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Miles Simon

A class of Riemannian manifolds that pinch when evolved by Ricci flow

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Abstract. The purpose of this paper is to construct a set of Riemannian metrics $\mathcal{C}(X)$ on a manifold X with the property that $g_0 \in \mathcal{C}(X)$ will develop a *pinching singularity* in finite time when evolved by Ricci flow. More specifically, let $X = \mathbf{R} \times N^n$, where N^n is an arbitrary closed manifold of dimension $n \geq 2$ which admits an Einstein metric of positive curvature. We construct a (non-empty) set of warped product metrics $\mathcal{C}(X)$ on the non-compact manifold X such that if $g_0 \in \mathcal{C}(X)$, then a smooth solution $g(t) \in \mathcal{C}(X)$, $t \in [0, T)$ to the Ricci flow equation exists for some maximal constant T, $0 < T < \infty$, with initial value $g(0) = g_0$, and

$$\sup_{x \in K, t \in [0,T)} |\operatorname{Riem}(g(t))| = \infty,$$
$$\sup_{x \in X - K, t \in [0,T)} |\operatorname{Riem}(g(t))| < \infty,$$

where K is some compact set $K \subseteq X$.

1. Introduction

Let X be a manifold with fixed differential structure. We will use the notation (X, g) to denote the manifold X equipped with the Riemannian metric g. Given g_0 on X we wish to find a smooth solution $(X, g(t))$ to the equation

$$\left.\begin{aligned} \frac{\partial}{\partial t} g(t)(v_p, w_p) &= -2\,{}^{g(t)}R(v_p, w_p), \forall p \in X, v_p, w_p \in T_pX, t \in [0, T), \\ g(0) &= g_0, \end{aligned}\right\} \tag{1.1}$$

where ${}^{g(t)}R(v_p, w_p)$ is the Ricci curvature of X with respect to the metric $g(t)$ in direction (v_p, w_p) at p, and $T > 0$ is some constant (possibly $T = \infty$). We say that $(X, g(t))$ evolves by *Ricci flow* with starting or initial metric g_0 if it satisfies (1.1). Ricci flow is a tool that helps us to examine the geometry of manifolds, and was introduced by Hamilton in [Ha 1].

In this paper we are interested in solutions $(X, g(t))$, $t \in [0, T)$ to (1.1) that exist for some finite constant T, $0 < T < \infty$, and for which $g(t)$ *pinches* as $t \to T$.

M. Simon: Universität Freiburg, Eckerstraße 1, 79104 Freiburg, Germany.
e-mail: msimon@bingo.mathematik.uni-freiburg.de

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The purpose of this paper is to show that pinching singularities can form under Ricci flow, and to obtain a better understanding of the properties of such singularities. That pinching can occur under Ricci flow was conjectured by Hamilton in [Ha 2]. If we understood pinching singularities then we could possibly extend Ricci flow in a weak sense past such singularities. (see [Ha 2] sec. 3, "Intuitive solutions").

For our purposes, the following simple definition of pinching will suffice.

Pinching definition 1.1. *Let X be a manifold with fixed differentiable structure, and let $g(t)$, $t \in [0, T)$ be a continuous 1-parameter family of Riemannian metrics on X, where $0 < T < \infty$ is some constant. We say that $(X, g(t))$ pinches on a set $K \subseteq X$ as $t \to T$ (or at T) if*

$$\sup_{p \in X} {}^{g(t)}|{}^{g(t)}\text{Riem}(p)|^2 < \infty \quad \textit{for all } \; t \in [0, T)$$

$$\sup_{(p,t) \in K \times [0,T)} {}^{g(t)}|{}^{g(t)}\text{Riem}(p)|^2 = \infty$$

$$\sup_{(p,t) \in (X - \bar{\Omega}) \times [0,T)} {}^{g(t)}|{}^{g(t)}\text{Riem}(p|^2 < \infty \quad \textit{for all} \quad \textit{open} \quad \Omega, \textit{satisfying} K \subseteq \Omega,$$

where ${}^{g(t)}|{}^{g(t)}\text{Riem}|$ is the norm of the full Riemannian curvature tensor at time t.

Here, the norm ${}^{g(t)}|\cdot|$ is the standard norm induced from $g(t)$ for tensors on X (see the definitions in chapter two for more details). If $(X, g(t))$, $t \in [0, T)$ pinches on K at time T, then we call $T > 0$ the blow up time (since curvature *blows up* somewhere at time $t = T$).

Hamilton showed in [Ha 1], that if X^3 is a three dimensional compact, closed manifold with metric g_0, and (X, g_0) has positive Ricci curvature everywhere, then there exists some maximal constant T, $0 < T < \infty$ such that (1.1) has a smooth solution $(X^3, g(t))$, $t \in [0, T)$ and

$$\sup_{t \in [0,T)} {}^{g(t)}|{}^{g(t)}\text{Riem}(p)|^2 \to \infty \quad \text{as } t \to T, \quad \text{for all } \; p \in X^3.$$

Hence pinching will never occur when we do not have negative curvature somewhere (in the compact case). For this reason we expect negative curvature to play an important role in pinching, as it does in this paper.

We shall illustrate pinching with an intuitive solution. Let $\tilde{X}^3 = F_0(\mathbf{R} \times S^2) \subseteq \mathbf{R}^4$ be the rotationally symmetric three dimensional hyper surface obtained from the imbedding $F_0 : \mathbf{R} \times S^2 \to \mathbf{R}^4$, given by

$$F_0(x, \alpha) = (x, r_0(x) i(\alpha)), \quad \text{for } \; (x, \alpha) \in \mathbf{R} \times S^2,$$
$$r_0(x) = \sqrt{a^2 x^2 + b^2},$$

where $i : S^2 \to \mathbf{R}^3$ is the standard embedding of S^2 into $\mathbf{R}^3$, and a^2, b^2 are constants satisfying $0 < a^2 << b^2 << 1$. We call $r_0 : \mathbf{R} \to \mathbf{R}$ the generating function of $\tilde{X}$. We note that $\tilde{X}$ is asymptotic to a cone as $|x| \to \infty$, and is close to a (3-dim) cylinder of radius $|b| > 0$ for $x \in [-N, N]$ for some constant $N = N(a, b) > 0$. Since we are only interested in the intrinsic properties of $\tilde{X}$, we consider the equivalent Riemannian manifold $(\mathbf{R} \times S^2, g_0)$, where g_0 is the pull back of the metric $\tilde{g}_0$ under the mapping F_0, and $\tilde{g}_0$ is the metric on $\tilde{X}$ induced from $\mathbf{R}^4$. Let $(\mathbf{R} \times S^2, g(t))$, $t \in [0, T)$ be the solution to (1.1) with $g(0) = g_0$, where $[0, T)$ is the maximal time interval for which the solution exists (theorem 3.4 guarantees that such a solution exists when the initial hyper surface is smooth enough). We will call $([-N, N] \times S^2, g(t))$ the *neck* of the Riemannian manifold $(\mathbf{R} \times S^2, g(t))$ at time t. Since the neck at time zero is close to a long thin cylinder of radius $|b|$, we expect that the high positive intrinsic curvature there should force the radius of the neck to shrink quickly towards zero under (1.1). Far away from the neck, the manifold is more like a cone with each slice of this cone being an S^2 with very large radius. Hence the intrinsic curvature there is relatively small, and so the radius of each slice of the cone should shrink very slowly under (1.1). We expect at some finite time $t = T > 0$ that the middle of the neck will *pinch* leaving (topologically speaking) cone like manifolds either side. As time approaches T, the radius of the S^2 slice in the middle of the neck will approach zero, and so the curvature there will approach infinity.

This picture is the basis for the following definition. Let N^n be any given closed manifold ($\dim N^n = n \geq 2$) that admits an Einstein metric γ (for example $N^n = S^n$ with the standard metric). The manifolds we shall be concerning ourselves with are $X = \mathbf{R} \times N^n$. We shall principally be interested in warped product metrics. Let $\mathcal{M}(X) = \{C^\infty \text{ Riemannian metrics on } X\}$.

Definition 1.2. *Let $N^n (n \geq 2)$ be a closed manifold that admits a smooth Einstein metric γ. We define the set of smooth warped product metrics $\mathcal{W}(N, \gamma) \subseteq \mathcal{M}(\mathbf{R} \times N^n)$ to be the set of $g \in \mathcal{M}(\mathbf{R} \times N^n)$ which can be written*

$$g(x, q) = h(x) \oplus r^2(x)\gamma(q),$$

for some arbitrary C^∞ metric h on $\mathbf{R}$, and some arbitrary C^∞ function $r : \mathbf{R} \to \mathbf{R}$. Here, γ is a fixed Einstein metric on N. We define $\mathcal{C}(N, \gamma) \subseteq \mathcal{W}(N, \gamma)$ to be the set of

$$g(x, q) = h(x) \oplus r^2(x)\gamma(q) \in \mathcal{W}(N, \gamma),$$

that satisfy

$$^{g}R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \leq 0, \ \frac{\partial}{\partial x} \in T_x\mathbf{R}, \quad \textit{for all } x \in \mathbf{R} \tag{1}$$

$$^{g}R(V_q, V_q) \geq 0 \quad \textit{for all } V_q \in T_qN, \quad \textit{for all } q \in N \tag{2}$$

$$\inf_{x \in \mathbf{R}} r(x) > 0. \tag{3}$$

The following derivative and lower order bounds will often be assumed for warped product metrics $g \in \mathcal{W}(N, \gamma)$. We say $g(x, q) = h(x) \oplus r^2(x)\gamma(q)$ satisfies (4) if

$$\begin{aligned} &\sup_{x \in \mathbf{R}} h_{xx} < \infty, \ \inf_{x \in \mathbf{R}} h_{xx} > 0, \ \inf_{x \in \mathbf{R}} r(x) > 0, \\ &\sup_{x \in \mathbf{R}} \left(\left|\left(\frac{\partial}{\partial x}\right)^j h\right| + \left|\left(\frac{\partial}{\partial x}\right)^j \log r\right|\right) < \infty, \quad \text{for all } j \in \{1, 2, ...\}, \end{aligned} \tag{4}$$

where here $|\cdot|$ is the standard dot product for $\mathbf{R}$, and $(\frac{\partial}{\partial x})^j$ is the operator $\frac{\partial}{\partial x}$ to the power j. We shall often talk of a smooth solution $(X, g(t))$, $t \in [0, T]$ to (1.1). This will mean that $g \in C^\infty(X \times [0, T])$. Condition (1) is somewhat non-standard, as most of the work on Ricci flow has been done for manifolds that satisfy some positive curvature conditions. There are a plethora of such conditions, and we refer the reader to [Ha 2], section 5, for a thorough survey of the sorts of conditions one initially assumes, and then shows are preserved by Ricci flow. Note that on the 2-sphere ([Ch]) curvature becomes positive after a short time, and so negative curvature will not be preserved by Ricci flow there.

1.1. Main results

The main results are presented in the following two theorems.

Conservation theorem (thm. 5.1). *Assume that* $g_0(x, q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{C}(N, \gamma)$*, and* g_0 *satisfies (4). Then there exists a maximal constant* $T > 0$ *such that (1.1) has a unique smooth solution*

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q), t \in [0, T)$$

satisfying (4). This solution satisfies $g(t) \in \mathcal{C}(N, \gamma)$ *for all* $t \in [0, T)$.

As a corollary to this theorem we see that

$$\begin{aligned} \frac{\partial}{\partial t}(r^2(x, t)\gamma(q)(V, V)) &= \frac{\partial}{\partial t} g(x, q, t)(V, V) \\ &= -2\,{}^{g}\!R(V, V) \le 0 \quad \text{for all } V \in TN^n, \end{aligned}$$

and hence for fixed x, $r(x, t)$ is non-increasing as a function of time.

In order to force the manifold to pinch, we need to assume certain things about the growth of r_0 as $|x| \to \infty$. More specifically, we need the quadratic growth condition

$$\begin{aligned} r_0^2(x) &\le a^2\rho_0^2(x) + b^2 \\ 0 < a^2 &< \frac{k}{(n-1)}, \\ b^2 &> 0, \end{aligned} \tag{5}$$

where here $\rho_0(x)$ is the distance function with respect to the metric h_0 from some fixed $x_0 \in \mathbf{R}$:

$$\rho_0(x) = {}^{h_0}\mathrm{dist}(x_0, x),$$

and a^2, b^2 are constants satisfying the stated conditions, and k is the constant of the Einstein metric γ,

$${}^{\gamma}\!R(V_q, W_q) = k\gamma(V_q, W_q) \quad \text{for all } q \in N, V_q, W_q \in T_qN.$$

It is worth while noting that when $(N, \gamma) = (S^n, d\Omega^2)$ is the n dimensional sphere with the standard metric, then $\frac{k}{(n-1)} = {}^{\gamma}\mathrm{sec} = 1$, where ${}^{\gamma}\mathrm{sec}$ is the sectional curvature of (S^n, γ) (of any plane). Clearly from (5) we must have $k > 0$. We will also need that

$$r_0(x) \to \infty \quad \text{as } |x| \to \infty. \tag{6}$$

Pinching theorem (thm 7.2). *Assume that* $g_0(x,q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{C}(N,\gamma)$ *satisfies (4), (5), and (6). Then there exists a constant,* $0 < A < \infty$, *such that the maximal warped product solution*

$$g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{W}(N,\gamma),\ t \in [0,T)$$

to (1.1) of the conservation theorem 5.1, satisfies

$$\inf_{x\in[-A,A],t\in[0,T)} r^2(x,t) = 0$$
$$\inf_{x\in(-\infty,-A]\cup[A,+\infty),t\in[0,T)} r^2(x,t) > 0.$$

This implies that the manifold pinches on the compact set $K = [-A,A] \times N \subseteq \mathbf{R} \times N^n$ *at time* T.

The study of singularities that can form under various different geometric flows has been extensively examined. Huisken studied the types of singularities that occur under mean curvature flow, and in particular their asymptotic behaviour [Hu 1]. In the last section of [Hu 1] he examines periodic rotationally symmetric surfaces with positive mean curvature (which always develop a singularity). He shows that the singularities satisfy a certain blow up estimate, and behave asymptotically like cylinders. The main tool of [Hu 1] is the parabolic maximum principle for tensors. See [Ha 1] or [Ha 3] for a reference. The author [Si] generalised some of the results of [Hu 1] to higher dimensions. Smoczyk [Sm] showed that certain periodic rotationally symmetric surfaces embedded in Euclidean space (Hyperbolic space) pinch when the mean curvature is positive (bigger than two). Grayson [Gr] has created a class of rotationally symmetric barrier surfaces, each of which is asymptotic to a cylinder in its middle region and grows exponentially outside this region. When we evolve one of these barriers by mean curvature flow, the resulting evolving surface will pinch at a finite time $t = T > 0$ in its middle region, where T depends on the length and radius of the initial cylindrical middle region. Ecker [Ec] showed the existence of a class of evolving symmetric barrier hyper-surfaces each of which is asymptotic to a cone at plus and minus infinity and pinches at a finite time $t = T > 0$ at its middle point. In this case, T depends on the diameter of the middle region of the barrier at time zero, and the angle of the cone to which the barrier is asymptotic. The barriers of [Gr] and [Ec] may be used to find a large class of manifolds that pinch under mean curvature flow. Dziuk and Kawohl [DK] showed that periodic rotationally symmetric surfaces of positive curvature, which have one minimum on each period and satisfy a certain monotonicity condition on the derivative of the curvature, pinch at exactly one point. Altschuler, Angenent and Giga [AAG], showed that any compact connected rotationally symmetric hypersurface that pinches under mean curvature flow does so at finitely many discrete points. For a discussion on manifolds that should "intuitively" pinch under Ricci flow, we refer the reader to [Ha 3], Sect. 3: "Intuitive solutions".

The similarities between hyper-surfaces that flow by mean curvature flow, and manifolds whose metric is evolving by Ricci flow are numerous. For example: an n dimensional sphere sitting in $\mathbf{R}^{n+1}$ evolving by mean curvature flow will shrink at the rate

$$\text{radius } R(t) = \sqrt{R_0^2 - 2nt}, \quad \text{where } R(0) = R_0.$$

An n dimensional sphere whose metric evolves by Ricci flow satisfies

$$g(t) = \left(R_0^2 - 2(n-1)t\right) d\Omega^2, \quad \text{where } g(0) = g_0 = R_0^2 d\Omega^2,$$

and $d\Omega^2$ is the standard metric on S^n. There are other self-similar hyper surfaces in mean curvature flow that have analogies in Ricci flow. This led the author to look for manifolds, similar to those described in [Ec], that will pinch when evolved by Ricci flow.

2. Definitions and curvature formulae

We state here the main relations for the Riemannian curvature tensor of a general warped product. We refer to the book by O'Neill [ON], for a reference. That is, we will express the curvatures of

$$\left(M \times N, g(x,q) = h(x) \oplus r^2(x)\gamma(q)\right)$$

in terms of curvatures of (M, h), curvatures of (N, γ) and derivatives of r. As was mentioned in the introduction, we will be interested in $M = \mathbf{R}$, however in this chapter we make no such restriction.

As will be standard in this paper, lower-case Roman letters $\{i, j, k, l\}$ refer to co-ordinates in M, lower-case Greek letters refer to co-ordinates in N, and lower-case Roman letters $\{a, b, c, d, p, q\}$ refer to general co-ordinates in $M \times N$:

$$\left\{\frac{\partial}{\partial i}\right\}_{i=1}^{m} \in TM, \left\{\frac{\partial}{\partial \alpha}\right\}_{\alpha=1}^{n} \in TN, \left\{\frac{\partial}{\partial a}\right\}_{a=1}^{n+m} \in T(N \times M).$$

For $M = \mathbf{R}$, the letter x refers to the standard co-ordinate

$$\frac{\partial}{\partial x} \in T\mathbf{R}.$$

${}^g\nabla$ refers to the gradient with respect to the metric g. Hence, for a function $f : M \times N \to \mathbf{R}$,

$${}^g\nabla f = g^{ab}\left(\frac{\partial}{\partial a} f\right)\frac{\partial}{\partial b}.$$

The Hessian of a function $f : M \times N \to \mathbf{R}$ with respect to the metric g is defined using co-ordinates by

$${}^g\nabla_a {}^g\nabla_b f = {}^g\text{Hess}\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\right) f = \frac{\partial}{\partial a}\frac{\partial}{\partial b} f - {}^g\Gamma_{ab}^c \frac{\partial}{\partial c} f,$$

where

$${}^g\Gamma_{ab}^c = g^{cd\,g}\Gamma_{ab,d}$$

and

$$ {}^g\Gamma_{ab,d} = \frac{1}{2}\left(\frac{\partial}{\partial a}g_{db} + \frac{\partial}{\partial b}g_{da} - \frac{\partial}{\partial d}g_{ba}\right) $$

are the Christoffel symbols with respect to the metric g. The Laplacian of a function $f : M \times N \to \mathbf{R}$ with respect to the metric g is defined using the Hessian:

$$ {}^g\Delta f = g^{ab}\,{}^g\mathrm{Hess}\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\right) f. $$

As is standard, the Riemannian curvature tensor on $M \times N$ with respect to the metric g is defined by

$$ {}^gR(u, v, w) = -{}^g\nabla_u {}^g\nabla_v w + {}^g\nabla_v {}^g\nabla_u w + {}^g\nabla_{[u,v]} w. $$

We use the convention

$$ {}^gR_{abcd} = g\left({}^gR\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}\right), \frac{\partial}{\partial d}\right) $$

for the Riemannian curvature tensor with respect to the metric g on $M \times N$. With this convention, the sectional curvature of the plane spanned by the vectors v and w is then given by

$$ {}^g\mathrm{sec}(v, w) = \frac{{}^gR_{abcd} v^a w^b v^c w^d}{{}^gQ(v, w)} = \frac{g({}^gR(v, w, v), w)}{{}^gQ(v, w)}, $$
$$ {}^gQ(v, w) = g(v, v)g(w, w) - g(v, w)^2, $$

which is the opposite from the classical indexing convention, but is the convention used by Hamilton in his Ricci flow papers. The Ricci curvature tensor is then denoted

$$ {}^gR_{ab} = {}^gR_{acbd} g^{cd}. $$

${}^g|\;|$ refers to the norm of a tensor with respect to the metric g. So for example, if we take a tensor $T = \{T_{ab}\}$, $a, b \in 1$ to $m + n$, then

$$ {}^g|T|^2 = g^{ac} g^{bd} T_{ab} T_{cd}. $$

Notice in these definitions that we have used a superscript g to show the dependence on the metric g. This notation shall be standard throughout this paper. For example

$$ {}^h\mathrm{Hess}\left(\frac{\partial}{\partial i}, \frac{\partial}{\partial j}\right)(f) = {}^h\nabla_i {}^h\nabla_j f $$

refers to the Hessian of a function $f : M \to \mathbf{R}$ with respect to the metric h on M. The end of a proof will be denoted by a $\diamondsuit$.

Proposition 2.1. *Let $M^m \times N^n$ be a manifold with smooth metric*

$$g(p,q) = h(p) \oplus r^2(p)\gamma(q).$$

Then

$$
\begin{aligned}
{}^{g}R_{ijkl} &= {}^{h}R_{ijkl} && (2.1)\\
{}^{g}R_{i\alpha j\beta} &= P_{ij} g_{\alpha\beta} \quad \textit{where} \quad P_{ij} = -\frac{{}^{h}\nabla_i {}^{h}\nabla_j r}{r}, && (2.2)\\
{}^{g}R_{\alpha\beta\tau\sigma} &= r^2({}^{\gamma}R_{\alpha\beta\tau\sigma}) + r^2 {}^{h}|{}^{h}\nabla r|^2(\gamma_{\alpha\sigma}\gamma_{\tau\beta} - \gamma_{\beta\sigma}\gamma_{\tau\alpha}) && (2.3)\\
{}^{g}R_{ij} &= {}^{h}R_{ij} + nP_{ij} && (2.4)\\
{}^{g}R_{i\alpha} &= 0 && (2.5)\\
{}^{g}R_{\alpha\beta} &= {}^{\gamma}R_{\alpha\beta} + g_{\alpha\beta}\frac{1}{r^2}(-r{}^{h}\Delta r + (1-n){}^{h}|{}^{h}\nabla r|^2)\\
\textit{which implies } {}^{g}R_{\alpha\beta} &= {}^{\gamma}R_{\alpha\beta} + g_{\alpha\beta}\frac{1}{2r^2}(-{}^{h}\Delta r^2 + (4-2n){}^{h}|{}^{h}\nabla r|^2). && (2.6)
\end{aligned}
$$

Hence if γ is an Einstein metric with ${}^{\gamma}R_{\alpha\beta} = k\gamma_{\alpha\beta}$ (k is a constant), then the last equation becomes

$$ {}^{g}R_{\alpha\beta} = f g_{\alpha\beta}, \tag{2.7}$$

where $f = \frac{1}{r^2}(k - r{}^{h}\Delta r + (1-n){}^{h}|{}^{h}\nabla r|^2) = \frac{1}{2r^2}(-{}^{h}\Delta r^2 + (4-2n){}^{h}|{}^{h}\nabla r|^2 + 2k)$.

Proof. The above formulae (2.1)–(2.6) can be found in a co-ordinate free form in the chapter on warped product metrics (204–211) in the book by O'Neill [ON]. □

Lemma 2.2. *Let*

$$g(x,q) = h(x) \oplus r^2(x)\gamma(q) \in \mathcal{W}(N,\gamma)$$

be a smooth warped product metric on the manifold $\mathbf{R} \times N$*, where γ is an Einstein metric with Einstein constant k , that satisfies (1) and (2). Then*

$$ {}^{h}|{}^{h}\nabla r|^2 \le \frac{k}{(n-1)} \tag{2.9}$$

$$ {}^{h}\Delta r \ge 0 \tag{2.10}$$

$$ {}^{h}\Delta r \le \frac{k}{r}. \tag{2.11}$$

Proof. Using condition (1), equation (2.4), and $M = \mathbf{R}$ we derive

$$0 \ge {}^{g}R_{xx} = nP_{xx} = -n\frac{{}^{h}\nabla_x {}^{h}\nabla_x(r)}{r} = -nh_{xx}{}^{h}\Delta r,$$

which implies (2.10).

In view of (2.7) and condition (2) we see that

$$ r{}^{h}\Delta r + (n-1){}^{h}|{}^{h}\nabla r|^2 \le k. \tag{2.12}$$

Both of the terms on the left-hand side of the above equation are positive (see 2.10), and so we obtain (2.9) and (2.11). □

Lemma 2.3. *Let*

$$g(x,q) = h(x) \oplus r^2(x)\gamma(q) \in \mathcal{W}(N,\gamma)$$

be a warped product metric satisfying the same conditions as in Lemma 2.2. Then

$$^{g}|^{g}\mathrm{Riem}(x,q)|^2 \leq \frac{1}{r^4(x)} c(n,\gamma), \tag{2.13}$$

where $c(n,\gamma)$ is a constant depending on n and γ.

Proof. Choose geodesic co-ordinates for g at a point (x,q) in $\mathbf{R} \times N$. So $h_{xx} = g_{xx} = 1$ and $r^2\gamma_{\alpha\beta} = g_{\alpha\beta} = \delta_{\alpha\beta}$. Then by (2.2), and (2.10) we have

$$^{g}|^{g}R_{x\alpha x\beta}| = {}^{g}|P_{xx}g_{\alpha\beta}| = n|P_{xx}| = \frac{^{h}\Delta r}{r} \leq \frac{k}{r^2}. \tag{2.14}$$

Using the formula 2.3 for $^{g}R_{\alpha\beta\tau\sigma}$ we see that

$$\begin{aligned} ^{g}|^{g}R_{\alpha\beta\tau\sigma}|^2 &= r^{4g}|(^{\gamma}R_{\alpha\beta\tau\sigma})|^2 + 2r^{4h}|^{h}\nabla r|^2 g\big(^{\gamma}R_{\alpha\beta\tau\sigma}, \gamma_{\alpha\sigma}\gamma_{\tau\beta} - \gamma_{\beta\sigma}\gamma_{\tau\alpha}\big) \\ &\quad + \frac{1}{r^4}{}^{h}|^{h}\nabla r|^{4g}|(g_{\alpha\sigma}g_{\tau\beta} - g_{\beta\sigma}g_{\tau\alpha})|^2 \\ &= \frac{1}{r^4}{}^{\gamma}|(^{\gamma}R_{\alpha\beta\tau\sigma})|^2 + \frac{2}{r^4}{}^{h}|^{h}\nabla r|^2\gamma\big(^{\gamma}R_{\alpha\beta\tau\sigma}, \gamma_{\alpha\sigma}\gamma_{\tau\beta} - \gamma_{\beta\sigma}\gamma_{\tau\alpha}\big) \\ &\quad + \frac{c(n)}{r^4}{}^{h}|^{h}\nabla r|^4, \end{aligned} \tag{2.15}$$

where we have used,

$$\begin{aligned} ^{g}|^{\gamma}R_{\alpha\beta\tau\sigma}|^2 &= g^{\alpha\alpha'}g^{\beta\beta'}g^{\tau\tau'}g^{\sigma\sigma'\gamma}R_{\alpha\beta\tau\sigma}{}^{\gamma}R_{\alpha'\beta'\tau'\sigma'} \\ &= \frac{1}{r^8}{}^{\gamma}|^{\gamma}R_{\alpha\beta\tau\sigma}|2. \end{aligned}$$

Since γ is a fixed, smooth metric on the compact manifold N, substituting (2.9) into (2.15), we get

$$^{g}|^{g}R_{\alpha\beta\tau\sigma}|^2 \leq \frac{1}{r^4}c(n,\gamma). \tag{2.26}$$

The curvature $^{g}R_{ijkl}$ is trivially bounded as $^{g}R_{ijkl} = {}^{h}R_{ijkl} \equiv 0$ on a one dimensional manifold. Writing

$$^{g}|^{g}\mathrm{Riem}|^2 = {}^{g}|^{g}R_{\alpha\beta\tau\sigma}|^2 + {}^{g}|^{g}R_{i\alpha j\beta}|^2 + {}^{g}|^{g}R_{ijkl}|^2 = {}^{g}|^{g}R_{\alpha\beta\tau\sigma}|^2 + {}^{g}|^{g}R_{i\alpha j\beta}|^2,$$

and substituting (2.16) and (2.15) into the above, the result then follows. □

3. Existence, uniqueness and a priori estimates in the warped product case

This chapter is concerned with showing that if g_0 is an arbitrary warped product metric $g_0 \in \mathcal{W}(N, \gamma)$ which satisfies (4), then there exists a unique warped product solution $g(t) \in \mathcal{W}(N, \gamma)$, $t \in [0, T]$ to (1.1), for some constant $T > 0$, which satisfies (4) and $g(0) = g_0$. We will also prove uniqueness and certain a priori bounds for derivatives of such a solution. To accomplish both of these tasks, we will need the following theorem of Shi [Sh].

Theorem [Shi (1.1)]. *Let (X, g_0) be a non-compact, complete manifold without boundary, with*

$$^{g_0}|^{g_0}\mathrm{Riem}|^2 \leq v,$$

where v is some constant, and g_0 is smooth. Then there exists some constant $T(dim(X), v) > 0$ such that (1.1) has a smooth solution $g(t)$ for all $t \in [0, T]$, and for all non-negative integers j we have

$$^{g(t)}|^{g(t)}\nabla^j \mathrm{Riem}|^2 \leq \frac{c(\dim(X), j, v)}{t^j} \quad \textit{for all } t \in (0, T],$$

where $^{g(t)}\nabla^j$ Riem is the tensor obtained by taking j co-variant derivatives of the Riemannian curvature tensor $^{g(t)}$Riem.

Theorem 3.1. *Let $g_0(x, q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{W}(N, \gamma)$ satisfy (4). Then there exists a constant $T > 0$, and a smooth solution*

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q) \in \mathcal{W}(N, \gamma), t \in [0, T]$$

to (1.1) , with $g(0) = g_0$, and $g(t)$ satisfies (4) for all $t \in [0, T]$.

Proof of Theorem 3.1. To show that there is a short time solution of (1.1), we may use theorem (1.1) of Shi, since (4) with (2.14) and (2.15) implies that

$$^{g_0}|^{g_0}\mathrm{Riem}|^2 \leq v$$

for some v depending on the bounds of (4).

To show that there is a *warped product solution* $g(t) \in \mathcal{W}(N, \gamma)$, $t \in [0, T]$ of (1.1) (for some $T > 0$) requires a closer examination of the paper by Shi [Sh]. Shi mostly examines a *modified* Ricci flow. This modified flow was first employed by DeTurck [De] to give a short proof of short time existence to the Ricci flow on compact manifolds without boundary, and is often referred to as Ricci DeTurck flow. (The first proof of short time existence was proved by Hamilton and can be found in [Ha1]). For the rest of this proof (and only for this proof) we shall denote a solution to the *modified* flow as $g(t)$ and the corresponding solution to the *unmodified* flow as $\hat{g}(t)$. In the first part of this proof we show that the *modified* flow has a maximal solution that is a warped product in $\mathcal{W}(N, \gamma)$. In the next part of the proof we show that the corresponding *unmodified* solution is also a warped product.

Lemma 2.1 in [Sh] gives us the evolution equation for the metric $g(t)$ in terms of the initial metric g_0 as follows.

$$
\begin{aligned}
\frac{\partial}{\partial t} g_{ab} = & g^{cd}\tilde{\nabla}_c\tilde{\nabla}_d g_{ab} - g^{cd} g_{ap}\tilde{g}^{pq}\tilde{R}_{bcqd} - g^{cd} g_{bp}\tilde{g}^{pq}\tilde{R}_{acqd} \\
& + \frac{1}{2} g^{cd} g^{pq}(\tilde{\nabla}_a g_{pc}\cdot\tilde{\nabla}_b g_{qd} + 2\tilde{\nabla}_c g_{ap}\cdot\tilde{\nabla}_q g_{bd} \\
& - 2\tilde{\nabla}_c g_{ap}\cdot\tilde{\nabla}_d g_{bq} - 4\tilde{\nabla}_a g_{pc}\cdot\tilde{\nabla}_d g_{bq}),
\end{aligned} \tag{3.1}
$$

$$\tilde{g} = g_0, \quad \tilde{\nabla} = {}^{g_0}\nabla, \quad \widetilde{\text{Riem}} = {}^{g_0}\text{Riem}.$$

Now if we assume that our solution $g(t) \in \mathcal{W}(N, \gamma)$ is a warped product with

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q),$$

and $g(x, q, 0) = g_0(x, q) = dx^2(x) \oplus r_0^2(x)\gamma(q)$, then this equation breaks into a coupled system of two evolution equations,

$$
\begin{aligned}
\frac{\partial}{\partial t} h_{xx} = & h^{xx}\frac{\partial}{\partial x}\frac{\partial}{\partial x} h_{xx} - 2\frac{\tilde{r}^2}{r^2} h_{xx}\tilde{h}^{xx}\tilde{R}_{xx} - \frac{3}{2} h^{xx} h^{xx}(\frac{\partial}{\partial x}(h_{xx}))^2 \\
& + \frac{n}{2r^4}(\frac{\partial}{\partial x} r^2)^2 \\
\frac{\partial}{\partial t}(r^2\gamma_{\alpha\beta}) = & h^{xx}\frac{\partial}{\partial x}\frac{\partial}{\partial x}(r^2\gamma_{\alpha\beta}) - 2k\frac{\gamma_{\alpha\beta} r^2}{r^2} \\
& + \frac{2}{n}\frac{\gamma_{\alpha\beta} r^2}{r^2}\tilde{R}_{xx}\left(\tilde{r}^2\tilde{g}^{xx} - g^{xx} r^2\right) - (r^2\gamma_{\alpha\beta})\frac{h^{xx}(\frac{\partial}{\partial x}(r^2))^2}{r^4}
\end{aligned}
$$

$$\tilde{h}(\cdot) = h_0(\cdot), \tilde{r}^2(\cdot) = r_0^2(\cdot), \tilde{R}_{xx} = {}^{g_0}R_{xx}. \tag{3.2}$$

In particular notice that as long as

$$(1-\zeta)\delta_{ij} \le h_{ij} \le (1+\zeta)\delta_{ij} \quad \text{for all } t \in [0, T], \text{ for some } 0 < \zeta < 1$$

this system is parabolic. We note also that a solution $g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q)$ to (3.2) also solves (3.1), and hence we may use the same techniques that Shi uses to obtain a priori estimates. Let $B_\delta(x_1, q_1) \in \mathbf{R} \times N$ be a ball of radius $\delta > 0$. We may use Theorem 2.5 and Lemma 4.1 of [Sh] to obtain the a priori estimates

$$
\begin{aligned}
(1-\zeta) g_0 \le g(\cdot, t) \le (1+\zeta) g_0, \\
{}^{g_0}|\tilde{\nabla}^j g(\cdot, t)| \le c_j(\cdot) \quad \text{for all } j \in \{1, 2, 3, ...\}
\end{aligned} \tag{3.3}
$$

etc. for some $T = T(n, v) > 0$ and some $\zeta(n, v, T, g_0) > 0$, and for all $t \in [0, T]$, $(x, q) \in B_\delta(x_1, q_1)$, where the dependence of c_j is

$$c_j = c_j\Big(n, v, T, \sup_{x \in B_{2\delta}(x_1, q_1), k \in \{1, \dots, j\}} {}^{g_0}|\tilde{\nabla}^k \widetilde{\text{Riem}}|\Big),$$

where $\widetilde{\text{Riem}} = {}^{g_0}\text{Riem}$. Note that Shi does not include the dependence of c_j on ${}^{g_0}|\tilde{\nabla}^k \widetilde{\text{Riem}}|$, he simply writes $c_j(n, v, T, \tilde{g}, \delta)$ (as this is all that he needs). For later use, we have written the explicit dependence here, and we refer the reader to the proof of lemma 4.1 of [Sh], particularly estimate (7), from which this dependence can be seen.

As we are in the special warped product setup, this implies that

$$\begin{aligned}(1-\zeta)\delta_{ij} \leq h_{ij}(\cdot,t) \leq (1+\zeta)\delta_{ij},\\ (1-\zeta)r_0 \leq r(\cdot,t) \leq (1+\zeta)r_0,\\ |(\frac{\partial}{\partial x})^j h(\cdot,t)_{xx}| \leq c_j(\cdot)|(\frac{\partial}{\partial x})^j \log \mathrm{r}^2(\cdot,\mathrm{t})| \leq \mathrm{c_j}(\cdot)\\ \text{for all } j \in \{1,2,3,\dots\}\end{aligned} \tag{3.4}$$

etc. for all $t \in [0,T]$, where $T = T(n,v) > 0$ is some constant (as above), and the dependence of c_j is

$$c_j = c_j\Big(n, v, T, \inf_{x\in\mathbf{R}} r_0(x), \sup_{x\in B_{2\delta}(x_1,q_1), k\in\{1,\dots,j+2\}} |(\frac{\partial}{\partial x})^k \log \mathrm{r}_0|\Big),$$

and ζ is as above. Here we have used that ${}^{g_0}|\tilde{\nabla}^k \widetilde{\mathrm{Riem}}|$ may be bounded from above by

$${}^{g_0}|\tilde{\nabla}^k \widetilde{\mathrm{Riem}}(x)| \leq c\Big(\sup_{i\in\{1,2,\dots,k+2\}} |(\frac{\partial}{\partial x})^i \log \mathrm{r}_0(\mathrm{x})|, \inf_{\mathrm{x}\in\mathbf{R}} \mathrm{r}_0(\mathrm{x})\Big),$$

when $(h_0)_{xx} = \delta_{xx}$. This follows from the formulae for the curvature tensor (see Proposition 2.1). Then following Shi, we use the same argument as in the proof of Theorem 8.1 [LSU 4 , Sect. 7, Ch. VII], for $l \in 0, 1, \dots$, to construct solutions to the Dirichlet problem

$$\begin{aligned}&\Big(\overset{l}{h}(x,t), \overset{l}{r}{}^2(x,t)\gamma(q)\Big) \text{ solves (3.2)} && \text{for all } (x,q) \in [-l,l]\times N,\\ &\Big(\overset{l}{h}(l,t), \overset{l}{r}{}^2(l,t)\gamma(q)\Big) = \Big(h_0(l), r_0^2(l)\gamma(q)\Big), && \text{for all } q \in N, \text{ for all } t \in [0,T],\\ &\Big(\overset{l}{h}(x,0), \overset{l}{r}{}^2(x,0)\gamma(q)\Big) = \Big(h_0(x), r_0^2(x)\gamma(q)\Big), && \text{for all } (x,q) \in [-l,l]\times N,\\ & && \text{for all } t \in [0,T],\end{aligned}$$

where $T > 0$ is some constant $T = T(n,v)$. The estimates (3.4) hold for each solution $\Big(\overset{l}{h}(x,t), \overset{l}{r}{}^2(x,t)\gamma(q)\Big)$, and the constants in (3.4) are clearly independent of l. Hence we may let $l \to \infty$, taking subsequences if necessary, to obtain a smooth solution to (3.2)

$$\Big(\overset{l}{h}(x,t), \overset{l}{r}{}^2(x,t)\gamma(q)\Big) \xrightarrow{C^\infty} \Big(h(x,t), r^2(x,t)\gamma(q)\Big) = g(x,q,t) \quad \text{as } l \to \infty$$

that satisfies (4).

So we have shown that there are solutions to the *modified* flow that are warped product metrics in $\mathcal{W}(N,\gamma)$. Actually we explicitly used that $(h_0)_{xx} = \delta_{xx}$, but the argument still works for any smooth h_0 that satisfies (4).

To show that a corresponding solution to the *unmodified* flow is a warped product metric in $\mathcal{W}(N,\gamma)$, we must examine the relationship between it and the *modified* flow. If $g(x,t)$ is a modified solution then there is a corresponding unmodified solution given by $\hat{g}(x,t)$ where

$$g(t) = \phi_t{}^* \hat{g}(t),$$

and

$$\phi_t : \mathbf{R} \times N \to \mathbf{R} \times N$$

is a diffeomorphism that satisfies

$$\frac{\partial}{\partial t}\phi_t = g^{cd}(\Gamma^a_{cd} - \tilde{\Gamma}^a_{cd})\frac{\partial}{\partial a}, \tag{3.5}$$

where here Γ^a_{cd} are the Christoffel symbols with respect to the metric $g(t)$, and $\tilde{\Gamma}^a_{cd}$ are the Christoffel symbols with respect to the metric g_0. Since we already know that the modified solution $g(t) \in \mathcal{W}(N,\gamma)$ is a warped product metric we may calculate the Christoffel symbols ${}^g\Gamma^a_{cd}$ using the formulae in the chapter on warped product metrics (204–211) in the book by O'Neill [ON].

Substituting the Christoffel symbol formulae into (3.5) we obtain

$$\frac{\partial}{\partial t}\phi_t = h^{xx}\Big({}^h\Gamma^x_{xx} - {}^{h_0}\tilde{\Gamma}^x_{xx}\Big)\frac{\partial}{\partial x} + \frac{n}{r^2}\Big(h^{xx}\frac{\partial}{\partial x}(r^2) - h_0{}^{xx}\frac{\partial}{\partial x}(r_0^2)\Big)\frac{\partial}{\partial x}. \tag{3.6}$$

That is, ϕ_t is only a diffeomorphism in x, which we express by

$$\phi_t(x,q) = (\psi_t(x), q),$$

where $\psi_t(x) = \phi_t(x,q)$ satisfies the evolution equation (3.6).

Then $g(t) = (\psi_{-t})^* \hat{g}(t)$ implies

$$\begin{aligned} \hat{g}_{xx}(x,q,t) &= \left(\frac{\partial}{\partial x}(\psi_{-t}(x))\right)^2 g_{xx}(\psi_{-t}(x),q,t) \\ \hat{g}_{\alpha\beta}(x,q,t) &= r^2(\psi_{-t}(x),t)\gamma_{\alpha\beta}(q), \end{aligned} \tag{3.7}$$

which is clearly a warped product metric of the form

$$\hat{g}(x,q,t) = \hat{h}(x,t) \oplus \hat{r}^2(x,t)\gamma(q) \in \mathcal{W}(N,\gamma).$$

We note for later use that (3.6) with (3.7) implies that the estimates (3.4) also hold for $\hat{g}(t)$, $t \in [0,T)$. □

Lemma 3.2. *Let*

$$g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{C}(N,\gamma),\ t \in [0,T]$$

be a smooth warped product metric on $(\mathbf{R} \times N^n)$ *evolving according (1.1). If*

$$r_0^2 \le a^2\rho_0^2 + b^2$$

for some constants $a^2, b^2 > 0$ *then*

$$r^2 \le a'^2\rho^2 + b'^2 \quad \text{for all } t \in [0,T] \tag{3.8}$$

for some constants $a'^2 = a'^2(a, b, \sup_{(x,q)\in\mathbf{R}\times N, t\in[0,T]} {}^{g(t)}|{}^{g(t)}\mathrm{Riem}|) > 0$ *and* $b'^2 > 0$*, (* b'^2 *having the same dependence as* a'^2*).*

Proof. By Lemma 2.3 and (3) we havehat

$$v = \sup_{(x,q)\in \mathbf{R}\times N, t\in[0,T]} |^{g(t)}{}^{g(t)}\mathrm{Riem}| \leq \sup_{(x,q)\in \mathbf{R}\times N, t\in[0,T]} \frac{c(n,\gamma)}{r^4(x,t)} < \infty,$$

and hence we may use the estimate (139) of [Sh], Theorem 7.7 to obtain

$$\frac{1}{C_1} g_0 \leq g(t) \leq C_1 g_0 \quad \text{for all } t \in [0,T]$$

where $0 < C_1 < \infty$ is a constant depending on g_0,n,v. Applying these estimates to our warped product metric, we obtain

$$\frac{1}{C_1} r_0^2(\cdot) \leq r^2(\cdot, t) \leq C_1 r_0^2(\cdot)$$
$$\frac{1}{C_1}(h_0)_{xx} \leq h_{xx} \leq C_1 (h_0)_{xx}$$

for some constant $C_1 > 0$. Combining this with the hypothesis of the theorem, we obtain

$$r^2 \leq C_1 a^2 \rho_0^2 + C_1 b^2. \tag{3.9}$$

Writing ρ in its integral form, and using the estimate (139) again, we see that

$$\rho_0(x_1) = \int_{x_0}^{x_1} \sqrt{(h_0)_{xx}}\, dx \leq \int_{x_0}^{x_1} \sqrt{C_1 h(t)_{xx}}\, dx = \sqrt{C_1}\rho(x).$$

Substituting this inequality into (3.9) we obtain the result. □

As our manifold X is non-compact, we may *not* use the standard uniqueness argument for Ricci flow which is obtained by examining the corresponding Ricci–DeTurck flow (see for example [De]). However...

Theorem 3.3. *Let* $g_0(x,q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{W}(N,\gamma)$ *be a warped product metric on the manifold* $\mathbf{R} \times N$*, satisfying (4). Then any smooth solution* $g(t) \in \mathcal{W}(N,\gamma)$*,* $t \in [0,T]$ *to (1.1) that satisfies (4) for each* $t \in [0,T]$ *and* $g(0) = g_0$*, is unique.*

Proof. Let $h(x,t) \oplus r^2(x,t)\gamma(q)$ and $k(x,t) \oplus s^2(x,t)\gamma(q)$ be two smooth solutions to (3.2) both satisfying (4) on some short time interval $t \in [0,T]$, whose initial values are the same

$$k_0(x) \oplus s_0^2(x)\gamma(q) = h_0(x) \oplus r_0^2(x)\gamma(q) \quad \text{for all } x \in \mathbf{R}, q \in N^n.$$

Using the estimates (4), and equation (3.2), we see in [Si 2] that the function

$$|P(x,t)|^2 = \Big(\log \mathrm{h_{xx}(x,t)} - \log \mathrm{k_{xx}(x,t)}\Big)^2 + \Big(\log \mathrm{r^2(x,t)} - \log \mathrm{s^2(x,t)}\Big)^2$$

satisfies the evolution equation

$$\frac{\partial}{\partial t}|P|^2 \leq {}^{k}\Delta|P|^2 + k(V, {}^{k}\nabla|P|^2) + c^2|P|^2, \tag{3.10}$$

where c^2 is some constant, $c^2 = c^2(n, T)$, and V is the vector field given by $V(x,t) = k^{xx}\,{}^{k}\Gamma^{x}_{xx}\frac{\partial}{\partial x}$. The estimates

$$\sup_{x\in\mathbf{R}, t\in[0,T]} {}^{k}|{}^{k}\nabla|P|^2| < \infty, \quad \sup_{x\in\mathbf{R}, t\in[0,T]} {}^{k}|V| < \infty, \tag{3.11}$$

follow immediately from (4). Also ,

$$\sup_{x\in\mathbf{R}, t\in[0,T]} {}^{k(t)}|\frac{\partial}{\partial t}k| = \sup_{x\in\mathbf{R}, t\in[0,T]} {}^{g(t)}|\frac{\partial}{\partial t}g| \leq \sup_{x\in\mathbf{R}, t\in[0,T]} {}^{g(t)}|{}^{g(t)}\mathrm{Riem}| < \infty, \tag{3.12}$$

due to Proposition (2.2) and (4). Since (3.10), (3.11) and (3.12) hold, we may apply the non-compact maximum principle of [EH] to the function $|P(x,t)|^2$, to obtain

$$|P(x,t)|^2 \equiv 0,$$

where here we have also used that $|P(x,0)|^2 \equiv 0$. This show us that given any initial warped product metric $g_0 \in \mathcal{W}(N,\gamma)$ that satisfies (4), then the modified Ricci–DeTurck flow has a unique smooth warped product solution $g(t) \in \mathcal{W}(N,\gamma)$ satisfying $g(0) = g_0$ and (4) for each $t \in [0,T)$. Now we show that the previous statement is also true for Ricci flow.

Let $\hat{g}(t)$ and $\hat{h}(t)$ be two warped product solutions to the Ricci flow equation (1.1) which satisfy (4) and have the same initial values. Then $g(t) = {\phi_t}^*(\hat{g}(t))$ is a solution to (3.2) which satisfies (4), where ϕ_t is the solution to (3.5) coming from $g(t)$. Similarly $h(t) = {\psi_t}^*(\hat{h}(t))$ is a solution to (3.2) which satisfies (4), where ψ_t is the solution to (3.5) coming from $h(t)$. Hence buniqueness of solutions to (3.2) satisfying (4) (proved above), we obtain that

$$g(t) = {\psi_t}^*(\hat{h}(t)) = {\phi_t}^*(\hat{g}(t)). \tag{3.13}$$

We also know that $\psi_t = \phi_t$ since a solution to the ODE (3.5) coming from $g(t)$is unique (see Shi, Sect. (7), p. 288). The result then follows. □

Remark. This argument can be used to prove uniqueness of the Ricci flow in a more general set up for non-compact manifolds, where g_0 is not necessarily a warped product, but satisfies similar estimates to (4). Note that any solution will continue to satisfy (4) for a short time by the estimates (3.4).

Theorem 3.4. *For any metric $g_0(x,q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{W}(N,\gamma)$ that satisfies (4), there exists a maximal constant $T > 0$ (possibly $T = \infty$) and a unique smooth solution*

$$g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{W}(N,\gamma),\, t \in [0,T)$$

to (1.1) satisfying (4) for all $t \in [0,T)$, and $g(0) = g_0$. If $T < \infty$, then

$$v = \sup_{x\in\mathbf{R}, q\in N, t\in[0,T)} {}^{g(t)}|{}^{g(t)}\mathrm{Riem}(x,q)| = \infty.$$

We shall call such a solution a maximal warped product solution.

Proof. By Theorem 3.1, there is a $S > 0$, such that (1.1) (and (3.2)) has a warped product solution $g(x,t) = h(x,t) \oplus r^2(x,t)\gamma(q)$ for $t \in [0,S]$. By Theorem 3.3, any two smooth solutions to (3.2) satisfying (4) must agree. Hence there is a unique smooth solution $g(t)$ on a maximal time interval $[0,T)$ that satisfies (4). Let $T < \infty$ and assume to the contrary that $v < \infty$. Choose $S > 0$ very close to T, and construct a new solution $\overset{+}{g}(t)$, $t \in [S, S+T']$, $T' = T'(v) > 0$, to (3.2) for which $S + T' > T$, $\overset{+}{g}(S) = g(S)$, and (4) holds. We define a new one parameter family of metrics $\overset{++}{g}(t)$, $t \in [0, S+T']$ by

$$\overset{++}{g}(t) = g(t) \quad \text{for all } t \in [0,S],$$
$$\overset{++}{g}(t) = \overset{+}{g}(t) \quad \text{for all } t \in [S, S+T'].$$

By uniqueness (theorem 3.3) the solutions agree on $[S,T)$, since they both solve (3.2) and satisfy (4) on this interval. We know that $\overset{++}{g}(t) \in C^\infty(\mathbf{R} \times [0,S])$ since $g(t) \in C^\infty(\mathbf{R} \times [0,S])$. Also $\overset{++}{g}(t) \in C^\infty(\mathbf{R} \times [S, S+T'])$ since $\overset{+}{g}(t) \in C^\infty(\mathbf{R} \times [S, S+T'])$. Hence we have constructed a smooth solution

$$\overset{++}{g}(t) \in C^\infty(\mathbf{R} \times [0, S+T']),\, t \in [0, S+T']$$

that satisfies (4) and $S + T' > T$. This contradicts $g(t), t \in [0,T)$ being the maximal smooth solution that satisfies (3.2), (4) and $g(0) = g_0$, and so the result follows. Note that as a consequence of (4), Lemma 2.2 and Lemma 2.3, $\sup_{x \in \mathbf{R}, t \in [0,S]} {}^{g(t)}|{}^{g(t)}\text{Riem}(x)| < \infty$ for all $S < T$. $\square$

4. Metric and curvature evolution equations

In this section we present evolution equations for an arbitrary family of warped product metrics $g(t) \in \mathcal{W}(N,\gamma)$, $t \in [0,T]$ that is evolving by Ricci flow. We also present evolution equations for the Ricci curvatures of such an evolving metric. We then apply a maximum principle to the curvature evolution equations to show that any $g_0 \in \mathcal{W}(N,\gamma)$ that satisfies (1) and (2) at time zero, will continue to do so for every later time when it is evolved by Ricci flow. In deriving the evolution equations for the Ricci curvature, we will use some formulae that are specific to warped product metrics of the form (0). We stress here that the high degree of symmetry of this warped product metric allows us to greatly simplify the general curvature evolution equations derived by Hamilton [Ha 1]. In particular the fact that γ is an Einstein metric and hence ${}^{g}R_{\alpha\beta} = f g_{\alpha\beta}$ (see 2.7) is perhaps most significant. That this symmetry is maintained under Ricci flow is perhaps the key to *seeing* how the curvature is evolving and behaving.

Proposition 4.1. *Let*

$$g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{W}(N,\gamma)$$

be a smooth warped product metric on $(\mathbf{R} \times N^n)$ *evolving according to (1.1), where* γ *is an Einstein metric (with constant k),* $t \in [0,T]$*, where* $T \geq 0$ *is some constant.*

Then we have that

$$\frac{\partial}{\partial t}h_{ij} = \frac{2n}{r}{}^h\nabla_i{}^h\nabla_j(r) \tag{4.1}$$

$$\frac{\partial}{\partial t}r^2 = -2fr^2 = {}^h\Delta r^2 + (2n-4)^h|{}^h\nabla r|^2 - 2k, \tag{4.2}$$

$$\frac{\partial}{\partial t}{}^gR_{xx} = {}^g\Delta{}^gR_{xx} + \frac{2}{n}g^{\beta\sigma\,g}R_{\beta\sigma}{}^gR_{xx} - 2g^{xx}({}^gR_{xx})^2 \tag{4.3}$$

$$\frac{\partial}{\partial t}{}^gR_{\alpha\beta} = {}^g\Delta{}^gR_{\alpha\beta} + \frac{2}{n}g^{xx}g^{xx}g_{\alpha\beta}({}^gR_{xx})^2 - \frac{2}{n}{}^gR_{\alpha\beta}g^{xx\,g}R_{xx}. \tag{4.4}$$

$$\frac{\partial}{\partial t}f = {}^g\Delta f + \frac{2}{n}g^{xx}g^{xx}({}^gR_{xx})^2 - \frac{2}{n}fg^{xx\,g}R_{xx} + 2nf^2, \quad \text{where} \tag{4.5}$$

$$f = \frac{1}{n}g^{\alpha\beta}R_{\alpha\beta}$$
$$= \frac{1}{2r^2}(-{}^h\Delta r^2 + (4-2n)^h|{}^h\nabla r|^2 + 2k) = (k - r{}^h\Delta r + (1-n)^h|{}^h\nabla r|^2).$$

To prove Proposition (4.1) we will need the following technical lemma.

Lemma 4.2. *Let*

$$g(x,q) = h(x) \oplus r^2(x)\gamma(q) \in \mathcal{W}(N,\gamma).$$

Then

$${}^gR_{x\alpha x\beta} = \frac{1}{n}{}^gR_{xx}g_{\alpha\beta} \tag{4.6}$$

$$g^{\tau\delta\,g}R_{\tau\alpha\delta\beta} - {}^gR_{\alpha\beta} = -\frac{1}{n}g^{xx\,g}R_{xx}g_{\alpha\beta}. \tag{4.7}$$

Proof of Lemma 4.2. Equation (4.6) follows immediately from (2.2) and (2.4). By definition of ${}^gR_{\alpha\beta}$ and the fact that $g_{\delta x} = {}^gR_{x\delta} = 0$, we obtain

$${}^gR_{\alpha\beta} = g^{ab\,g}R_{a\alpha b\beta} = g^{xx\,g}R_{\alpha x\beta x} + g^{\tau\delta\,g}R_{\tau\alpha\delta\beta}.$$

Substituting (4.6) into the above gives us (4.7). □

Proof of Proposition 4.1. Remembering the equation for Ricci flow (1.1) and using the formula for ${}^gR_{ij}$ (2.4) we derive the evolution equation

$$\frac{\partial}{\partial t}h_{ij} = \frac{\partial}{\partial t}g_{ij} = -2{}^gR_{ij} = -2{}^hR_{ij} + \frac{2n}{r}{}^h\nabla_i{}^h\nabla_j r,$$

which is (4.1). Substituting the formula (2.7) into (1.1) we obtain

$$\frac{\partial}{\partial t}(r^2\gamma_{\alpha\beta}) = \frac{\partial}{\partial t}(g_{\alpha\beta}) = -2{}^gR_{\alpha\beta} = -2fg_{\alpha\beta} = -2fr^2\gamma_{\alpha\beta}.$$

Since by assumption γ is independent of t (4.2) follows. In [Ha1] the equation for the evolution of the Ricci-curvature is derived to be

$$\frac{\partial}{\partial t}{}^gR_{ab} = {}^g\Delta{}^gR_{ab} + 2g^{pc}g^{qd\,g}R_{paqb}{}^gR_{cd} - 2g^{pq\,g}R_{pa}{}^gR_{qb},$$

where $p, q, c, d \in \{1, \ldots, n+1\}$. Hence

$$\frac{\partial}{\partial t}{}^g R_{xx} = {}^g\Delta {}^g R_{xx} + 2g^{pc} g^{qd\,g} R_{pxqx}{}^g R_{cd} - 2g^{pq\,g} R_{px}{}^g R_{qx}. \tag{4.8}$$

By the curvature formulae of chapter (2), we see that the only non-zero ${}^g R_{pxqx}$ are of the form ${}^g R_{\alpha x \beta x}$. Combining this with $g^{x\alpha} \equiv {}^g R_{\alpha x} \equiv 0$, we get

$$\frac{\partial}{\partial t}{}^g R_{xx} = {}^g\Delta {}^g R_{xx} + 2g^{\alpha\beta} g^{\gamma\sigma\,g} R_{\alpha x \gamma x}{}^g R_{\beta\sigma} - 2g^{xx\,g} R_{xx}{}^g R_{xx}.$$

Substituting identity (4.6) into the above equation we see that

$$\frac{\partial}{\partial t}{}^g R_{xx} = {}^g\Delta {}^g R_{xx} + \frac{2}{n} g^{\beta\sigma\,g} R_{\beta\sigma}{}^g R_{xx} - 2g^{xx}({}^g Rxx)^2$$

which proves (4.3).

Now we calculate the evolution of the Ricci curvature in the N^n direction. From the general equation for the evolution of the Ricci curvature we get

$$\frac{\partial}{\partial t}{}^g R_{\alpha\beta} = {}^g\Delta {}^g R_{\alpha\beta} + 2g^{pr} g^{qs\,g} R_{p\alpha q\beta}{}^g R_{rs} - 2g^{pq\,g} R_{p\alpha}{}^g R_{q\beta},$$

where $p, q, r, s \in \{1, \ldots, n+1\}$. Once again using that $g^{x\alpha} \equiv 0$, and ${}^g R_{\alpha x} \equiv 0$ we see that

$$\begin{aligned}\frac{\partial}{\partial t}{}^g R_{\alpha\beta} =& {}^g\Delta {}^g R_{\alpha\beta} + 2g^{xx} g^{xx\,g} R_{x\alpha x\beta}{}^g R_{xx} \\ &+ 2g^{\sigma\delta} g^{\eta\rho\,g} R_{\sigma\alpha\eta\beta}{}^g R_{\delta\rho} - 2g^{\sigma\delta\,g} R_{\sigma\alpha}{}^g R_{\delta\beta}.\end{aligned}$$

Using identity (4.6) and (2.7) in the above we see that

$$\begin{aligned}\frac{\partial}{\partial t}{}^g R_{\alpha\beta} =& {}^g\Delta {}^g R_{\alpha\beta} + \frac{2}{n} g^{xx} g^{xx\,g} R_{xx} g_{\alpha\beta}{}^g R_{xx} \\ &+ 2f g^{\sigma\delta} g^{\eta\rho\,g} R_{\sigma\alpha\eta\beta} g_{\delta\rho} - 2g^{\sigma\delta} f g_{\sigma\alpha}{}^g R_{\delta\beta} \\ =& {}^g\Delta {}^g R_{\alpha\beta} + \frac{2}{n} g^{xx} g^{xx\,g} R_{xx} g_{\alpha\beta}{}^g R_{xx} + 2f\Big(g^{\eta\sigma\,g} R_{\sigma\alpha\eta\beta} - {}^g R_{\alpha\beta}\Big).\end{aligned} \tag{4.9}$$

Substituting (4.7) into (4.9) we obtain

$$\frac{\partial}{\partial t}{}^g R_{\alpha\beta} = {}^g\Delta {}^g R_{\alpha\beta} + \frac{2}{n} g^{xx} g^{xx\,g} R_{xx} g_{\alpha\beta}{}^g R_{xx} + 2f\Big(-\frac{1}{n} g^{xx\,g} R_{xx} g_{\alpha\beta}\Big),$$

in view of (4.6). Finally, we wish to turn our $f g_{\alpha\beta}$ back into a curvature term, and so we use (2.7) again to obtain

$$\frac{\partial}{\partial t}{}^g R_{\alpha\beta} = {}^g\Delta {}^g R_{\alpha\beta} + \frac{2}{n} g^{xx} g^{xx\,g} R_{xx}{}^g R_{xx} g_{\alpha\beta} - \frac{2}{n} g^{xx\,g} R_{xx}{}^g R_{\alpha\beta}$$

which is (4.4).

$$\frac{\partial}{\partial t} f = \frac{\partial}{\partial t}\Big(\frac{1}{n} g^{\alpha\beta\,g} R_{\alpha\beta}\Big) = \frac{1}{n}\Big(\frac{\partial}{\partial t} g^{\alpha\beta}\Big){}^g R_{\alpha\beta} + \frac{1}{n} g^{\alpha\beta}\Big(\frac{\partial}{\partial t}{}^g R_{\alpha\beta}\Big). \tag{4.10}$$

Using (1.1), and the fact that cross terms of g and the Ricci curvature tensor are zero, we obtain

$$\frac{\partial}{\partial t} g^{\alpha\beta} = 2g^{\alpha\sigma} g^{\beta\tau\, g} R_{\sigma\tau}.$$

Substituting this equation and (4.3) into (4.10) we get

$$\begin{aligned}\frac{\partial}{\partial t} f = \frac{1}{n} g^{\alpha\beta} \Big({}^{g}\Delta\, {}^{g}R_{\alpha\beta} + \frac{2}{n} g^{xx} g^{xx} g_{\alpha\beta} ({}^{g}R_{xx})^2 - \frac{2}{n} {}^{g}R_{\alpha\beta} g^{xx\, g} R_{xx} \Big) \\ + \frac{2}{n} {}^{g}R_{\alpha\beta} g^{\sigma\alpha} g^{\tau\beta\, g} R_{\sigma\tau}.\end{aligned}$$

Since ${}^{g}\Delta g_{ab} \equiv {}^{g}\Delta g^{ab} \equiv 0$ and $g^{\alpha\beta} g_{\alpha\beta} = n$, this implies

$$\frac{\partial}{\partial t} f = {}^{g}\Delta f + \frac{2}{n} g^{xx} g^{xx} ({}^{g}R_{xx})^2 - \frac{2}{n} f g^{xx\, g} R_{xx} + \frac{2}{n} {}^{g}R_{\alpha\beta} g^{\sigma\alpha} g^{\tau\beta\, g} R_{\sigma\tau}.$$

Equation (4.5) then follows from

$$\frac{2}{n} {}^{g}R_{\alpha\beta} g^{\gamma\alpha} g^{\eta\beta\, g} R_{\gamma\eta} = 2nf^2. \qquad \square$$

5. Conservation of curvature sign

We now show that any warped product metric $g_0 \in \mathcal{W}(N, \gamma)$ that satisfies (1) and (2) at time zero, will continue to do so for every later time when it is evolved by Ricci flow. We do this by combining the evolution equations of chapter five with a maximum principle.

Theorem 5.1. *Let $g_0(x, q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{C}(N, \gamma)$ satisfy (4), and let*

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q) \in \mathcal{W}(N, \gamma),\ t \in [0, T)$$

be the maximal warped product solution to (1.1) satisfying (4) for each $t \in [0, T)$, with initial values $g(0) = g_0$ (see Theorem 3.4). Then $g(t) \in \mathcal{C}(N, \gamma)$, for all $t \in [0, T)$.

Proof. From the definition of a maximal solution, we have

$$v = \sup_{x \in \mathbf{R}, t \in [0,S]} {}^{g(t)}|{}^{g(t)}\mathrm{Riem}(x)|^2 < \infty \quad \text{for all } S < T. \tag{5.1}$$

Hence (4.3) d (4.5) imply

$$\frac{\partial}{\partial t} {}^{g}R_{xx} = {}^{g}\Delta({}^{g}R_{xx}) + A\, {}^{g}R_{xx} \quad \text{for all } t \in [0, S] \tag{5.2}$$

$$\frac{\partial}{\partial t} f \geq {}^{g}\Delta f + Bf \quad \text{for all } t \in [0, S], \tag{5.3}$$

where $A, B : \mathbf{R} \times [0, S] \to \mathbf{R}$ are functions bounded in terms of v. By (4), we see that any gradient of the Riemannian curvature tensor is bounded for $t \in [0, S]$. Hence,

$$\sup_{x\in\mathbf{R}} {}^g|{}^g\nabla R_{xx}|^2 < \infty \quad \text{for all } t \in [0, S] \tag{5.4}$$

and

$$\sup_{x\in\mathbf{R}} {}^g|{}^g\nabla f|^2 < \infty \quad \text{for all } t \in [0, S]. \tag{5.5}$$

We also have

$$\sup_{(x,q)\in\mathbf{R}\times N, t\in[0,S]} {}^g\left|\frac{\partial}{\partial t} g(x, q)(t)\right| = \sup_{x\in\mathbf{R}, t\in[0,S]} {}^{g(t)}|-2\,{}^{g(t)}\text{Ricci}\,(x, q)| < \infty \tag{5.6}$$

in view of (5.1). In view of (5.2), (5.4), (5.6) and ${}^{g_0}R_{xx} \le 0$ (since $g_0 \in \mathcal{C}(N, \gamma)$) we may apply the non-compact maximum principle of [EH] to the function ${}^gR_{xx}$ on the time interval $[0, S]$ to obtain

$${}^{g(t)}R_{xx} \le 0 \quad \text{for all } t \in [0, S].$$

Letting $S \to T$ gives us (1). In view of (5.3), (5.5), (5.6) and $f_0 \ge 0$ (since $g_0 \in \mathcal{C}(N, \gamma)$), we may apply the non-compact maximum principle of [EH] to the function f to infer

$$\frac{1}{n} g^{\alpha\beta} R_{\alpha\beta} = f(\cdot, t) \ge 0 \quad \text{for all } t \in [0, S], \quad \text{for all } S < T.$$

Letting $S \to T$ gives us (2). □

6. The formation of singularities

In this chapter we show that any maximal warped product solution

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q) \in \mathcal{C}(N, \gamma), t \in [0, T)$$

to (1.1) that satisfies (4) and (5) for each $t \in [0, T)$, will also satisfy $T < \infty$, and

$$\inf_{x\in\mathbf{R}, t\in[0,T)} r^2(x, t) = 0.$$

We say such a solution has collapsed in finite time. We shall see later (Theorem 7.2) that this implies that

$$\sup_{(x,q)\in\mathbf{R}\times N, t\in[0,T)} {}^{g(t)}|{}^{g(t)}\text{Riem}(x, q)|^2 = \infty$$

and so $(X, g(t))$ has formed a singularity in finite time.

Notice that (so far) we have not shown that $r(x, t)$ will actually approach zero in finite time. In order to force this to happen, we need to assume (5).

Theorem 6.1. *Let* $g_0(x, q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{C}(N, \gamma)$ *satisfy (5). Assume* $g(t)$ *is a smooth solution to (1.1) defined on* $[0, T)$, *and that*

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q) \in \mathcal{C}(N, \gamma),$$

for each $t \in [0, T)$. *Then we also have*

$$r^2 \leq a^2\rho^2 + b^2 - c^2 t,$$

as long as $b^2 - c^2 t \geq 0$, *where* c *is the constant* $c^2 = 2\Big(k - a^2(n-1)\Big)$, *and* $\rho(x, t) = {}^{h(t)}\mathrm{dist}(x_0, x)$ *(*x_0 *is fixed and comes from (5)). Note* $c^2 > 0$ *by (5).*

Corollary 6.2. *Let* $g_0(x, q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{C}(N, \gamma)$ *satisfy (4) and (5). Then the maximal warped product solution*

$$g(x, q, t) = h(x, t) \oplus r^2(x, t)\gamma(q) \in \mathcal{C}(N, \gamma),\ t \in [0, T)$$

satisfying (4) for each $t \in [0, T)$ *and* $g(0) = g_0$ *(see Theorem 5.1) must have* $T < \infty$, *and*

$$\inf_{x \in \mathbf{R}, t \in [0, T)} r^2(x, t) = 0.$$

Proof of Corollary 6.2. Assume that $T = \infty$ and assume to the contrary that

$$\inf_{x \in \mathbf{R}, t \in [0, S]} r^2(x, t) > 0, \quad \text{for all } S < \infty.$$

Then by Theorem 6.1 we have

$$r^2 \leq a^2\rho^2 + b^2 - c^2 t$$

for all $t \in [0, S]$. In particular if $x_0 \in \mathbf{R}$ is the fixed point from which ρ is measured, then

$$\rho(x_0, t) = {}^{h(t)}\mathrm{dist}(x_0, x_0) = 0,$$

and hence, letting $S \to \infty$,

$$r^2(x_0, t) \leq b^2 - c^2 t \quad \text{for all } t \in [0, \infty).$$

Hence $r^2(x_0, t)$ must reach zero in finite time, which contradicts the assumption made at the beginning of the proof. So we may assume $T < \infty$. Assume to the contrary that

$$\inf_{x \in \mathbf{R}, t \in [0, T)} r^2(x, t) > 0.$$

Then by Lemma 2.2, and Lemma 2.3, we obtain

$$\sup_{(x,q) \in \mathbf{R} \times N, t \in [0, T)} {}^{g(t)}|{}^{g(t)}\mathrm{Riem}(x, q)| < \infty,$$

which contradicts $g(t)$, $t \in [0, T)$ being maximal. □

Proof of Theorem 6.1. We recall that ρ is defined for a fixed x_0 to be

$$\rho(x,t) = \rho_{x_0}(x,t) = {}^{h(t)}\mathrm{dist}(x_0,x) = \int_{x_0}^{x} \sqrt{h(t)_{xx}}.$$

Then (1.1) with (2) tells us

$$\frac{\partial}{\partial t}\rho(x,t) = \int_{x_0}^{x} -\frac{{}^{g}R_{xx}}{\sqrt{h_{xx}}}dx \geq 0. \tag{6.1}$$

Using the fact that ${}^{h}\Delta\rho \equiv 0$ for a 1-dimensional manifold and ${}^{h}|{}^{h}\nabla\rho| = 1$ (see [SY]), we obtain

$$(\frac{\partial}{\partial t} - {}^{h}\Delta)\rho^2(x,t) \geq -2. \tag{6.2}$$

The evolution equation for r^2 (4.2) implies

$$\begin{aligned}(\frac{\partial}{\partial t} - {}^{h}\Delta)r^2 &= (2n-4){}^{h}|{}^{h}\nabla r|^2 - 2k \\ &= \frac{(n-2)}{2r^2}{}^{h}|{}^{h}\nabla r^2|^2 - 2k.\end{aligned} \tag{6.3}$$

Let F be defined by

$$F(x,t) = r^2(x,t) - a^2\rho^2(x,t) - b^2 + c^2 t.$$

Combining (6.2) and (6.3) we obtain

$$(\frac{\partial}{\partial t} - {}^{h}\Delta)F \leq -2k + 2a^2 + c^2 + \frac{(n-2)}{2r^2}{}^{h}|{}^{h}\nabla r^2|^2. \tag{6.4}$$

We wish to write the term

$$\frac{(n-2)}{2r^2}{}^{h}|{}^{h}\nabla r^2|^2$$

as a term involving ∇F. To do this we use the algebraic identity,

$${}^{h}|{}^{h}\nabla r^2|^2 = h({}^{h}\nabla(r^2-e), {}^{h}\nabla(r^2+e)) + {}^{h}|{}^{h}\nabla e|^2, \tag{6.5}$$

where $e : \mathbf{R} \to \mathbf{R}$ is an arbitrary function. Let $e = r^2 - F$. Then

$$\begin{aligned}{}^{h}|{}^{h}\nabla r^2|^2 &= h({}^{h}\nabla F, {}^{h}\nabla(2r^2 - F)) + a^4\,{}^{h}|{}^{h}\nabla\rho^2|^2 \\ &= -{}^{h}|\nabla F|^2 + 2h({}^{h}\nabla F, {}^{h}\nabla r^2) + a^4\,{}^{h}|{}^{h}\nabla\rho^2|^2.\end{aligned} \tag{6.6}$$

Upon substituting this into (6.4), we obtain

$$\begin{aligned}(\frac{\partial}{\partial t} - {}^{h}\Delta)F \leq &-2k + 2a^2 + c^2 + \frac{(n-2)}{r^2}h({}^{h}\nabla F, {}^{h}\nabla r^2) \\ &+ \frac{(n-2)a^4}{2r^2}{}^{h}|{}^{h}\nabla\rho^2|^2 \\ =& (-2k + 2a^2 + c^2) + h({}^{h}\nabla F, \frac{(n-2)}{r^2}{}^{h}\nabla r^2) \\ &+ 2a^4\frac{(n-2)}{r^2}\rho^2,\end{aligned} \tag{6.7}$$

since ${}^{h}|{}^{h}\nabla\rho| = 1$. We may calculate, using the definition of F, that

$$2a^4\frac{(n-2)}{r^2}\rho^2 = \frac{2(n-2)}{r^2}a^2F - \frac{2(n-2)}{r^2}a^2(b^2-c^2t) + 2(n-2)a^2 \\ \leq \frac{2(n-2)}{r^2}a^2F + 2a^2(n-2),$$

as long as $b^2 - c^2t \geq 0$. Substituting this inequality into (6.7) we get

$$(\frac{\partial}{\partial t} - {}^{h}\Delta)F \leq \Big(-2k + c^2 + 2a^2(n-1)\Big) + h({}^{h}\nabla F, w) + GF, \text{ where}$$

$$w(x,t) = \frac{2(n-2)}{r(x,t)}{}^{h}\nabla r(x,t) \tag{6.8}$$
$$G(x,t) = \frac{2(n-2)}{r^2(x,t)}a^2.$$

By the definition of c, we have $\Big(-2k + c^2 + 2a^2(n-1)\Big) = 0$, and hence we obtain

$$(\frac{\partial}{\partial t} - {}^{h}\Delta)F \leq h({}^{h}\nabla F, w) + GF, \tag{6.9}$$

as long as $b^2 - c^2t \geq 0$.

Fix $S < T$. We know, by the hypothesis of the theorem, that $g(t) \in \mathcal{C}(N,\gamma)$, and hence $g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q)$ satisfies (3) and (2.9). This implies

$$\sup_{x\in\mathbf{R}, t\in[0,S]} G(x,t) = \sup_{x\in\mathbf{R}, t\in[0,T]} \frac{2(n-2)a^2}{r^2(x,t)} < \infty,$$
$$\sup_{x\in\mathbf{R}, t\in[0,S]} w(x,t) = \sup_{x\in\mathbf{R}, t\in[0,T]} {}^{h}|\frac{2(n-2)}{r(x,t)}{}^{h}\nabla r(x,t)| < \infty. \tag{6.10}$$

By construction

$$F_0 = F(\cdot,0) \leq 0. \tag{6.11}$$

One readily checks that

$${}^{h}|{}^{h}\nabla F(x,t)|^2 = 4r^2\,{}^{h}|{}^{h}\nabla r|^2 + 4a^4\rho^2. \tag{6.12}$$

Substituting (2.9), and then (3.8), into (6.12), we obtain

$${}^{h}|{}^{h}\nabla F|^2 \leq (a')^2\rho^2 + (b')^2$$

for some constants a', b'. In particular this implies

$$\int_0^S \int_{\mathbf{R}} {}^{h}|{}^{h}\nabla F|^2 e^{-\rho^2} d\mu_{h(t)}dt < \infty, \tag{6.13}$$

since the exponential function dominates any polynomial.

Conditions (5.6), (6.9), (6.10) and (6.13) are the conditions needed to apply the non-compact maximum principle of [EH] to the function F. Hence $F(\cdot,t) \leq 0$ for all $t \in [0,S]$ as long as $b^2 - c^2t \geq 0$. Letting $S \to T$ we obtain the result. □

7. Neck pinching

In this chapter we combine the results of the previous chapters to show that if $g_0 \in \mathcal{W}(N,\gamma)$ satisfies (1)–(6), then the maximal warped product solution $g(t) \in \mathcal{W}(N,\gamma)$, $t \in [0,T)$ to (1.1) (see Theorem 5.1) will pinch at time T. The pinching will occur on a compact set $[-A,A] \times N \subseteq \mathbf{R} \times N$, where $A > 0$ is some finite constant.

An important part of the pinching argument is to show that the manifold $\mathbf{R} \times N$ with a warped product metric $h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{C}(N,\gamma)$, $t \in [0,T]$, evolving by Ricci flow does not completely collapse away at the first time a singularity occurs. In our warped product set-up this means that if

$$\inf_{x\in\mathbf{R}, t\in[0,T)} r^2(x,t) = 0,$$

then we also have

$$\inf_{x\in(-\infty,-A]\cup[A,+\infty), t\in[0,T)} r^2(x,t) > 0$$

for some constant A, $0 < A < \infty$. We essentially attain this result for free from Theorem 7.1.

Theorem 7.1. *Let $g_0(x,q) = h_0(x) \oplus r_0^2(x)\gamma(q) \in \mathcal{C}(N,\gamma)$ satisfy (4) and let*

$$g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{C}(N,\gamma),\ t \in [0,T)$$

be the maximal warped product solution to (1.1) satisfying (4) for each $t \in [0,T)$ and $g(0) = g_0$ (see Theorem 5.1). Then

$$r^2(x,t) \geq r_0^2(x,t) - 2kt \quad \text{for all } x \in \mathbf{R},\ t \in [0,T). \tag{7.1}$$

Proof. Equation (6.3) implies

$$\begin{aligned}\frac{\partial}{\partial t} r^2 &= \overset{h}{\Delta}(r^2) + (2n-4)\overset{h}{|}\overset{h}{\nabla} r|^2 - 2k \\ &= 2r\overset{h}{\Delta} r + (2n-2)\overset{h}{|}\overset{h}{\nabla} r|^2 - 2k \\ &\geq 2r\overset{h}{\Delta} r - 2k,\end{aligned}$$

and hence, in view of condition (1), we obtain

$$\frac{\partial}{\partial t} r^2 \geq -2k.$$

□

Theorem 7.2. *Let $g_0(x,q) = h_0(x) \oplus r_0^2(x)\gamma \in \mathcal{C}(N,\gamma)$ satisfy (4) and let*

$$g(x,q,t) = h(x,t) \oplus r^2(x,t)\gamma(q) \in \mathcal{C}(N,\gamma),\ t \in [0,T)$$

be the maximal warped product solution to (1.1) satisfying (4) for each $t \in [0,T)$, and $g(0) = g_0$ (see Theorem 5.1). Then $T < \infty$ and there exists some $A > 0$ such that

$$\inf_{x\in[-A,A], t\in[0,T)} r^2(x,t) = 0 \tag{7.2}$$

and

$$\inf_{x\in(-\infty,-A]\cup[A,+\infty),t\in[0,T)} r^2(x,t) > 0, \tag{7.3}$$

which implies that $\Big(\mathbf{R}\times N, g(t)\Big)$ *pinches on the set* $[-A, A]\times N$ *at time* T.

Proof. We have $T < \infty$ and (7.2) from corollary 6.2. We now show that the manifold does not completely collapse away, that is that (7.3) is true. Theorem 7.1 implies that

$$r^2(x,t) \geq r_0^2(x) - 2kt.$$

By (6), there exists an $A > 0$ such that

$$r_0^2(x) \geq 2kT + 1 \quad \text{for all } x \in (-\infty, -A]\cup[A, +\infty).$$

Hence,

$$r^2(x,t) \geq 2kT + 1 - 2kt \geq 1 \quad \text{for all } x \in (-\infty, -A]\cup[A,+\infty),\, t\in[0,T),$$

and hence (7.3) is also true. We must also show that (7.2) and (7.3) imply that the manifold has pinched. We know from theorem 5.1, that (1) and (2) remain true for the evolving metric $g(t)$ under Ricci-flow. From Lemma 2.2, and inequalities (2.15), (2.16) and (7.3) we see that

$$\sup_{x\in(-\infty,-A]\cup[A,+\infty),t\in[0,T)} {}^{g(t)}|{}^{g(t)}\mathrm{Riem}(x,t)|^2 \leq v, \tag{7.4}$$

for some constant $v < \infty$. By (7.2) and (7.3), for t close to T, there exists $x_t \in [-A, A]$ such that the infimum of r is attained:

$$r(x_t, t) = \inf_{x\in\mathbf{R}} r(x,t).$$

Then since ${}^h\nabla r(x_t, t) = 0$, we may substitute this into (2.3) to obtain

$${}^g R_{\alpha\beta\tau\sigma}(x_t,t) = r^2(x_t,t)\, {}^\gamma R_{\alpha\beta a\sigma}.$$

Then

$$g^{\alpha\tau} g^{\beta\sigma}\, {}^g R_{\alpha\beta\tau\sigma}(x_t,t) = \frac{1}{r^2(x_t,t)} \gamma^{\alpha\tau}\gamma^{\beta\sigma}\, {}^\gamma R_{\alpha\beta\tau\sigma} = \frac{kn}{r^2(x_t,t)},$$

where here we have used that γ is an Einstein metric with Einstein constant $k(> 0)$. Hence

$${}^{g(t)}|{}^g\mathrm{Riem}(x_t,t)| \geq g^{\alpha\tau} g^{\beta\sigma}\, {}^g R_{\alpha\beta\tau\sigma}(x_t,t) = \frac{kn}{r^2(x_t,t)} \to \infty \quad \text{as } t\to T,$$

in view of (7.2), which together with (7.4) implies that $g(t)$ pinches at time T. □

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