. As a consequence the term $\| \operatorname{Rm}(g(0))\|_{g(0)} \cdot T$ is invariant under parabolic rescaling. Then choosing $t_l > 0$ as close to zero as we like as our new initial time and performing the parabolic rescaling just described shows the claim. ■

The next application is a doubling time estimate for the Ricci curvature along homogeneous Ricci flows. It shows that the Ricci curvature cannot grow too quickly.

Proposition 4.4. Let $(M^n, g(t))_{t\in[0,b]}$ be a homogeneous Ricci flow solution. If $\| \operatorname{Ric}(g(0))\| = 1$, then

$$\| \operatorname{Ric}(g(t))\| \leq 2,$$

for all $0 \leq t \leq 1/C(n)$. □

Proof. According to the evolution equation for $\| \operatorname{Ric} \|^2$ along Ricci flow [25], Lemma 2.40, together with the Gap Theorem, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\| \operatorname{Ric} \|^2 \leq C_1(n) \cdot \| \operatorname{Rm} \| \cdot \| \operatorname{Ric} \|^2 \leq C(n) \cdot \| \operatorname{Ric} \|^3.$$

Recall that in the homogeneous case $\| \operatorname{Ric} \|^2$ is a constant function. By a standard comparison argument, one obtains $\| \operatorname{Ric}(g(t))\| \leq \frac{2}{2-C(n)\cdot t}$ since $\rho(t) = \frac{2}{2-C(n)\cdot t}$ is the solution to $\frac{\mathrm{d}}{\mathrm{d}t}\rho^2 = C(n) \cdot \rho^3$, $\rho(0) = 1$. The proposition now follows. ■

5 Non-Collapsed Homogeneous Ancient Solutions

In this section, we prove Theorem 5.2 which is essentially Theorem 3, and we show in Lemma 5.4 that for any unstable homogeneous Einstein metric there exists a non-collapsed homogeneous ancient solution emanating from it.

Since non-trivial ancient solutions $(g(t))_{t\in(-\infty,0]}$ to the Ricci flow have positive scalar curvature, non-trivial ancient homogeneous solutions must develop a Type I singularity close to their extinction time by Corollary 2. By [52] and [28] the blow-up of such a solution will subconverge to a non-flat homogeneous gradient shrinking soliton. These homogeneous limit solitons were classified in [61]. Up to finite coverings they are the Riemannian product of a compact homogeneous Einstein space and a flat factor. Notice that the flat factor might be absent.

Also by Corollary 2, ancient homogeneous solutions develop a Type I behavior in the past. It is then natural to consider the corresponding blow-downs

$$g_i(t) := \tfrac{1}{s_i} \cdot g(s_i \cdot t)$$

for a sequence $\{s_i\}_{i\in\mathbb{N}}$ with $s_i \to \infty$ and all $t \in (-\infty, 0]$. In the non-collapsed situation, it follows by [52] and [21] that the sequence $(g_i(t))_{i\in\mathbb{N}}$ subconverges to a non-flat asymptotic soliton.

A compact homogeneous space has a presentation $M^n = G/H$, where G is a compact Lie group acting transitively on M^n with compact isotropy group H. Notice that G and H are not necessarily connected. Since G is compact, there exists an $\mathrm{Ad}(G)$-invariant scalar product Q on the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{m}$ denote the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then the set $\mathcal{M}^G$ of G-homogeneous metrics on G/H can be viewed as the set of $\mathrm{Ad}(H)$-invariant scalar products on $\mathfrak{m}$. This set in turn can be viewed as the Euclidean space $S^2(\mathfrak{m})^{\mathrm{Ad}(H)}$ of symmetric, positive-definite, $\mathrm{Ad}(H)$-equivariant linear endomorphisms of $\mathfrak{m}$ as follows $g(x,y) = Q|_{\mathfrak{m}}(g \cdot x, y)$, where $x, y \in \mathfrak{m}$. Recall, that a G-homogeneous Einstein metric on G/H is a critical point of the total scalar curvature functional

$$\mathcal{S} : \mathcal{M}_1^G \to \mathbb{R}\ ;\ g \mapsto \mathrm{scal}(g)$$

restricted to the space $\mathcal{M}_1^G$ of G-homogeneous metrics of volume one, and that the gradient flow

$$g'(t) = -2 \cdot g(t) \cdot \mathrm{Ric}_0(g(t))$$

of $\mathcal{S}$ is nothing but the volume-normalized Ricci flow for G-homogeneous metrics. Here we consider the Ricci-endomorphism $\mathrm{Ric}(g(t))$, defined by

$$\mathrm{Ric}(g(t))(x,y) = g(t)(\mathrm{Ric}(g(t)) \cdot x, y) \tag{21}$$

and then also $g(t)$ as an endomorphism as above. Since the space of G-homogeneous metrics is finite-dimensional, we have existence and uniqueness of homogeneous Ricci flow solutions also backwards in time.

Lemma 5.1. The gradient flow of $\mathcal{S}$ on $\mathcal{M}^G_1$ is analytic. □

Proof. The set $\mathcal{M}^G_1$ is an algebraic subvariety of the Euclidean space $\mathcal{M}^G$, and at the same time a smooth submanifold. Moreover, when fixing a Q-orthonormal basis of $\mathfrak{m}$ any G-homogeneous metric can be considered an $\mathrm{Ad}(H)$-equivariant matrix. Now the Ricci endomorphism $\mathrm{Ric}(g)$ of $g \in \mathcal{M}^G$ can be written down explicitly as in Proposition 1.5 in [16]. It is a rational map and consequently the same is true for its traceless part $\mathrm{Ric}_0(g)$. This shows the claim. ■

Theorem 5.2. Let G/H be a compact homogeneous space. Then any non-collapsed homogeneous ancient solution has a unique compact asymptotic soliton, which is a homogeneous Einstein metric on G/H. □

Proof. As mentioned above, by [52] and [33] for any sequence of blow-downs $(g_i(t))_{t\in(-\infty,0]}$ of a non-collapsed ancient solution $(g(t))_{t\in(-\infty,0]}$ there exists a non-flat asymptotic soliton to which they subconverge. Now by [61] we know, that up to a finite covering this asymptotic soliton is a product of a compact homogeneous Einstein space and a flat factor. We have to exclude the flat factor.

If there were a flat factor $\mathbb{R}^k$, then for large i the volume of the metrics $g_i(-1)$ would be unbounded, whereas $\mathrm{scal}(g_i(-1)) \to c_\infty > 0$ for $i \to \infty$. As a consequence, for the unit volume normalization $\bar{g}(t)$ of $g(t)$ the scalar curvature would be unbounded as $t \to -\infty$. This is a contradiction, since the scalar curvature is increasing along the volume-normalized Ricci flow on a compact homogeneous space.

Thus, the asymptotic soliton is compact. Next, we claim, that along the backward volume normalized homogeneous Ricci flow the scalar curvature of the solution $\bar{g}(\tau)$, $\tau = -t$, cannot converge to zero. This is clear, since otherwise the asymptotic limit soliton would be flat.

It follows, that for any sequence $(\tau_i)_{i\in\mathbb{N}}$ converging to $+\infty$, we will find a subsequence, such that $\bar{g}(\tau_{i_j})$ is a Palais–Smale-sequence C for $\mathcal{S}$ in $\{\mathcal{S} \geq \varepsilon\}$ for some $\varepsilon > 0$. Then by Theorem A in [16] there exists a subsequence, which converges to a homogeneous limit metric g_∞ on G/H.

Uniqueness of the limit follows from Lemma 2.1, since it is well-known that if the ω-limit set of an analytic gradient flow solution is non-empty, then it consists of a single point, see for instance [48]. ■

Remark 5.3. If a compact homogeneous space G/H, with G, H connected, is not a homogeneous torus bundle, that is if there exists no compact intermediate subgroup

$H < K < G$ such that K/H is a torus, then any ancient solution on G/H is non-collapsed. This is seen as follows. For a collapsed solution to the backward volume-normalized Ricci flow the scalar curvature must tend to zero. That means that on that space there exists a zero-Palais–Smale-sequence. Now by Theorem 2.1 in [16] the claim follows. □

Next, we show that for any unstable homogeneous Einstein metric there are always ancient solutions emanating from it. Recall that a homogeneous Einstein metric is called *unstable*, if it is not a local maximum of $\mathcal{S}$.

Recall that the Milnor fibre of a critical point g_E is defined as follows

$$F_+(g_E) := \{g \in \mathcal{M}_1^G : \mathcal{S}(g) = \mathcal{S}(g_E) + r^N\} \cap B_r(g_E)$$

where $B_r(g_E)$ is a ball in $\mathcal{M}_1^G$ of a very small radius $r > 0$ and N is large. Clearly, it is empty if and only if g_E is a local maximum of $\mathcal{S}$.

If an Einstein metric g_E is non-degenerate and unstable, then by the unstable manifold theorem there exists solutions to the gradient flow of $\mathcal{S}$ emanating from it. Since the scalar curvature is still increasing along such flow lines, it follows that these solutions have positive scalar curvature. Hence, by [45] they are ancient. By standard Morse theory the Milnor fibre is homotopy equivalent to S^{k-1}, k being the dimension of the positive eigenspace of the Hessian of $\mathcal{S}$ at the critical point g_E. In general however, the unstable Einstein metric g_E might be degenerate and even not isolated.

Lemma 5.4. Let g_E be a G-homogeneous unstable Einstein metric on a compact homogeneous space G/H. Then there exists a G-homogeneous ancient solution emanating from g_E. Moreover, the dimension of such solutions can be estimated from the below by the cohomological dimension of the Milnor fibre of $\mathcal{S}$ at g_E. □

Proof. This follows from [54]. It is shown there, that if the Milnor fibre is not empty, then there exists a solution to the negative gradient flow of $\mathcal{S}$ with omega limit set g_E. Moreover, the authors show that the Cech–Alexander cohomology groups of the set of such solutions and the Milnor fibre agree. ■

We should point out that in the above lemma we do not claim that all these ancient solutions are pairwise non-isometric.

Remark 5.5. An Einstein metric g_E on a compact manifold is the Yamabe metric in its conformal class. Moreover, by Theorem C in [16] any nearby metric with constant scalar curvature is a Yamabe metric as well, provided g_E is not in the conformal class of the round metric. It follows, that if a homogeneous Einstein metric is unstable, then it is not a local maximum of the Yamabe functional. Hence, by Theorem 1.6 in [44] there exists an ancient solution emanating from it. It may be possible to adjust the proof in [44] to show that this ancient solution can also be chosen to be homogeneous.

Let us also mention that there are general existence results for unstable homogeneous Einstein metrics on compact homogeneous spaces relying on min-max principles [13, 16, 32], to which Lemma 5.4 can be applied. For instance, if a compact homogeneous space G/H satisfies a certain algebraic property (its graph having two non-toral components, see [16] for details), then there exists $C > 0$ such that $\mathcal{S}^{-1}([C,\infty))$ is disconnected. Moreover, there is also a smooth path of metrics in $\{0 < \mathcal{S} \leq C\}$ joining two connected components of $\mathcal{S}^{-1}([C,\infty))$. By a standard mountain pass lemma the existence of a critical point of $\mathcal{S}$ follows: see Proposition 3.7 in [16]. It also follows that there must exist a critical point which is not a local maximum of $\mathcal{S}$, since by Proposition 1.5 in [16] the set of all critical points of $\mathcal{S}$ is a disjoint union of finitely many compact, connected, semialgebraic sets on each one of which $\mathcal{S}$ is constant. □

6 Examples of Collapsed Ancient Solutions

For collapsed homogeneous ancient solitons there will of course be no asymptotic gradient shrinking soliton in the category of smooth manifolds. However, when working with Riemannian groupoids, as introduced by Lott in [50], one might hope to prove the existence of a locally homogeneous asymptotic gradient shrinking soliton. That means that after considering the blow-downs of a collapsed ancient solution, we pull back these metrics to a ball in the tangent space. Due to the curvature estimates provided by Theorem 1 these balls can be chosen to have a uniform radius. The convergence in the category of Riemannian groupoids then only means that these locally homogeneous metrics converge in C^∞-topology to a locally homogeneous limit metric on this fixed ball (cf. Section 2). In other words, one considers the convergence of the corresponding *geometric models*, defined in Section 2. Now an asymptotic soliton, would be a locally homogeneous product metric of an Einstein metric with positive Ricci curvature and a flat factor. In the collapsed case the flat factor cannot be absent. Notice, that if a collapsed ancient homogeneous solution does admit a locally homogeneous asymptotic soliton, it must be non-flat by the curvature estimates provided in Corollary 2.

We would like to mention that by [62] there exist locally homogeneous Einstein metrics on $(S^3 \times S^3)/S^1_r$ of positive scalar curvature, which by [43] cannot be extended to globally homogeneous compact Einstein spaces (cf. [16], p. 725). Here S^1_r is embedded into the maximal torus of $S^3 \times S^3$ with irrational slope r. Whether such locally homogeneous spaces occur as compact factors in asymptotic solitons is unknown.

In order to provide an example of a collapsed ancient solution with non-compact singularity model, we cannot work with homogeneous spaces whose isotropy representation admits only two summands: see [27] for a classification. Instead, we are looking for homogeneous spaces G/H not admitting any G-invariant Einstein metric, to prevent a compact singularity model, but which in addition are homogeneous S^1-bundles.

We recall, how to compute the Ricci curvature of a compact homogeneous space G/H. As mentioned above, every G-invariant metric on G/H is uniquely determined by an $Ad(H)$-invariant scalar product on $\mathfrak{m}$. It follows that for any G-invariant metric g on G/H there exists a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\ell$ of $\mathfrak{m}$ into $Ad(H)$-irreducible summands, such that g is diagonal with respect to Q, that is

$$g = x_1 \cdot Q|_{\mathfrak{m}_1} \perp \cdots \perp x_\ell \cdot Q|_{\mathfrak{m}_\ell}$$

with $x_1, ..., x_\ell > 0$. By [55, 69], the diagonal entries r_m of the Ricci endomorphism $\mathrm{Ric}(g)$ of g are given by

$$r_m = \frac{b_m}{2x_m} - \frac{1}{2d_m} \sum_{j,k=1}^{\ell} [jkm] \frac{x_k}{x_m x_j} + \frac{1}{4d_m} \sum_{j,k=1}^{\ell} [jkm] \frac{x_m}{x_j x_k}. \tag{22}$$

Here, $-B|_{\mathfrak{m}_m} = b_m \cdot Q|_{\mathfrak{m}_m}$ and $d_m = \dim \mathfrak{m}_m$, where B denotes the Killing form of $\mathfrak{g}$. The structure constants $[ijk]$ with respect to the above decomposition of $\mathfrak{m}$ are defined as follows:

$$[ijk] = \sum Q([e_\alpha, e_\beta], e_\gamma)^2$$

where the sum is taken over $\{e_\alpha\}$, $\{e_\beta\}$, and $\{e_\gamma\}$, Q-orthonormal bases for $\mathfrak{m}_i$, $\mathfrak{m}_j$ and $\mathfrak{m}_k$, respectively. Notice that $[ijk]$ is invariant under permutation of i, j, k.

Example 6.1. On $G/H = (\mathrm{SU}(n)\mathrm{SU}(n))/(\Delta\mathrm{SU}(n-1)\Delta\mathrm{U}(1))$ there exist for $n \geq 3$ a one-parameter family of homogeneous ancient solutions with the same non-compact asymptotic soliton $(E_-, g^1_-) \times \mathbb{R}$, where

$$E_- = (\mathrm{SU}(n)\mathrm{SU}(n))/(\Delta(\mathrm{SU}(n-1))\mathrm{U}(1)\mathrm{U}(1)),$$

and the same non-compact singularity model $(E_+, g_+) \times \mathbb{R}^{4(n-1)+1}$, where

$$E_+ = (\mathrm{SU}(n-1)\mathrm{SU}(n-1))/\Delta\mathrm{SU}(n-1).$$

The Einstein metric g_-^1 on E_- is the unstable homogeneous Einstein metric on E_- and (E_+, g_+) is a compact symmetric space. Furthermore, there exists one further homogeneous ancient solution in the closure of the above family with the asymptotic soliton (E_-, g_-^2) and the same singularity model. Here g_-^2 is now the stable homogeneous Einstein metric on E_-. Finally, all the above ancient solutions have positive Ricci curvature. □

Proof. We set $G = G_1 G_2$, with $G_1 = G_2 = \mathrm{SU}(n)$ and $H = \Delta\mathrm{SU}(n-1)\Delta\mathrm{U}(1)$ for $n \geq 3$. Here H is embedded into G as follows: Consider the subgroup $\mathrm{SU}(n-1)$ of $\mathrm{SU}(n)$, embedded as an upper $(n-1) \times (n-1)$-block. Then the semisimple part of H is embedded diagonally in $\mathrm{SU}(n-1)\mathrm{SU}(n-1) \subset G$. The subgroup $\mathrm{SU}(n-1)$ commutes with its centralizer $\mathrm{U}(1)$ in $\mathrm{SU}(n)$. This $\mathrm{U}(1)$ is embedded diagonally into $\mathrm{SU}(n)$, the first $(n-1)$ diagonal entries being equal. Now $\Delta\mathrm{U}(1)$ is embedded diagonally into the product $\mathrm{U}(1)\mathrm{U}(1)$.

We choose the $\mathrm{Ad}(G)$-invariant scalar product $Q(X, Y) = \frac{1}{2}\operatorname{tr}(X \cdot Y^*)$ on $\mathfrak{g}$. Then $b_m = 4n$ for all m (see [68], p. 583). Next, let $\mathfrak{m}'$ denote the orthogonal complement of $\mathfrak{u}(n-1) \oplus \mathfrak{u}(1)$ in $\mathfrak{u}(n)$, considered as a subspace in the first factor $\mathfrak{su}_1(n)$ and let $\mathfrak{m}''$ be defined accordingly. We set $\mathfrak{m}_1 = \mathfrak{m}' \oplus \mathfrak{m}''$ and conclude $d_1 = \dim \mathfrak{m}_1 = 4(n-1)$. The space $\mathfrak{m}_2$ is the orthogonal complement of $\Delta\mathfrak{su}(n-1)$ in $\mathfrak{su}_1(n-1) \oplus \mathfrak{su}_2(n-1)$, thus $d_2 = n(n-2)$. Finally, $\mathfrak{m}_3$ is the orthogonal complement of $\Delta\mathfrak{u}(1)$ in $\mathfrak{u}_1(1) \times \mathfrak{u}_1(1)$, hence $d_3 = 1$.

The group $G = G_1 G_2$ admit the involution $f(g_1, g_2) = (g_2, g_1)$. Clearly, we have $f(H) = H$. We set now $\hat{G} = \mathbb{Z}_2 \ltimes G$ and $\hat{H} = \mathbb{Z}_2 \ltimes H$. Then $G/H = \hat{G}/\hat{H}$ as manifolds. Moreover, the modules $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are now $\mathrm{Ad}(\hat{H})$-irreducible and of course inequivalent, since there dimensions are different.

We will consider the three-parameter family of homogeneous metrics

$$g = x_1 \cdot Q|_{\mathfrak{m}_1} \perp x_2 \cdot Q|_{\mathfrak{m}_2} \perp x_3 \cdot Q|_{\mathfrak{m}_3}$$

and compute their Ricci curvatures for $x_1, x_2, x_3 > 0$.

The only non vanishing structure constants are [112] and [113]. To this end notice that $\mathrm{U}(n)/(\mathrm{U}(n-1)\mathrm{U}(1))$ is a symmetric space, hence $[111] = 0$. Next, $\mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{h}$ is a subalgebra, and therefore $[122] = [123] = [133] = 0$. Since $(\mathrm{SU}(n-1)\mathrm{SU}(n-1))/\Delta(\mathrm{SU}(n-1)$ is also a symmetric pair, we have $[222] = 0$. Moreover, $[333] = 0$, since $\mathfrak{m}_3$ is an abelian subalgebra. Finally $[\mathfrak{m}_2, \mathfrak{m}_3] = 0$.

We have that $4n = [311]$ by the identity $d_3 b_3 = \sum_{i,j=1}^{3}[3ij]$ from Lemma 1.5 in [69], since $d_3 = 1$. Next, we claim that $[211] = 4d_2$. To this end we choose a standard orthonormal basis of $\mathfrak{m}_1$ consisting of $2(n-1)$ skew-symmetric elements with only two non-vanishing entries ± 1 and $2(n-1)$ symmetric elements with two non-vanishing entries i. For each of these basis vectors e there exist precisely $1+2(n-2)$ other basis elements not commuting with e. The first special basis element e^* has its non-vanishing entries at the same spot as e does. The Lie-bracket $[e, e^*]$ is diagonal and has two non-vanishing entries $\pm 2i$. When computing the projection of $[e, e^*]$ onto $\mathfrak{su}(n-1)$ one deduces that $\|[e, e^*]_{\mathfrak{su}(n-1)}\|^2 = 2\frac{n-2}{n-1}$. The computation of the other $2(n-2)$ non-vanishing brackets is standard and we obtain

$$[211] = 4(n-1) \cdot \left(2\tfrac{n-2}{n-1} + 2(n-2)\right) \cdot \tfrac{1}{2} = 4(n-2)n = 4d_2,$$

noticing that all the brackets in question had to be projected to the diagonally embedded $\Delta\mathfrak{su}(n-1)$ in $\mathfrak{su}(n-1) \oplus \mathfrak{su}(n-1)$.

Since the three modules $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$ are inequivalent, it follows from Schur's Lemma that the Ricci tensor of the metric $g = g(x_1, x_2, x_3)$ is diagonal as well and by (22) we deduce

$$r_1 = \tfrac{2n}{x_1} - \tfrac{n(n-2)}{2(n-1)} \cdot \tfrac{x_2}{x_1^2} - \tfrac{n}{2(n-1)} \cdot \tfrac{x_3}{x_1^2} \tag{23}$$

$$r_2 = \tfrac{2(n-1)}{x_2} + \tfrac{x_2}{x_1^2} \tag{24}$$

$$r_3 = n \cdot \tfrac{x_3}{x_1^2} \tag{25}$$

Recall that the Ricci flow on the homogeneous space $\hat{G}/\hat{H}$ is given by $x_i' = -2x_i \cdot r_i$, $i = 1, 2, 3$. Furthermore, we can reduce dimension by one considering the volume normalized Ricci flow $x_i' = -2x_i \cdot r_i^0$, where $r_i^0 = r_i - \frac{1}{N} \cdot \mathrm{scal}$ denotes the entries of the traceless part of the Ricci endomorphism, $N = \dim \hat{G}/\hat{H}$. To understand this normalized Ricci flow it is convenient to introduce new coordinates $\alpha = \frac{x_2}{x_1}$ and $\beta = \frac{x_3}{x_1}$. We have that $\frac{\alpha'}{\alpha} = \frac{x_2'}{x_2} - \frac{x_1'}{x_1} = 2(r_1 - r_2)$ and similarly $\frac{\beta'}{\beta} = 2(r_1 - r_3)$. The volume constraint $x_1^{d_1} x_2^{d_2} x_3 = 1$ reads in these new coordinates $(\alpha^{d_2} \cdot \beta)^{\frac{1}{N}} = \frac{1}{x_1}$. We deduce that the volume normalized Ricci flow is equivalent to

$$\alpha' = p(\alpha, \beta) \cdot \left(4(n-1) - \tfrac{n^2-2}{n} \cdot \alpha - \tfrac{4(n-1)^2}{n} \cdot \tfrac{1}{\alpha} - \beta\right) \tag{26}$$

$$\beta' = q(\alpha, \beta) \cdot (4(n-1) - (n-2) \cdot \alpha - (2n-1) \cdot \beta), \tag{27}$$

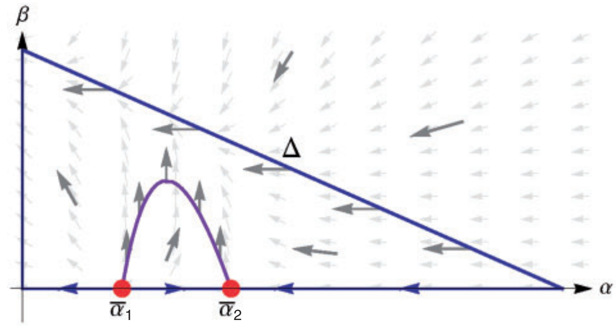

Fig. 2. Volume normalized Ricci flow on $\hat{G}/\hat{H}$ for $n = 4$.

where $p(\alpha,\beta) = \alpha \cdot \frac{n}{n-1} \cdot \alpha^{\frac{d_2}{N}} \cdot \beta^{\frac{1}{N}}$ and $q(\alpha,\beta) = \beta \cdot \frac{n}{n-1} \cdot \alpha^{\frac{d_2}{N}} \cdot \beta^{\frac{1}{N}}$. Now, notice that if we divide the right hand side of the system (26) and (27) by the positive function $\frac{n}{n-1} \cdot \alpha^{\frac{d_2}{N}} \cdot \beta^{\frac{1}{N}}$ only the time-parametrization of solutions does change, but the integral curves do not. Consequently, the volume normalized Ricci flow of $\hat{G}/\hat{H}$ is a up to time reparametrization equivalent to

$$\alpha' = -\frac{4(n-1)^2}{n} + 4(n-1) \cdot \alpha - \frac{n^2-2}{n} \cdot \alpha^2 - \alpha \cdot \beta \tag{28}$$

$$\beta' = \beta \cdot \big(4(n-1) - (n-2) \cdot \alpha - (2n-1) \cdot \beta\big), \tag{29}$$

restricted to the domain $\{\alpha, \beta > 0\}$.

We turn to the qualitative behavior of this system, however on the slightly larger domain $D := \{\alpha > 0\} \cup \{\beta \geq 0\}$ (cf. Figure 2). First notice, that the positive α-axis in invariant under this system. In fact this restriction is up to reparametrization nothing but the volume normalized Ricci flow on E_-. Moreover, the above system admits precisely two constant solutions $(\bar{\alpha}_1, 0)$ and $(\bar{\alpha}_2, 0)$ with

$$\bar{\alpha}_1 = \tfrac{2(n-1)}{n+\sqrt{2}} \quad \text{and} \quad \bar{\alpha}_2 = \tfrac{2(n-1)}{n-\sqrt{2}}.$$

To this end, for $\alpha > 0$ the condition $\alpha' = 0$ implies

$$\beta = -\tfrac{4(n-1)^2}{n} \cdot \tfrac{1}{\alpha} + 4(n-1) - \tfrac{n^2-2}{n} \cdot \alpha.$$

Plugging this into (29) yields a quadratic equation for α, which does not have real solutions. This shows in particular that the space $\hat{G}/\hat{H}$ does not admit a $\hat{G}$-invariant Einstein metric.

Next, we consider the right hand side of the above system as a smooth vector field X. Its differential $(DX)_{(\bar{\alpha}_i,0)}$, $i = 1,2$, is upper triangular, with eigenvalues

$$\lambda_1^i = (-1)^{i+1} \cdot \tfrac{4(n-1)\sqrt{2}}{n} \quad \text{and} \quad \lambda_2^i = 4(n-1) - (n-2) \cdot \bar{\alpha}_i > 0.$$

This shows, that $(\bar{\alpha}_1, 0)$ is a node, while $(\bar{\alpha}_2, 0)$ is a saddle point, whose unstable manifold intersects the α-axis transversally.

The set $\{\beta' \geq 0\} \cup \{\beta > 0\}$ is a right triangle Δ, depicted in Figure 2. Actually the hypotenuse belongs to Δ, but the two other sides do not. A computation shows now, that X intersects its hypotenuse transversally pointing into its interior. It follows that a maximal solution in D, starting in Δ, cannot leave Δ. Moreover, since in Δ we have $r_1 \geq r_3$ and since $r_2, r_3 > 0$, we conclude that any metric in Δ has positive Ricci curvature.

As we saw above, $(\bar{\alpha}_1, 0)$ is a node. Therefore, there exist a one-parameter family of solutions in D, which emanate from it. Since these solutions cannot leave Δ, they have positive Ricci curvature, hence they are ancient by [45]. Clearly, the compact factor of the asymptotic soliton of these ancient solutions is the unstable Einstein metric g_-^1 of E_-.

The second ancient solution is given by the unstable manifold of the stable Einstein metric $(\bar{\alpha}_2, 0)$ of the compact factor E_-. It lies in the closure of the above one-parameter family of solutions, but has an ancient soliton, whose compact factor is isometric to g_-^2. It is not hard to show, using the Ricci curvature formulas (23), (24) and (25), that these solutions cannot be isometric.

For all these ancient solutions $(\alpha(t), \beta(t))_{t \in (-\infty, T)} \in D$ it remains to compute their singularity model. Let us mention at this point, that they reach the β-axis. However this point does not correspond to a Riemannian metric on $\hat{G}/\hat{H}$. It is clear that $\lim_{t\to T} = \alpha(t) = \bar{\alpha} = 0$, while $\lim_{t\to T} \beta(t) = \bar{\beta} > 0$. We rescale now these metrics such that $x_2(t) \equiv 1$. Since $\bar{\alpha} = 0$, we deduce $\lim_{t\to T} x_1(t) = \infty$ and since $\bar{\beta} > 0$ it follows that also $\lim_{t\to T} x_3(t) = \infty$. As a consequence these metrics converge to a limit product metric on the singularity model $E_+ \times \mathbb{R}^{4(n-1)+1}$. Since E_+ is isotropy irreducible, the limit metric on E_+ is Einstein. ■

7 Bounds on the Gap

According to the decomposition of the space of curvature operators into irreducible $\mathrm{O}(n)$-modules in dimensions $n \geq 4$, the operator Rm decomposes as

$$\mathrm{Rm} = \mathrm{Rm}_I + \mathrm{Rm}_{\mathrm{Ric}_0} + \mathrm{W}\,,$$

where $\mathrm{Rm}_I = \frac{\mathrm{scal}}{n(n-1)}\,\mathrm{id}\wedge\mathrm{id}$, $\mathrm{Rm}_{\mathrm{Ric}_0} = \frac{2}{n-2}\,\mathrm{Ric}_0\wedge\mathrm{id}$, $\mathrm{Ric}_0 = \mathrm{Ric} - \frac{\mathrm{scal}}{n}\,\mathrm{id}$ is the traceless Ricci tensor and W is the Weyl tensor. A standard computation shows that

$$\|\mathrm{Rm}_I\|^2 = \frac{\mathrm{scal}^2}{2n(n-1)} \quad \text{and} \quad \left\|\mathrm{Rm}_{\mathrm{Ric}_0}\right\|^2 = \frac{\|\,\mathrm{Ric}_0\,\|^2}{n-2}. \tag{30}$$

The aim of this section is to prove the following

Lemma 7.1. For $n \geq 4$ there exist homogeneous spaces (M^n, g) such that

$$\|\,\mathrm{W}(g)\|_g \geq \sqrt{\tfrac{n-2}{n-3}}\cdot\|\,\mathrm{Rm}(g)\|_g.$$

□

Proof. An n-dimensional real Lie algebra $\mathfrak{g}$ is called *almost abelian* if it admits a codimension-one abelian ideal $\mathfrak{n}$. In other words, there exist a basis $\{e_i\}_{i=0}^{n-1}$ for $\mathfrak{g}$ and an endomorphism $A \in \mathfrak{gl}_{n-1}(\mathbb{R})$ such that the Lie bracket is given by

$$[e_0, e_i] = -[e_i, e_0] = A\, e_i, \qquad [e_i, e_j] = 0, \qquad i,j \neq 0.$$

Let us consider an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ that makes $\{e_i\}$ orthonormal, and denote by (S_A, G) the corresponding simply-connected Lie group with left-invariant Riemannian metric. We denote by $D = \frac{1}{2}\left(A + A^t\right)$ and $Q = \frac{1}{2}\left(A - A^t\right)$ the symmetric and skew-symmetric parts of A, respectively. It follows from [51] that the curvature operator $\mathrm{Rm} : \Lambda^2\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is given by

$$\mathrm{Rm}\,(e_0\wedge e_i) = -e_0\wedge\left(D^2 + [D,Q]\right)e_i, \qquad i\neq 0,$$
$$\mathrm{Rm}\,\left(e_i\wedge e_j\right) = De_j\wedge De_i, \qquad i,j\neq 0\,,$$

the Ricci endomorphism $\mathrm{Ric} : \mathfrak{g}\to\mathfrak{g}$ by

$$\mathrm{Ric}(e_0) = -\left(\mathrm{tr}\,D^2\right)e_0, \quad \mathrm{Ric}(e_i) = \left([Q,D] - (\mathrm{tr}\,D)\,D\right)e_i, \quad i\neq 0$$

and the scalar curvature by $\mathrm{scal} = -\,\mathrm{tr}\,D^2 - (\mathrm{tr}\,D)^2$. If in particular one takes $Q = 0$ and D such that $\mathrm{tr}\,D = 0$, then

$$\mathrm{scal}^2 = \|D\|^4, \quad \|\,\mathrm{Ric}_0\,\|^2 = \frac{n-1}{n}\cdot\|D\|^4 \quad \text{and} \quad \|\,\mathrm{Rm}\,\|^2 = \tfrac{1}{2}\left(\|D\|^4 + \|D^2\|^2\right).$$

For getting a concrete estimate we choose D diagonal, with eigenvalues $\lambda_1 = n-2$ and $\lambda_2 = \ldots = \lambda_{n-1} = -1$. After using (30) and a straightforward computation we obtain

$$\|\mathrm{Rm}_I\|^2 + \left\|\mathrm{Rm}_{\mathrm{Ric}_0}\right\|^2 = \tfrac{1}{2}(n-1)(n-2)(2n-3)$$
$$\leq \frac{(n-1)(n-2)(2n^2-8n+9)}{2(n-3)} = \frac{1}{n-3} \cdot \|\,\mathrm{Rm}\,\|^2,$$

which shows the claim. ■

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