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Ricci Flow of Regions with Curvature Bounded Below in Dimension Three

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Abstract We consider smooth complete solutions to Ricci flow with bounded curvature on manifolds without boundary in dimension three. Assuming an open ball at time zero of radius one has sectional curvature bounded from below by -1, then we prove estimates which show that compactly contained subregions of this ball will be smoothed out by the Ricci flow for a short but well-defined time interval. The estimates we obtain depend only on the initial volume of the ball and the distance from the compact region to the boundary of the initial ball. Versions of these estimates for balls of radius r follow using scaling arguments.

Keywords Ricci flow · Geometric evolution equations · Local results · Smoothing properties

Mathematics Subject Classification 53C44 · 35B65

1 Introduction

In this paper, we consider smooth solutions $(M, g(t))_{t\in[0,T)}$ to Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ricci}(g)$$

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as introduced and first studied in R. Hamilton's paper [7]. The solutions $(M, g(t))_{t\in[0,T)}$ we consider are smooth (in space and time), connected, complete for all $t \in [0, T)$, and M has no boundary. We usually assume that the solution $(M, g(t))_{t\in[0,T)}$ *has bounded curvature*, that is that $\sup_{M\times[0,T)} |\operatorname{Riem}(x,t)| < \infty$. The value $k_0 := \sup_{M\times[0,T)} |\operatorname{Riem}(x,t)| < \infty$ will play no role in the estimates we obtain.

In the paper [12], G. Perelman proved a Pseudolocality Theorem for solutions of the type described above: if a ball ${}^0B_r(p_0)$ of radius $r > 0$ in an n-dimensional manifold $(M^n, g(0))$ at time zero is *almost Euclidean* (see sect. 10 in [12]), and $(M^n, g(t))_{t\in[0,T)}$ is a complete solution to the Ricci flow with bounded curvature, then for small times $t \in [0, \varepsilon^2(n)r^2))$, we have estimates on how the curvature behaves on balls ${}^tB_{\varepsilon(n)r}(p_0)$. There are a number of versions of this theorem: see the introduction in the paper [13] for references and further remarks. In the paper [13], we generalised this result in the two-dimensional setting. In particular, we allow regions at time zero which are not necessarily almost Euclidean: see Theorem 1.1 in [13] and the remarks before and after the statement of Theorem 1.1 there. The purpose of this paper is to generalise this result to the three-dimensional setting.

Notation 1.1 In this paper, $\mathcal{R}(g)$ always refers to curvature operator. When we write $\mathcal{R}(g) \geq c$ for a constant $c \in \mathbb{R}$, then we mean that

$\operatorname{Riem}(g)^{ikjl}\omega_{ik}\omega_{jl} \geq c g^{ij}g^{kl}\omega_{ik}\omega_{jl}$ on M for all two forms $\omega = \omega_{ij}dx^i \otimes dx^j$, $\omega_{ij} = -\omega_{ji}$, where $\operatorname{Riem}^{ijkl}$ is the full Riemannian curvature tensor. A two form ω has *length one*, if $|\omega|_g^2 := g^{ij}g^{kl}\omega_{ik}\omega_{jl} = 1$.

We show the following in this paper.

Theorem 1.2 *Let $r, v_0 > 0$ and $0 < \alpha < 1$ be given. Let $(M^3, (g(t))_{t\in[0,T)}$ be a smooth complete solution to Ricci flow with bounded curvature and no boundary, and let $p_0 \in M$ be a point such that*

- $\operatorname{vol}({}^0B_r(p_0)) \geq v_0 r^3$ *and*
- $\mathcal{R}(g(0)) \geq -\frac{1}{r^2}$ *on* ${}^0B_r(p_0)$.

Then there exist an $N = N(v_0, \alpha)$ and a $\tilde{v}_0 = \tilde{v}_0(v_0) > 0$ such that

(a) $\operatorname{vol}({}^tB_r(p_0)) \geq \tilde{v}_0 r^3$,
(b) $\mathcal{R}(g(t)) \geq -\frac{N^2}{r^2}$ *on* ${}^tB_{r(1-\alpha)}(p_0)$ *and*
(c) $|\operatorname{Riem}| \leq \frac{N^2}{t}$ *on* ${}^tB_{r(1-\alpha)}(p_0)$

as long as $t \leq \frac{r^2}{N^2}$ *and* $t \in [0, T)$.

Remark 1.3 By scaling, it suffices to prove the theorem for $r = 1$.

Remark 1.4 The regions which are considered are not necessarily *almost Euclidean* at time zero (see the introduction in the paper [13] for further remarks and comments).

Remark 1.5 This localises the global results of Theorem 1.7 of [14] and Theorem 1.9 of [15] which proved a similar result for the case that the curvature operator is bounded from below by minus one on the whole manifold, and that the solution has bounded curvature and $\operatorname{vol}({}^0B_1(x)) \geq v_0 > 0$ for all x in the manifold at time zero.

The above result (Theorem 1.2) is obtained as a corollary of the following theorem (Theorem 1.6) combined with Theorem 3.1 (which is a modified version of Theorem 2.2 of [13]), as we explain in the last section of this paper.

Theorem 1.6 *Let $r, v_0 > 0$ be given and $(M^3, g(t))_{t\in[0,T)}$ be a smooth complete solution to Ricci flow with bounded curvature and no boundary. Let $p_0 \in M$ be a fixed point and assume that*

- $\mathrm{vol}({}^0B_s(x)) \geq v_0 s^3$ *for all $s > 0$ and $x \in M^3$ which satisfy ${}^0B_s(x) \subseteq {}^0B_r(p_0)$ and*
- $\mathcal{R}(g(0)) \geq -\frac{1}{r^2}$ *on ${}^0B_r(p_0)$.*

Then there exist a (large) $K = K(v_0)$ and a (small) $\sigma_0 = \sigma_0(v_0) > 0$ such that

(i) $\mathcal{R}(g(t))(x)(r - d_t(x, p_0))^2 > -K^2$

for all $x \in \overline{{}^tB_{r-(K\sqrt{t}/\sqrt{\sigma_0})}(p_0)}$ and $t \leq \frac{r^2\sigma_0}{K^2}$ and $t \in [0, T)$. Here $d_t(x, p_0) = d(g(t))(x, p_0)$ is the distance from x to p_0 measured using $g(t)$.

Remark 1.7 We may change the result of the theorem to the statement ‘Then there exists a (large) $N = N(v_0)$ and a (small) $\sigma_0 = \sigma_0(v_0) > 0$ such that

(i) $\mathcal{R}(g(t))(x)(r - d_t(x, p_0))^2 > -\sigma_0 N^2$

for all $x \in \overline{{}^tB_{r-N\sqrt{t}}(p_0)}$ which satisfy $t \leq \frac{r^2}{N^2}$ and $t \in [0, T)$’, by setting $N^2 = \frac{K^2}{\sigma_0}$. This is the statement that we shall prove.

2 Comments on the Proof of Theorem 1.2 and the Use of Perelman’s Pseudolocality Theorem Therein

The main ingredients of the proof of Theorem 1.2 are as follows: (i) Theorem 1.5 of [13], (ii) The Pseudolocality Theorem of G. Perelman (sect. 10 in [12]) and (iii) Theorem 3.1 of this paper (which is a modified version of Theorem 2.2 of [13]).

The idea is essentially as follows: Theorem 1.2 follows from Theorem 1.6 and the results (slightly modified) of [13]. So we have to prove Theorem 1.6. We use the notation from Remark 1.7. We choose $N(v_0) > 0$ large and $\sigma(v_0)$ small: they are specified in the proof. The first part of the proof of Theorem 1.6 is a scaling argument which gets us into a setting where the scaled radius r is now L, and the first valid time and point $t = t_0$ and $x = z_0$ where $\mathcal{R}(g(t))(x)(L - d_t(x, p_0))^2 > -\sigma_0 N^2$ fails to hold satisfies (after scaling): $t_0 \in [0, 1]$, and the new radius L is very large, in particular $L \geq N$,

(a) $\mathrm{vol}({}^0B_s(x)) \geq v_0 s^3$ for all ${}^0B_s(x) \subseteq {}^0B_L(p_0)$,
(b) $\mathcal{R}(g(t))(x)\,\mathrm{dist}^2_{L,t}(x) \geq -\sigma_0 N^2$ for all $x \in \overline{{}^tB_{L-N\sqrt{t}}(p_0)}$ which satisfy $t \leq t_0 \leq 1$,
(c) $\mathrm{dist}_{L,t_0}(z_0) = N$, $\mathcal{R}(g(t_0))(z_0)(\omega, \omega) = -\frac{\sigma_0 N^2}{\mathrm{dist}^2_{L,t_0}(z_0)} = -\sigma_0$ for a two form ω of length one (w.r.t to $g(t_0)$) and
(d) $\mathcal{R}(g(0))(x) \geq -\frac{1}{L^2} \geq -\frac{1}{N^2}$ on ${}^0B_L(p_0)$,

where $\mathrm{dist}_{L,t}(x) = (L - d_t(x, p_0))$ for $x \in {}^tB_L(p_0)$ and is 0 otherwise.

We next show that (a),(b),(c) and (d) lead to a contradiction if the constant $\sigma(v_0)$ is small enough, and the constant $N(v_0)$ is large enough. Here we give a **rough** sketch of the proof idea. There are two cases that need to be considered, when obtaining this contradiction:

Case (i): t_0 is *not too near to* 1 ($t_0 \le 1 - 10\beta_0$ *with* $\beta_0 = \sigma_0^{1/4}$ *suffices).*

In this case, we see (see the proof) that ${}^tB_{\beta_0 N}(z_0) \subseteq {}^tB_{L-N\sqrt{t}}(p_0)$ for all $t \le t_0$, and hence we have the estimate (b), at any point in the space-time cylinder of radius $\beta_0 N$ centred at z_0 with base time 0 and top time t_0, $\cup_{t\in[0,t_0]}{}^tB_{\beta_0 N}(z_0) \times \{t\}$, and hence $\mathcal{R}(\cdot,\cdot) \ge -1$ on the same space-time cylinder of half the radius, in view of (b). Note that, the radius $\beta_0 N$ is very large by assumption (see proof). Regularity estimates of the previous papers (in particular the paper [13]), which do not rely on Perelman's Pseudolocality result, show us that the norm of the full curvature tensor is bounded by $\frac{C(v_0)}{t}$ for some constant $C(v_0)$, at any point (x, t) in a space-time cylinder of a smaller (but still large enough) radius, with the same centre point, and base resp. top time. Then, (d) and a regularity result from [13], tells us that $\mathcal{R}(z_0, t_0) \ge -\varepsilon(N) \to 0$ as $N \to \infty$: this contradicts (c) if $N = N(v_0)$ is chosen large enough initially. That is: the case $t_0 \le 1 - 10\beta_0$ follows from this scaling or 'blow up' argument and the (slightly modified) results of [13], and the use of Perelman's Pseudolocality Theorem is **not** necessary in this case.

Case (ii): The case that t_0 is near to one, that is $1 \ge t_0 \ge 1 - 10\beta_0$.

This case cannot be immediately handled in the same way as Case (i). The reason is: it could be that we cannot find a large enough radius $R > 0$ such that ${}^tB_R(z_0) \subseteq {}^tB_{L-N\sqrt{t}}(p_0)$ for all $t \le t_0$, and hence we **do not** have the estimate (b) on some space-time cylinder with large (enough) radius centred at z_0 with base time 0 and top time t_0. In the extreme case we have $t_0 = 1$, and hence $d_1(p_0, z_0) = L - N$ (since $\mathrm{dist}_{L,1}(z_0) = N$) and hence z_0 is in the boundary of ${}^{t_0}B_{L-\sqrt{t_0}N}(p_0) = {}^1B_{L-N}(p_0)$ at time t_0. To get around this problem, we proceed as follows: Using the method described in Case (i), we see that $|\,\mathrm{Riem}(x, s)| \le \frac{C(v_0)}{s}$ for all points (x, s) on some space-time cylinder with large radius centred at z_0 with base time 0 and top time t, as long as $t \le 1 - 10\beta_0$. The (second) Pseudolocality Theorem of G. Perelman for times $t \in [1 - 10\beta_0, t_0]$, combined with the estimates which were obtained for $t \le 1 - 10\beta_0$, allows us to extend this estimate to $|\,\mathrm{Riem}(\cdot, t)| \le \frac{\tilde{C}(v_0)}{t}$ for all $t \in [0, t_0]$ on some space-time cylinder with large radius centred at z_0 with base time 0 and top time t_0. Now we use the regularity result of [13], as in Case (i), and get that $\mathcal{R}(z_0, t_0) \ge -\varepsilon(N) \to 0$ as $N \to \infty$, which contradicts (c) if $N = N(v_0)$ is chosen large enough initially.

We write '**rough** sketch' above, because many of the difficulties which occur in the proof are avoided in this sketch. In particular, we actually prove estimates on cylinders of the type explained above with arbitrary centre points $y_0 \in {}^0B_{L-N+\frac{3}{4}\beta_0 N}(p_0)$ instead of z_0, and then we show, by proving estimates on how distances can change, that z_0 is in fact a point in ${}^0B_{L-N+\frac{3}{4}\beta_0 N}(p_0)$, and hence the estimates of the type explained above (for z_0) do hold.

Note that the use of G. Perelman's Pseudolocality Theorem is very necessary for this proof. The difficult case is $t_0 \ge 1 - 10\beta_0$. As we pointed out above, this corresponds to z_0 being close to or in the boundary of $B_{L-N\sqrt{t_0}}(p_0)$. In all such blow up arguments in geometric analysis, this is the difficult case and there is no guarantee that this case can be dealt with. Whether this case can be dealt with or not will depend on the flow being considered. For the Ricci flow, the Pseudolocality Theorem enables us to deal with this case.

Note that Theorem 1.2 (respectively, Theorem 1.6) of this paper is *almost* a truly local theorem: we only need assumptions at time zero along with the assumption that the solution we are considering has bounded curvature and is complete. We say *almost*, because we require an assumption on the solution itself, namely that the solution has bounded curvature and is complete, in order to apply the Pseudolocality Theorem. The regularity theorem, Theorem 1.5, of the paper [13] is not a truly local theorem: one of the requirements of that theorem is that $|\,\text{Riem}\,| \le c/t$ for some $c > 0$ on the ball ${}^tB_r(p_0)$ (for all $t \in [0, T]$) that we are considering. This is a strong assumption on the solution.

3 Some Local Results

In this section we prove some lemmata, which follow readily from previously proved results. These results will be required in the proof of Theorem 1.6.
First we prove a modified version of Theorem 2.2 of [13]. The result of the theorem below and that of Theorem 2.2 of [13] differ in the following way: In Theorem 2.2 of [13] condition (a), there was '[a] : $\text{vol}({}^tB_r(p_0)) \ge v_0 r^n$ for all $t \in [0, T)$'. Here we only require $\text{vol}({}^0B_r(p_0)) \ge v_0 r^n$ at time zero.

Theorem 3.1 *Let $r, V, v_0 > 0$, $1 > \alpha > 1/2$ and $(M^n, g(t))_{t\in[0,T)}$ be a smooth, complete solution to Ricci flow with no boundary which satisfies*

(a) $\text{vol}({}^0B_r(p_0)) \ge v_0 r^n$,
(b) $\mathcal{R}(x, t) \ge -\frac{V}{r^2}$ *for all* $t \in [0, T)$, $x \in {}^tB_r(p_0)$.
Then, there exist $0 < m_0 = m_0(n, v_0, \alpha, V)$, $c_0 = c_0(n, v_0, \alpha, V) < \infty$ *and* $\tilde{v}_0 = \tilde{v}_0(n, v_0, V) > 0$ *such that*
(c) $|\,\text{Riem}(x, t)| < \frac{c_0}{t}$ *for all* $x \in {}^tB_{r(1-\alpha)}(p_0)$, $t \in [0, m_0 r^2) \cap [0, T)$ *and*
(d) $\text{vol}({}^tB_s(p_0)) > \tilde{v}_0 s^n$ *for all* $t \in [0, m_0 r^2) \cap [0, T)$, *for all* $s < r$.

Remark 3.2 Note that here we do not require that $(M^n, g(t))_{t\in[0,T)}$ is a solution with bounded curvature.

Proof Let $(M, g(t))_{t\in[0,T)}$ be as in the statement of the theorem. Without loss of generality, after scaling, we have $r = 1$. We prove the case $s = 1$: the general statement in (d) then follows from the Bishop–Gromov comparison principle.

We know that $\text{vol}({}^0B_{1/800}(p_0)) \ge V_0(v_0, V, n) > 0$ due to the Bishop–Gromov volume comparison principle. From the Appendix, Theorem 1.1, we see that the following is true: there exists an $\varepsilon_0 = \varepsilon_0(V_0, n) = \varepsilon_0(v_0, V, n) > 0$ such that if $d_{GH}({}^tB_{1/800}(p_0), {}^0B_{1/800}(p_0)) \le \varepsilon_0$ for some $t \in [0, T)$, then $\text{vol}({}^tB_{1/800}(p_0)) > \varepsilon_0$. Assume there is a first time $S \in (0, T)$ where $\text{vol}({}^tB_{1/800}(p_0)) > \varepsilon_0$ is violated: $\text{vol}({}^tB_{1/800}(p_0)) > \varepsilon_0$ for all $0 \le t < S$ and $\text{vol}({}^SB_{1/800}(p_0)) = \varepsilon_0$. From Theorem 2.2 of [13], we have $|\,\text{Riem}\,| < \frac{N^2}{t}$ on ${}^tB_{1-\alpha}(p_0)$ for all $t \le \min(\hat{T}(\varepsilon_0, n, \alpha, V), S) =$

$\min(\hat{T}(v_0, V, n, \alpha), S)$ for some $N = N(\varepsilon_0, n, \alpha, V) = N(v_0, V, n, \alpha)$ and $\hat{T} = \hat{T}(\varepsilon_0, n, \alpha, V) = \hat{T}(v_0, V, n, \alpha) > 0$. But then, this estimate, (b) and [9] (Lemma 17.3 combined with Theorem 17.4) imply (for such t) that $e^{ta(n)V} d_0(x, y) \geq d_t(x, y) \geq d_0(x, y) - a(n)N\sqrt{t}$ for all $x, y \in {}^tB_{1/200}(p_0)$ (see sect. 4 of [8]), since any geodesic at time t between such x and y must lie in ${}^tB_{1/2}(p_0)$, due to the triangle inequality. This means that ${}^tB_{1/700}(p_0) \subseteq {}^0B_{1/400}(p_0) \subseteq {}^tB_{1/200}(p_0)$ and

$$(1 + \varepsilon_0^2) d_0(x, y) \geq d_t(x, y) \geq d_0(x, y) - \varepsilon_0^2 \text{ on } {}^tB_{1/700}(p_0) \tag{3.1}$$

for all such t which also satisfy $t \leq \tilde{T}(v_0, V, \alpha, n)$, where $\tilde{T}(v_0, V, \alpha, n) > 0$ is small enough. Assume $S \leq \min(\tilde{T}(v_0, V, \alpha, n), \hat{T}(v_0, V, n, \alpha), T)$. Then we have $d_{GH}({}^SB_{1/800}(p_0), {}^0B_{1/800}(p_0)) < \varepsilon_0$ (a Gromov–Hausdorff approximation $f : {}^SB_{1/800}(p_0) \to {}^0B_{1/800}(p_0)$ is given by $f(x) = x$ for $x \in {}^SB_{1/800}(p_0) \cap {}^0B_{1/800}(p_0)$ and $f(x) = \tilde{x}$ for $x \in {}^SB_{1/800}(p_0) \backslash {}^0B_{1/800}(p_0)$ where $\tilde{x} \in {}^0B_{1/800}(x_0)$ is an arbitrary point with $d_0(x, \tilde{x}) \leq 10\varepsilon_0^2$: such a point exists in view of the inequalities (3.1)). This is a contradiction to the definition of ε_0. □

The next lemma is an integrated version of Lemma 8.3 (b) of Perelman, [12], in the case that the curvature behaves like a constant divided by time.

Lemma 3.3 *For any $j_0, \ell > 0, n \in \mathbb{N}$, there exists a constant $a(n, j_0, \ell)$ such that the following is true. Let $(M^n, g(t))_{t\in[0,T]}$ be a complete smooth solution to Ricci flow with bounded curvature, and no boundary, and let $s_0 \leq \min(1, T)$. Assume that $y_0, x_0 \in M$ and that $|\mathrm{Ricci}(\cdot, t)| \leq \frac{\ell}{t}$ on both ${}^tB_{j_0}(x_0)$ and ${}^tB_{j_0}(y_0)$ for all $t \in [0, s_0]$. Then $d_s(y_0, x_0) \geq d_0(y_0, x_0) - a(n, j_0, \ell)$ for all $0 \leq s \leq s_0$.*

Proof For $t \leq j_0^2$, we have $|\mathrm{Ricci}(x, t)| \leq \frac{\ell}{t}$ for any $x \in {}^tB_{\sqrt{t}}(x_0)$ and for any $y \in {}^tB_{\sqrt{t}}(y_0)$, since $\sqrt{t} \leq j_0$. Hence we may apply Lemma 8.3 (b) of [12] to this with t_0, K, r_0 of Lemma 8.3 (b) of [12] given by $t_0 = t$, $K = \frac{\ell}{t}$, $r_0 = \sqrt{t}$ to obtain

$$\begin{aligned}\frac{\partial}{\partial t} d_t(x_0, y_0) &\geq -2(n-1)\left(\frac{2}{3}Kr_0 + r_0^{-1}\right)\\ &= -2(n-1)\left(\frac{2\ell\sqrt{t}}{3t} + \frac{1}{\sqrt{t}}\right)\\ &=: \frac{1}{\sqrt{t}}\left(-\frac{4(n-1)\ell}{3} + 1\right),\end{aligned} \tag{3.2}$$

for $t \leq j_0^2$, where the time derivative is to be understood in the sense of forward difference quotients. For $t \in [j_0^2, s_0]$, we have that $|\mathrm{Ricci}(\cdot, t)| \leq \frac{\ell}{j_0^2}$ and hence applying Lemma 8.3 (b) of [12] with $t_0 = t$, $K = \frac{\ell}{j_0^2}$, $r_0 = j_0$, we get

$$\begin{aligned}\frac{\partial}{\partial t} d_t(x_0, y_0) &\geq -2(n-1)\left(\frac{2}{3}Kr_0 + r_0^{-1}\right)\\ &= -2(n-1)\left(\frac{2j_0\ell}{3j_0^2} + \frac{1}{j_0}\right)\end{aligned} \tag{3.3}$$

for $t \in [j_0^2, s_0]$, where the time derivative is to be understood in the sense of forward difference quotients. Integrating first Equation (3.2) from 0 to j_0^2 and then Equation (3.3) from j_0^2 to s gives us the result (if $s \leq j_0^2$ then we merely integrate Equation (3.2) from 0 to s). □

The last lemma of this section is a technical lemma, which uses some facts from differential geometry.

Lemma 3.4 *For every $\tilde{v}_0, \ell > 0$ and $\delta \in (0, 1)$, the following is true. Let (M^n, g) be a smooth Riemannian manifold, $x_0 \in M$, with no boundary such that the closure of $B_1(x_0)$ is compactly contained in M and*

(i) $|\operatorname{Riem}| \leq \ell$ *on* $B_1(x_0)$ *and*
(ii) $\operatorname{vol}(B_1(x_0)) \geq \tilde{v}_0$.

Then there exists an $R_0(n, \ell, \tilde{v}_0, \delta) > 0$ such that

$$|\operatorname{Riem}|(\cdot) \leq \frac{1}{R_0^2} \text{ on } B_{R_0}(x_0)$$
$$\operatorname{vol}(B_{R_0}(x_0)) \geq \omega_n(1-\delta)R_0^n,$$

where ω_n is the volume of an n-dimensional Euclidean ball of radius one.

Proof The inequalities (i) and (ii) imply that $\operatorname{inj}(g)(x_0) \geq i_0(n, \tilde{v}_0, \ell) > 0$ for some $i_0(n, \tilde{v}_0, \ell) > 0$ in view of the estimate of J. Cheeger/M. Gromov/M. Taylor, (4.22) in Theorem 4.3 of [5]. Hence, using Riemannian normal coordinates (see Theorem 1.53 and the proof thereof in [2]), we see that

$$\operatorname{vol}(B_r(x_0)) \geq \omega_n(1-\delta)r^n$$

for all $r \leq R_0(n, \ell, i_0(n, \tilde{v}_0, \ell), \delta) = R_0(n, \ell, \tilde{v}_0, \delta)$, if $R_0(n, \ell, \tilde{v}_0, \delta) > 0$ is small enough, where ω_n is the volume of the unit ball in n-dimensional Euclidean space. Without loss of generality, we also have

$$|\operatorname{Riem}(y)| \leq \frac{1}{R_0^2}$$

on $B_{R_0}(x_0)$, since without loss of generality $\frac{1}{R_0^2(n,\ell,\tilde{v}_0,\delta)} \geq \ell$: if not, decrease $R_0(n, \ell, \tilde{v}_0, \delta)$ until it is. □

4 Proof of Theorem 1.6

In order to obtain local estimates, we introduce the following distance function for balls which are evolving in time under the Ricci flow. Let $(M, g(t))_{t \in [0,T)}$ be a solution to Ricci flow. Let ${}^tB_r(p_0)$ be the open ball of radius r at time t centred at $p_0 \in M$. Notice that $x \in {}^tB_r(p_0)$ does not necessarily guarantee that $x \in {}^sB_r(p_0)$ for a different time s. For $x \in {}^tB_r(p_0)$, we define

$$\mathrm{dist}_{r,t}(x) := (r - d_t(p_0, x))$$

where $d_t(p_0, x)$ is the distance from x to p_0 measured using the evolving metric $g(t)$. Cut-off functions of this type were used in the papers [6,16] and [17] in combination with Ricci flow to prove that local estimates hold, **if** one a priori assumes that the curvature satisfies a bound of the type $|\mathrm{Riem}(\cdot, t)| \leq c/t$.

Notice that $0 < \mathrm{dist}_{r,t}(x) \leq r$ for all $x \in {}^tB_r(p_0)$. $\mathrm{dist}_{r,t}(x)$ is a measure of how far the point x at time t is from the boundary of ${}^tB_r(p_0)$. In the case that $g(t) = \delta$ the Euclidean metric on $\mathbb{R}^n$, then we see that $\mathrm{dist}_{r,t}(x) := (r - d_t(p_0, x)) = (r - |x - p_0|)$ is the distance from x to the boundary of $B_r(p_0)$. Due to scaling, it will be sufficient to consider the case $r = 1$. Let ${}^0B_1(p_0)$ be a ball at time zero with curvature bounded from below by minus one. The following theorem implies a lower bound on the curvature at $x \in {}^tB_1(p_0)$ depending on $\mathrm{dist}_{1,t}(x)$ at later times for a well-defined time interval, as long as $\mathrm{dist}^2_{1,t}(x) \geq N^2 t$ where $N^2 = N^2(v_0)$ is sufficiently large, and v_0 is a lower bound (at time zero) on the volume quotient of balls contained in the ball we are considering, and the curvature of ${}^0B_1(p_0)$ at time zero is bounded from below by -1. Combining this theorem with the results of sect. 3 will imply the result of Theorem 1.2 stated in the introduction (see sect. 5 for the proof of Theorem 1.2). Here we restate Theorem 1.6 for the case $r = 1$ using the notation that we just introduced, and Remark 1.7.

Theorem 4.1 *Let $(M^3, (g(t))_{t\in[0,T)}$ be a smooth complete solution to Ricci flow with bounded curvature and no boundary and $v_0 > 0$. Let $p_0 \in M$ be a point such that*

- $\mathrm{vol}({}^0B_r(x)) \geq v_0 r^3$ *for all $x \in M^3$, and $r > 0$ which satisfy ${}^0B_r(x) \subseteq {}^0B_1(p_0)$ and*
- $\mathcal{R}(g(0)) \geq -1$ *on ${}^0B_1(p_0)$*

Then there exists an $N = N(v_0)$, $\sigma_0 = \sigma_0(v_0) > 0$, such that

(i) $\mathcal{R}(g(t))(x)\,\mathrm{dist}^2_{1,t}(x) > -\sigma_0 N^2$

for all $x \in \overline{{}^tB_{1-N\sqrt{t}}(p_0)}$ which satisfy $t \leq \frac{1}{N^2}$ and $t \in [0, T)$.

Proof v_0 is fixed throughout the proof and $\sigma_0 = \sigma_0(v_0) > 0$ is a small constant determined in the proof.

Assuming the theorem is false for some given $N = N(v_0, \sigma(v_0)) = N(v_0)$ large and $\sigma_0 = \sigma_0(v_0) > 0$ small (to be determined in the proof), then there must be a first time $t_0 \leq \frac{1}{N^2} < T$ where the theorem fails. That is (i) is violated at t_0. We show that if $\sigma(v_0) > 0$ is chosen small enough, and $N = N(v_0) > 0$ is chosen large enough, that this leads to a contradiction. Let $\beta_0 = \sigma_0^{1/4}$ throughout the proof. At the end of the theorem, see Remark 4.2, we give a precise explanation of how N and σ can be chosen at this point of the theorem.

(i) is violated at some first time t_0 means that we can find a $z_0 \in {}^{t_0}B_1(p_0)$ and $0 < t_0 \leq \frac{1}{N^2} < T$ with $\mathrm{dist}^2_{1,t_0}(z_0) \geq N^2 t_0$ such that $\mathcal{R}(g(t_0))(z_0)(\psi, \psi)\,\mathrm{dist}^2_{1,t_0}(z_0) = -\sigma_0 N^2$ for some two form ψ of length one (measured with respect to $g(t_0)$), and the conclusions of the theorem are correct for $0 < t < t_0$. Let $L^2 = \frac{N^2}{\mathrm{dist}_{1,t_0}(z_0)^2}$. Remembering that $\mathrm{dist}_{1,t_0}(z_0)^2 \leq 1$, we see that $L \geq N$. We scale our solution by an appropriate constant, so that the new solution has $\tilde{\text{dist}}^2_{L,\tilde{t}_0}(z_0) = N^2$ at the new time $\tilde{t}_0$ which corresponds to the old time t_0 in the original solution: define $\tilde{g}(\cdot, \tilde{t}) := L^2 g(\cdot, \frac{\tilde{t}}{L^2})$. This solution is defined for $\tilde{t} \in [0, \tilde{T} = L^2 T \geq N^2 T)$. Then define for $x \in {}^{\tilde{t}}B_L(p_0)$

$$\tilde{\text{dist}}^2_{L,\tilde{t}}(x) := (L - \tilde{d}_{\tilde{t}}(x, p_0))^2 \tag{4.1}$$
$$= L^2(1 - d_t(x, p_0))^2$$

$$= L^2 \, \text{dist}^2_{1,t}(x) \tag{4.2}$$

where $t = \frac{\tilde{t}}{L^2}$ and $\tilde{d}_{\tilde{t}}(x, p_0)$ is the distance measured with respect to $\tilde{g}(\tilde{t})$.

This value is positive since $x \in {}^{\tilde{t}}B_L(p_0)$ if and only if $\tilde{d}_{\tilde{t}}(x, p_0) < L$. Using the definition of $\tilde{t}$ and $\tilde{\text{dist}}_{L,\tilde{t}}$, we see that

$$\text{dist}^2_{1,t}(x) \geq N^2 t \iff \tilde{\text{dist}}^2_{L,\tilde{t}}(x) \geq N^2 \tilde{t}. \tag{4.3}$$

Also,

$$\tilde{\text{dist}}^2_{L,\tilde{t}_0}(z_0) = L^2 \, \text{dist}^2_{1,t_0}(z_0) = \frac{N^2}{\text{dist}_{1,t_0}(z_0)^2} \text{dist}^2_{1,t_0}(z_0) = N^2 \tag{4.4}$$

and

$$\tilde{t}_0 = t_0 L^2 = t_0 \frac{N^2}{\text{dist}_{1,t_0}(z_0)^2} \leq t_0 \frac{N^2}{N^2 t_0} = 1.$$

Notice that

$$\mathcal{R}(\tilde{g}(\tilde{t}))(x) \tilde{\text{dist}}^2_{L,\tilde{t}}(x) = \mathcal{R}(g(t))(x) \, \text{dist}^2_{1,t}(x), \tag{4.5}$$

in view of the definition of $\tilde{\text{dist}}_{L,\tilde{t}}$ and the way curvature changes under scaling.

For ease of reading, we will denote the solution $\tilde{g}(x, \tilde{t})$ by $g(x, t)$. Also $\tilde{t}_0$ will be denoted by t_0, $\tilde{t}$ by t, and $\tilde{\text{dist}}_{L,\tilde{t}}$ by dist_t (L is now fixed). Then we now have

(a) $\text{vol}({}^0B_s(x)) \geq v_0 s^3$ for all ${}^0B_s(x) \subseteq {}^0B_L(p_0)$,
(b) $\mathcal{R}(g(t))(x) \, \text{dist}^2_t(x) \geq -\sigma_0 N^2$ for all $x \in {}^tB_{L-N\sqrt{t}}(p_0)$ which satisfy $t \leq t_0 \leq 1$,
(c) $\mathcal{R}(g(t_0))(z_0)(\omega, \omega) = -\frac{\sigma_0 N^2}{\text{dist}^2_{t_0}(z_0)} = -\sigma_0$ for the two form $\omega = L^2 \psi$ which has length one with respect to (the new) $g(t_0)$ and
(d) $\mathcal{R}(g(0))(x) \geq -\frac{1}{L^2} \geq -\frac{1}{N^2}$ on ${}^0B_L(p_0)$.

The first two inequalities are scale invariant (if they hold for some solution, then they hold for a scaling of the Ricci flow after adjusting the delimiters, assuming that we have defined the new dist_t for the scaled solution as in (4.1): cf. (4.3) and (4.5)). In the third equality, we used the fact that (after scaling) $\text{dist}^2_{t_0}(z_0) = N^2$

(see (4.4)): after scaling, we also have $\mathcal{R}(g(t_0))(z_0)(\omega,\omega)\,\mathrm{dist}^2_{t_0}(z_0) = -\sigma_0 N^2$ and hence $\mathcal{R}(g(t_0))(z_0)(\omega,\omega) = -\frac{\sigma_0 N^2}{\mathrm{dist}^2_{t_0}(z_0)} = -\sigma_0$. The last inequality, (d), follows since we are scaling by L^2 and we showed $L \geq N$. For all $0 \leq t \leq t_0$, we have

$$\{x \in \overline{{}^t B_L(p_0)} \mid \mathrm{dist}^2_t(x) \geq N^2 t\} = \overline{{}^t B_{L-N\sqrt{t}}(p_0)}.$$

Let $x_0 \in \overline{{}^0 B_{L-N+N\beta_0}(p_0)}$ be an arbitrary point. Clearly ${}^0 B_{\beta_0 N}(x_0) \subseteq \overline{{}^0 B_L(p_0)}$, in view of the triangle inequality (we are using that $\beta_0 \leq 1/2$, which we always assume).

The rest of the proof is broken up into three steps.

Step 1 For an arbitrary $x_0 \in \overline{{}^0 B_{L-N(1-\beta_0)}(p_0)} = \overline{{}^0 B_{L-N+N\beta_0}(p_0)}$, we show that ${}^t B_{\beta_0 N}(x_0) \subseteq \overline{{}^t B_{L-N\sqrt{t}}(p_0)}$ for all $t \leq 1-10\beta_0$ as long as $t \leq t_0$, and $N = N(v_0) > 0$ is sufficiently large. Using the estimates of Theorem 3.1, we will then see that this guarantees that $|\,\mathrm{Riem}(\cdot,t)| \leq \frac{c_0(v_0)}{t}$ on ${}^t B_{\frac{1}{\beta_0}}(x_0)$, and that $\mathrm{vol}({}^t B_1(x_0)) \geq \tilde{v}_0(v_0) > 0$ for all $t \leq \min(1-10\beta_0, t_0)$, for some constants $c_0(v_0), \tilde{v}_0(v_0) > 0$.

Now we present the details of Step 1.

Let $x_0 \in \overline{{}^0 B_{L-N+N\beta_0}(p_0)}$ be arbitrary. We know that $x_0 \in \overline{{}^t B_{L-N\sqrt{t}}(p_0)}$ is valid, if and only if

$$d_t(x_0, p_0) \leq L - N\sqrt{t}. \tag{4.6}$$

In the following we only consider t such that $0 \leq t \leq t_0$, where $t_0 \leq 1$ was defined at the beginning of the proof. Hence, starting at time zero and going forward to time t, as long as $x_0 \in \overline{{}^t B_{L-N\sqrt{t}}(p_0)}$ remains valid, any length minimising geodesic (with respect to the metric at time t) from p_0 to x_0 must also completely lie in $\overline{{}^t B_{L-N\sqrt{t}}(p_0)}$. At all points y on such a geodesic, we have

$$\mathcal{R}(g(t))(y)\,\mathrm{dist}^2_t(y) \geq -\sigma_0 N^2,$$

in view of (b). Using $\mathrm{dist}_t(y) = L - d_t(y, p_0)$, we see that this means

$$\begin{aligned}\mathcal{R}(g(t))(y) &\geq -\frac{\sigma_0 N^2}{\mathrm{dist}^2_t(y)}\\ &= -\frac{\sigma_0 N^2}{(L-d_t(y,p_0))^2}\end{aligned}$$

for such y.

Using this inequality in the evolution equation for the distance (Lemma 17.3 of [8]), we see (as long as $x_0 \in \overline{{}^t B_{L-N\sqrt{t}}(p_0)}$ remains valid)

$$\begin{aligned}\frac{\partial}{\partial t} d_t(x_0,p_0) &\leq \sup_{\gamma\in X_t} \int_0^{d_t(x_0,p_0)} -2\,\mathrm{Ricci}(\gamma(s),t)ds\\ &\leq \sup_{\gamma\in X_t} \int_0^{d_t(x_0,p_0)} \frac{\sigma_0 20 N^2}{(L-s)^2} ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{20\sigma_0 N^2}{L-s}\Big|_{s=0}^{s=d_t(x_0,p_0)}\\
&= \frac{20\sigma_0 N^2}{L-d_t(x_0,p_0)} - \frac{20\sigma_0 N^2}{L}\\
&\le \frac{20\sigma_0 N^2}{L-d_t(x_0,p_0)}\\
&\le \frac{20\sigma_0 N^2}{N\sqrt{t}}\\
&= \frac{20\sigma_0 N}{\sqrt{t}}\\
&= \frac{20\beta_0^4 N}{\sqrt{t}}
\end{aligned}
$$

where X_t is the set of distance minimising geodesics from p_0 to x_0 at time t (that is, measured with respect to the metric $g(t)$ at time t) parameterised by arc length, and we have used inequality (4.6). Here, $\frac{\partial}{\partial t}$ is to be understood in the sense of forward difference quotients: see chapter 17 of [8] for more details. Integrating in time from 0 to t, we see that this means

$$
\begin{aligned}
d_t(x_0,p_0) &\le d_0(x_0,p_0) + 40\beta_0^4 N t^{1/2}\\
&\le d_0(x_0,p_0) + \beta_0^2 N\\
&\le (L - N + N\beta_0) + \beta_0^2 N\\
&\le L - (1-2\beta_0)N
\end{aligned} \tag{4.7}
$$

for all $t \le t_0 (\le 1)$ as long as $x_0 \in \overline{{}^tB_{L-N\sqrt{t}}(p_0)}$ remains true, where we have used that $x_0 \in \overline{{}^0B_{L-N+N\beta_0}(p_0)}$ (and $\beta_0^2 \le \frac{1}{40}$ which we will always assume). Restrict now only to $t \le 1-10\beta_0$ in the above argument. Using the fact that $-(1-2\beta_0) \le -\sqrt{t}-\beta_0$ for such times [1], and inequality (4.7) , we see that

$$
\begin{aligned}
d_t(x_0,p_0) &\le L - (1-2\beta_0)N\\
&\le L - N\sqrt{t} - \beta_0 N
\end{aligned} \tag{4.8}
$$

for all $t \le \min(t_0, 1-10\beta_0)$ as long as $x_0 \in \overline{{}^tB_{L-N\sqrt{t}}(p_0)}$ remains true, and hence $x_0 \in \overline{{}^tB_{L-N\sqrt{t}}(p_0)}$ will not be violated as long as $t \le \min(t_0, 1-10\beta_0)$. Furthermore, the triangle inequality combined with (4.8) implies

$$
{}^tB_{\beta_0 N}(x_0) \subseteq \overline{{}^tB_{L-N\sqrt{t}}(p_0)}
$$

will not be violated as long as $t \le \min(1-10\beta_0, t_0)$: $y \in {}^tB_{\beta_0 N}(x_0)$ implies

$$
\begin{aligned}
d_t(y,p_0) &\le d_t(y,x_0) + d_t(x_0,p_0) \le \beta_0 N + L - N\sqrt{t} - \beta_0 N\\
&= L - N\sqrt{t}
\end{aligned}
$$

[1] Note that $-(1-2\beta_0) \le -\sqrt{t}-\beta_0$ if and only if $\sqrt{t} \le 1-3\beta_0$ if and only if $t \le (1-3\beta_0)^2 = 1-6\beta_0+9\beta_0^2$ and hence $t \le 1-10\beta_0$ implies $t \le 1-6\beta_0+9\beta_0^2$ implies $-(1-2\beta_0) \le -\sqrt{t}-\beta_0$ as claimed.

for such t, in view of the inequality (4.8)

The lower bound on the curvature, (b), is therefore valid on ${}^tB_{\beta_0 N}(x_0)$ as long as $t \le 1 - 10\beta_0$ and $t \le t_0$, and hence, for x in the ball of half the radius, $x \in {}^tB_{\frac{\beta_0 N}{2}}(x_0)$, we have

$$\begin{aligned} \mathcal{R}(g(t))(x) &\ge -\frac{\sigma_0 N^2}{\mathrm{dist}_t^2(x)} \\ &\ge -\frac{\sigma_0 N^2}{(N\sqrt{t} + \frac{N\beta_0}{2})^2} \\ &\ge -\frac{4\sigma_0}{\beta_0^2} \\ &= -4\beta_0^2 \ (\ge -1) \end{aligned} \tag{4.9}$$

($\sigma_0 > 0$ was chosen to be $\sigma_0 = \beta_0^4$) for all $t \le 1 - 10\beta_0$, $t \le t_0$, in view of the fact that

$$\begin{aligned} \mathrm{dist}_t(x) &= (L - d_t(x, p_0)) \\ &\ge (L - d_t(x, x_0) - d_t(x_0, p_0)) \\ &\ge (L - \frac{\beta_0 N}{2} - d_t(x_0, p_0)) \\ &\ge (L - \frac{\beta_0 N}{2} - L + N\sqrt{t} + \beta_0 N) \\ &= (\frac{\beta_0 N}{2} + N\sqrt{t}) \end{aligned}$$

for $x \in {}^tB_{\frac{\beta_0 N}{2}}(x_0)$, which follows from the definition of $\mathrm{dist}_t(x)$, the triangle inequality and inequality (4.8). Choosing $V = 16$, $\alpha = 1/2$, $r = \frac{2}{\beta_0}$ in Theorem 3.1 (this gives us $-\frac{V}{r^2} = -4(\beta_0^2)$), we see that

$$\begin{aligned} &|\,\mathrm{Riem}(\cdot, t)| \le \frac{c_0(v_0)}{t} \text{ on } {}^tB_{\frac{1}{\beta_0}}(x_0), \quad \text{and} \\ &\mathrm{vol}({}^tB_s(x_0)) \ge \tilde{v}_0(v_0)s^3, \quad \text{for all } s \le \frac{1}{\beta_0}, \\ &\text{for all } t \le \min(1 - 10\beta_0, t_0, \frac{m(v_0)}{\beta_0^2}), \end{aligned}$$

since N is large enough: we are assuming that $\frac{N\beta_0}{2} \ge \frac{2}{\beta_0}$, and so ${}^tB_{\frac{2}{\beta_0}}(x_0) \subseteq {}^tB_{\frac{\beta_0 N}{2}}(x_0)$ and so the conditions of Theorem 3.1 are satisfied in view of (4.9) and (a). Note that the dependency of the constants $c_0, m_0, \tilde{v}_0$ from Theorem 3.1 is $c_0 = c_0(n, v_0, \alpha, V) = c_0(3, v_0, 1/2, 16) = c_0(v_0)$, $m_0 = m_0(n, v_0, \alpha, V) = m_0(v_0) > 0$, and $\tilde{v}_0 = \tilde{v}_0(n, v_0, V) = \tilde{v}_0(v_0) > 0$ and $c_0, m_0, \tilde{v}_0$ **do not** depend on N or σ_0: decreasing σ_0 or increasing N will not affect $c_0(v_0), m_0(v_0)$ or $\tilde{v}_0(v_0)$. We assume that $\beta_0^2 = \sigma_0^{1/2} \le m_0(v_0)$, so that

$$
\begin{aligned}
&|\operatorname{Riem}(\cdot,t)| \le \frac{c_0(v_0)}{t} \text{ on } {}^tB_{\frac{1}{\beta_0}}(x_0) \text{ and}\\
&\operatorname{vol}({}^tB_s(x_0)) \ge \tilde{v}_0(v_0)s^3, \text{ for all } s \le \frac{1}{\beta_0},\\
&\quad \text{for all } t \le \min(1-10\beta_0, t_0)
\end{aligned} \tag{4.10}
$$

in view of the fact that $t_0 \le 1$. Let $\varepsilon(3)$, $\delta(3)$ be the constants appearing in the second Pseudolocality Theorem of G. Perelman, Theorem 10.3 in [12], in the case that $n = 3$ (as it is here). From Lemma 3.4 with $n = 3$, $\ell = 2c_0(v_0)$, $\tilde{v}_0 = \tilde{v}_0(v_0)$, $\delta = \delta(3)$, and $T_0 = \min(1-10\beta_0, t_0)$, we see that there exists an $R_0 = R_0(3, \ell, \tilde{v}_0, \delta) = R_0(3, 2c_0(v_0), \tilde{v}_0(v_0), \delta(3)) = R_0(v_0) > 0$ such that

$$
|\operatorname{Riem}|(\cdot,t) \le \frac{1}{R_0^2} \text{ on } {}^tB_{R_0}(x_0) \tag{4.11}
$$

$$
\operatorname{vol}({}^tB_{R_0}(x_0)) \ge \omega_3(1-\delta)R_0^3 \tag{4.12}
$$

for all $\frac{1}{2} \le t \le \min(1-10\beta_0, t_0)$.

It is helpful to notice the following at this stage: at the moment we have the freedom to choose $\beta_0 = \sigma_0^{1/4} > 0$ as small as we like. Decreasing σ_0 (and hence β_0) or increasing N will not change the constant $c_0(v_0)$ we obtained above, and hence will not change $R_0(v_0) = R_0(3, c_0(v_0), v_0, \delta(3))$ we obtained above. **This finishes Step 1**.

Step 2

In Step 2 we use the estimates from Step 1 and the (second) Pseudolocality result of G. Perelman to show that $|\operatorname{Riem}(\cdot,t)| \le \frac{\tilde{c}(v_0)}{t}$ on ${}^tB_{r_0}(x_0)$ for all $0 \le t \le t_0$, for some small $r_0 = r_0(v_0) > 0$ and some large $\tilde{c}(v_0)$, if $\sigma_0(v_0)$ is chosen sufficiently small, and x_0 is an arbitrary point in $\overline{{}^0B_{L-N(1-\beta_0)}(p_0)}$. That is, the estimate of Step 1 for $0 \le t \le \min(1-10\beta_0, t_0)$ can be extended to $0 \le t \le t_0$ (after changing $c_0(v_0)$ to a larger constant $\tilde{c}(v_0)$) on a small time-dependent neighbourhood of x_0 if necessary: it is only necessary to do this if $t_0 > 1-10\beta_0$. Using these estimates, we then show that $|\operatorname{Riem}(\cdot,t)| \le \frac{\tilde{c}(v_0)}{t}$ on the very large ball ${}^tB_{\frac{\beta_0 N}{64}}(y_0)$ for $0 \le t \le t_0$, for all $y_0 \in \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$, and hence, using Theorem 5.1 of [13] combined with (a) and (d), we see that $\mathcal{R} \ge -\frac{1}{N}$ on ${}^tB_{\frac{\sqrt{N}}{2}}(y_0)$ for all $0 \le t \le t_0$ for all $y_0 \in \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$.

Now we present the details of Step 2.

Let $\varepsilon(3)$, $\delta(3)$ be the constants introduced in Step 1: $\varepsilon(3)$, $\delta(3)$ are the constants which appear in the second Pseudolocality Theorem of G. Perelman, Theorem 10.3 in [12], in the case that $n = 3$ (as it is here). Assume $t_0 > 1-10\beta_0$. We know $t_0 \le 1$. Using Theorem 10.3 of [12], combined with the estimates (4.12), and (4.11), we get

$$
\begin{aligned}
|\operatorname{Riem}(x,t)| \;&\le \frac{1}{(\varepsilon(3)R_0)^2} \text{ for all } x \in {}^tB_{\varepsilon(3)R_0}(x_0),\\
&\quad \text{for all } t \in [1-10\beta_0, t_0) \cap [1-10\beta_0, 1-10\beta_0 + R_0^2(v_0)\varepsilon^2(3)).
\end{aligned}
$$

If we choose $\beta_0 = \beta_0(v_0) = \sigma_0^{1/4}(v_0) > 0$ small enough, so that $R_0^2(v_0)\varepsilon^2(3) > 10\beta_0$, then we have $1 - 10\beta_0 + R_0^2(v_0)\varepsilon^2(3) > 1$ und hence $[1 - 10\beta_0, t_0) \cap [1 - 10\beta_0, 1 - 10\beta_0 + R_0^2(v_0)\varepsilon^2(3)) = [1 - 10\beta_0, t_0)$, since $t_0 \le 1$. This means that

$$|\operatorname{Riem}(x,t)| \le \frac{1}{(\varepsilon(3)R_0)^2} \text{ for all } x \in {}^t.B_{\varepsilon(3)R_0}(x_0), \quad t \in [1 - 10\beta_0, t_0)$$

Combining this with (4.10), we see that

$$|\operatorname{Riem}(x,t)| \le \frac{\tilde{c}(v_0)}{t} \text{ for all } x \in {}^tB_{r_0}(x_0), \quad 0 \le t \le t_0 \tag{4.13}$$

for some small $r_0(v_0) = \varepsilon(3)R_0(v_0) > 0$ for all $x_0 \in \overline{{}^0B_{L-N+N\beta_0}(p_0)}$, where $\tilde{c}(v_0) = \max(\frac{1}{\varepsilon^2(3)R_0^2(v_0)}, c_0(v_0))$. That is, we have extended the estimates (4.10) up to time t_0 on a *small* time-dependent ball of fixed radius with middle point x_0, for arbitrary $x_0 \in \overline{{}^0B_{L-N+N\beta_0}(p_0)}$.

This is the point where we determine $\beta_0(v_0) = \sigma_0^{1/4}(v_0)$: it is now fixed for the rest of the argument. We stress the following point. The constants $\tilde{c}(v_0)$ from (4.13) and the constant $\sigma_0(v_0)$, and hence $\beta_0(v_0) = (\sigma_0(v_0))^{1/4}$) are now fixed. They only depend on $v_0 > 0$. They do not depend on N: we still have the freedom to choose N as large as we like without changing $\tilde{c}(v_0)$, $R_0(v_0)$, $c_0(v_0)$, $\beta_0(v_0)$, or $\sigma_0(v_0)$. In fact decreasing $\sigma(v_0)$ (and hence $\beta_0(v_0) = (\sigma_0(v_0))^{1/4}$) and increasing N would not change $\tilde{c}(v_0)$, $R_0(v_0)$ or $c_0(v_0)$ from above, in view of the definitions of $c_0(v_0)$, $R_0(v_0)$ and $\tilde{c}(v_0)$.

In order to get estimates on a *large* time-dependent ball, we restrict to points y_0 in $\overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)} \subseteq \overline{{}^0B_{L-N+N\beta_0}(p_0)}$ and use the estimates that we have just obtained. Let y_0 in $\overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$ be arbitrary.

Let $z \in \partial({}^0B_{\frac{\beta_0 N}{32}}(y_0))$. Then, using the estimate (4.13), we see that $|\operatorname{Riem}(\cdot,t)| \le \frac{\tilde{c}(v_0)}{t}$ on ${}^tB_{r_0}(z)$ for all $0 \le t \le t_0$ for some small fixed $r_0(v_0) > 0$, and the same is true on ${}^tB_{r_0}(y_0)$, since $z, y_0 \in \overline{{}^0B_{L-N+N\beta_0}(p_0)}$ due to the triangle inequality. Hence, using Lemma 3.3, we see that $d_t(y_0,z) \ge d_0(y_0,z) - a_0(v_0) = \frac{\beta_0 N}{32} - a_0(v_0) > \frac{\beta_0 N}{64}$ for all $t \in [0,t_0]$, where $a_0(v_0) = a(3, r_0(v_0), \tilde{c}(v_0))$ is the constant coming from Lemma 3.3 (with $n = 3$, $\ell = \tilde{c}(v_0)$ and $j_0 = r_0(v_0)$ there), and we assume without loss of generality that $\frac{N\beta_0}{64} > a_0(v_0)$. Hence, since $z \in \partial({}^0B_{\frac{\beta_0 N}{32}}(y_0))$ was arbitrary, it must be that $\overline{{}^tB_{\frac{\beta_0 N}{64}}(y_0)} \subseteq {}^0B_{\frac{\beta_0 N}{32}}(y_0)$ remains true for all $0 \le t \le t_0$: if there exists a first $t \in [0,t_0]$ where $\overline{{}^tB_{\frac{\beta_0 N}{64}}(y_0)} \subseteq {}^0B_{\frac{\beta_0 N}{32}}(y_0)$ is violated, then there must exist a point $z \in \partial({}^0B_{\frac{\beta_0 N}{32}}(y_0)) \cap \overline{{}^tB_{\frac{\beta_0 N}{64}}(y_0)}$ for this t, which contradicts the inequality $d_t(y_0,z) > \frac{\beta_0 N}{64}$ that we just showed.

This implies that

$$|\operatorname{Riem}(x,t)| \le \frac{\tilde{c}(v_0)}{t} \tag{4.14}$$

for all $x \in {}^tB_{\frac{\beta_0 N}{64}}(y_0)$ for all $y_0 \in \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$ in view of (4.13) and the fact that ${}^tB_{\frac{\beta_0 N}{64}}(y_0) \subseteq \overline{{}^0B_{\frac{\beta_0 N}{32}}(y_0)} \subseteq \overline{{}^0B_{L-N+N\beta_0}(p_0)}$ remains true for all $0 \le t \le t_0$. Let $r = \sqrt{N}$. Then we have

$$\begin{aligned}\mathcal{R} &\ge -\frac{1}{L^2}\\ &\ge -\frac{1}{N^2}\\ &= -\frac{1}{\tilde{c}400}(\frac{\tilde{c}400}{N^2})\\ &\ge -\frac{1}{\tilde{c}400}(\frac{1}{N})\\ &= -\frac{1}{400\tilde{c}r^2}\end{aligned}$$

at time zero on ${}^0B_r(y_0)$ since, without loss of generality, $\tilde{c}(v_0)400 \le N$ and $\sqrt{N} \le \frac{\beta_0 N}{64}$ which tells us that ${}^0B_r(y_0) = {}^0B_{\sqrt{N}}(y_0) \subseteq {}^0B_{\frac{\beta_0 N}{64}}(y_0) \subseteq \overline{{}^0B_{L-N+N\beta_0}(p_0)} \subseteq {}^0B_L(p_0)$. Now using Theorem 5.1 of the paper [13] , we see that

$$\mathcal{R}(x,t) \ge -\frac{1}{r^2} = -\frac{1}{N} \tag{4.15}$$

for all $x \in {}^tB_{\frac{r}{2}}(y_0) = {}^tB_{\frac{\sqrt{N}}{2}}(y_0)$, for all $y_0 \in \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$, for all $0 \le t \le \min(\hat{\delta}^2(v_0)r^2, t_0) = t_0$, where $\hat{\delta}(v_0) = \delta(v_0, \tilde{c}(v_0), \frac{1}{2})$ is the constant coming from Theorem 5.1 of [13], since without loss of generality $\hat{\delta}^2(v_0)r^2 = \hat{\delta}^2(v_0)N \ge 1$, and $|\operatorname{Riem}(\cdot,t)| \le \frac{\tilde{c}(v_0)}{t}$ on ${}^tB_{\sqrt{N}}(y_0)$ in view of equation (4.14), since ${}^tB_{\sqrt{N}}(y_0) \subseteq {}^tB_{\frac{\beta_0 N}{64}}(y_0)$, where here we are using again the fact that $\frac{\beta_0 N}{32} \ge \sqrt{N}$. To apply Theorem 5.1 of [13] here, scale so that $r = 1$ and then scale the conclusion of Theorem 5.1 of [13] back to the case $r = \sqrt{N}$, to obtain the estimate claimed here (the N appearing in Theorem 5.1 of [13] is $N := \tilde{c}$, where $\tilde{c}$ is the $\tilde{c}$ appearing in the current proof: the N of the theorem we are proving has nothing to do with the N of Theorem 5.1 of [13]).

This finishes Step 2

Step 3

In Step 3 we use the estimates from above to show that the contradiction point z_0 from the beginning of this argument must in fact be in $\overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$. This along with the fact that $\mathcal{R}(z_0)(t_0) \ge -\frac{1}{N}$ for such points (Step 2) and (c) leads to a contradiction if $N = N(v_0) > 0$ is large enough.

Now we present the details of Step 3.

Consider once again elements $y_0 \in \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)} \subseteq \overline{{}^0B_{L-N+N\beta_0}(p_0)}$, where $\beta_0 = \beta_0(v_0)$ is as defined in the Steps 1,2 above.

The estimate (4.14) above combined with Lemma 3.3 shows that for y in $\partial({}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0))$, that is for y with $d_0(y, p_0) = L - N + \frac{3}{4}\beta_0 N$, we have

$$\begin{aligned} d_t(y,p_0) \geq d_0(y,p_0) - a_1(v_0) &= L - N + \frac{3}{4}\beta_0 N - a_1(v_0) \\ &\geq L - N + \frac{N\beta_0}{2} \end{aligned}$$

for all $t \leq t_0$, since without loss of generality, $\frac{\beta_0 N}{4} \geq a_1(v_0)+1$, where we used that p_0 is also contained in $\overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$, and $a_1(v_0) = a(3,1,\tilde{c}(v_0))$ is the constant from Lemma 3.3 (with $n=3$, $\ell = \tilde{c}(v_0)$ and $j_0 = 1$ there). This implies that

$$d_t(y,p_0) \geq L - N + \frac{1}{2}\beta_0 N$$

for all $y \in (\ \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}\ )^c$, for all $0 \leq t \leq t_0$ (every length minimising geodesic with respect to $g(t)$ which joins p_0 to y, where y is outside of $\overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$, must intersect $\partial({}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0))$), and hence $L - d_t(y,p_0) \leq N - \frac{1}{2}\beta_0 N$, which means $\mathrm{dist}_t(y) = \max(0, L - d_t(y,p_0)) \leq \max(0, N - \frac{1}{2}\beta_0 N) = N - \frac{1}{2}\beta_0 N$, for all $t \leq t_0$ for such points $y \in (\ \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}\ )^c$. In particular z_0 is not in $(\ \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}\ )^c$ since $\mathrm{dist}_{t_0}(z_0) = N$ (we scaled so that this is true), and hence $z_0 \in \overline{{}^0B_{L-N(1-\frac{3}{4}\beta_0)}(p_0)}$. Now using the fact (inequality (4.15)) that $\mathcal{R}(\cdot,t) \geq -\frac{1}{N}$ on ${}^tB_{\frac{\sqrt{N}}{2}}(z_0)$ for all $t \in [0,t_0]$, we obtain a contradiction to (c), $\mathcal{R}(z_0,t_0)(\omega,\omega) = -\sigma_0$, if N is chosen large enough, for example $N \geq \frac{2}{\sigma_0}$.

This finishes Step 3 and the proof of the Theorem. □

Remark 4.2 The following important constants appeared, in this order, in the proof of the theorem: $c_0(v_0) = c_0(3, v_0, 1/2, 16)$, $m_0(v_0) = m_0(3, v_0, 1/2, 16)$, where $c_0(3, v_0, 1/2, 16)$, $m_0(3, v_0, 1/2, 16)$ are the constants coming from Theorem 3.1, $R_0(v_0) = R_0(3, 2c_0(v_0), \tilde{v}_0(v_0), \delta(3))$, where $R_0(3, 2c_0(v_0), \tilde{v}_0(v_0), \delta(3))$ is the constant coming from Theorem 3.4 and $\delta(3)$ and $\varepsilon(3)$ are the constants coming from G. Perelman's Pseudolocality Theorem 10.3 in [12], $r_0(v_0) = \varepsilon(3)R_0(v_0)$, $\tilde{c}(v_0) = \max(\frac{1}{\varepsilon^2(3)R_0^2(v_0)}, c_0(v_0))$, $a_0(v_0) = a(3, r_0(v_0), \tilde{c}(v_0))$, where $a(\cdot,\cdot,\cdot)$ is the constant coming from Lemma 3.3, $\hat{\delta}(v_0) := \delta(v_0, \tilde{c}(v_0), \frac{1}{2})$, where $\delta(\cdot,\cdot,\cdot)$ is the constant coming from Theorem 5.1 of [13], $a_1(v_0) = a(3, 1, \tilde{c}(v_0))$, where $a(\cdot,\cdot,\cdot)$ is the constant coming from Lemma 3.3. The following assumptions on the largeness of $N(v_0)$ and smallness of $\sigma_0(v_0)$ and $\beta_0(v_0)$ were used in the proof: $\beta_0^2 \leq m_0(v_0)$, $\frac{N\beta_0}{2} \geq \frac{2}{\beta_0}$, $\frac{\beta_0 N}{4} \geq a_0(v_0)$, $1 - 10\beta_0 + \varepsilon(3)R_0^2(v_0) > 0$. This fixes the constants β_0 and σ_0. We further require $N \geq a_0(v_0)\frac{64}{\beta_0}$, $N \geq 400\tilde{c}(v_0)$, $\sqrt{N}\beta_0 \geq 64$, $N\hat{\delta}^2(v_0) \geq 1$, $\beta_0 N \geq 4a_1(v_0) + 1$, $N \geq \frac{2}{\sigma}$. This determines N. Hence it is possible to determine σ (and hence $\beta_0 = \sigma_0^{1/4}$) and N in the first line of the above given proof.

5 Proof of Theorem 1.2

Proof of Theorem 1.2 Scale so that $r = 1$. Then we have due to the Bishop–Gromov volume comparison theorem,

(i) $\text{vol}({}^0B_r(x)) \geq v(\alpha, v_0)r^3$ for all ${}^0B_r(x) \subseteq {}^0B_{1-\alpha}(x_0)$,

- $\mathcal{R}(g(0)) \geq -\frac{1}{(1-\alpha)^2}$ on ${}^0B_{1-\alpha}(x_0)$.

See the Appendix in Version 1 of 'Local Smoothing Results for the Ricci flow in dimensions two and three', M. Simon, arXiv:1209.4274v1 for a reference: since the points x are not at the centre of the ball ${}^0B_{1-\alpha}(x_0)$, $v(\alpha, v_0)$ can depend on α. Hence, Theorem 1.6 is valid for $r = 1 - \alpha$, and we find that there exist $K = K(v(\alpha, v_0)) = K(\alpha, v_0)$ and $\sigma_0 = \sigma_0(v(\alpha, v_0)) = \sigma_0(\alpha, v_0) > 0$ such that

$$\mathcal{R}(g(t))(x)(1 - \alpha - d_t(x, x_0))^2 > -K^2$$

for all $x \in {}^tB_{1-\alpha}(x_0)$ which satisfy $(1 - \alpha - d_t(x, x_0))^2 \geq \frac{K^2}{\sigma_0}t$ and $t \leq \frac{\sigma_0(1-\alpha)^2}{K^2}$ and $t \in [0, T)$. In particular, $\mathcal{R}(g(t))(x) > -\frac{K^2}{\alpha^2}(*)$ for all $x \in {}^tB_{1-2\alpha}(x_0)$ for all $t \leq \min(\frac{\sigma_0\alpha^2}{K^2}, \frac{\sigma_0(1-\alpha)^2}{K^2})$ with $t \in [0, T)$. Now we may use Theorem 3.1, with $r = 1-2\alpha$ to further conclude that $|\text{Riem}(x, t)| \leq \frac{c_0(\alpha,v_0)}{t}(**)$ for all $x \in {}^tB_{1-4\alpha}(x_0)$, $t \leq S(\alpha, v_0)$. Choosing $\alpha = 1/10$ in the above argument, we see that we also get $\text{vol}({}^tB_1(x_0)) \geq \tilde{v}_0(v_0)(4/5)^3(***)$ for all $t \leq S(\alpha, v_0)$, for some $\tilde{v}_0(v_0) > 0$. The estimates (*),(**) and (***) are the desired estimates. □

Appendix: Dimension of Gromov–Hausdorff Limits of Collapsing and Non-collapsing Spaces

We explain why some certain well-known properties of collapsing, respectively, non-collapsing manifolds, with curvature bounded from below hold. These properties follow from the results contained in [3] (see also [4]). Note that the definition of *Alexandrov space with curvature bounded from below* in [3] (Definition 2.3) and [4] (Proposition 10.1.1) agree.

Theorem 1.1 *Let* $(B_1(p_i), g_i)$, $(B_1(q_i), h_i)$, $i \in \mathbb{N}$ *be balls whose closure is compactly contained in smooth Riemannian manifolds without boundary of dimension* $n \in \mathbb{N}$ *fixed. Assume that* $\sec \geq -V$ *on these balls and that* $d_{GH}((B_1(p_i), g_i), (B_1(q_i), h_i)) \to 0$ *as* $i \to \infty$, *and* $\text{vol}((B_1(p_i), g_i)) \geq v_0 > 0$ *for all* $i \in \mathbb{N}$. *Then it cannot be that* $\text{vol}(B_1(q_i), h_i) \to 0$ *as* $i \to \infty$.

Proof Assume the theorem is false. We know that $(B_1(p_i), g_i, p_i)$ and $(B_1(q_i), h_i, q_i)$ Gromov–Hausdorff converge, after taking a subsequence, to the same space $(X = B_1(p), d, p)$ by the theorem of M. Gromov, and that (X, d, p) is an Alexandrov space (see *Notes on Alexandrov Spaces* below). Without loss of generality, we may assume that $\sec \geq -k^2$ on the balls we are considering, where $k^2 > 0$ is as small as we like. This can be seen as follows. Without loss of generality (renumber the indices i), we have $\mathrm{vol}(B_{1/i}(q_i), h_i) \leq \mathrm{vol}(B_1(q_i), h_i) \leq \frac{1}{i^{n+1}}$. The Bishop–Gromov Comparison principle implies that $\mathrm{vol}(B_{1/i}(p_i), g_i) \geq c(v_0, n)\frac{1}{i^n}$. Scaling both Riemannian metrics by i^2, we have (we also call the rescaled metrics g_i and h_i) $\mathrm{vol}(B_1(p_i), g_i) \geq c(v_0, n) > 0$ and $\mathrm{vol}(B_1(q_i), h_i) \leq \frac{1}{i}$ and $\sec \geq -\frac{V}{i^2}$. So we assume $\sec \geq -k^2$ with $k > 0$ arbitrarily small.

Let $\overline{B_a(y)} \subseteq B_{1-10a}(p)$ and let $\{B_{R/3}(s_j)\}_{j\in\{1,\ldots,N\}}$ be any maximally pairwise disjoint collection of balls with $R << a < 1/(10)$ and centres s_j in $B_a(y)$. By *maximally pairwise disjoint* we mean that if we try and add a ball $B_{R/3}(z)$ to the collection, where $z \in B_a(y)$, then the new collection is not pairwise disjoint. Then clearly $\{B_R(s_j)\}_{j\in\{1,\ldots N\}}$ must cover $B_a(y)$. Let $\tilde{s}_j$, respectively, $\tilde{p} = p_i$, $\tilde{y}$ be the corresponding points in $(B_1(p_i), g_i, p_i)$ which one obtains by mapping s_j, respectively, p,y back to $(B_1(p_i), g_i, p_i)$ using the Gromov–Hausdorff approximation $f_i : (B_1(p), d, p) \to (B_1(p_i), g_i, p_i)$: we write $p_i = \tilde{p}$, and so on, suppressing the dependence of the points on i sometimes, in order to make this explanation more readable. For i large enough, $\{B_{2R}(\tilde{s}_j)\}_{j\in\{1,\ldots N\}}$ must cover $B_a(\tilde{y})$ and $\{B_{R/4}(\tilde{s}_j)\}_{j\in\{1,\ldots,N\}}$ must be pairwise disjoint and contained in $B_{2a}(\tilde{y}) \subseteq B_1(\tilde{p})$. The Bishop–Gromov volume comparison principle implies that $\frac{c_2(v_0,a,n)}{R^n} \leq N \leq \frac{c_1(v_0,a,n)}{R^n}$, for some fixed $0 < c_0(v_0, V, a, n), c_1(v_0, a, V, n) < \infty$ and hence the rough dimension of $B_a(y)$ (see Definition 6.2 in [3]) must be n.

This means that the Hausdorff dimension and burst index of $B_s(p)$ is also n for all $s < 1$ (see Lemma 6.4 and Definition 6.1 in [3]). Assume $\varepsilon \leq \frac{1}{1000n}$ in all that follows. Now let $z \in B_{1/4}(p)$ be a point for which there is an (n, ε) explosion (Definition 5.2 in [3]: an (n, ε) explosion is called an (n, ε) strainer in [4], see Definition 10.8.9 there). Note that for any $\frac{1}{1000n} \geq \varepsilon > 0$ such a point exists (see Corollary 6.7 in [3]). Let $(a_k, b_k)_{k\in\{1,\ldots n\}}$ be such an (n, ε) explosion at z and assume that $a_k, b_k \in B_s(z)$ for all $k = 1, \ldots, n$ with $s << 1$: as pointed out in [3] (just after Definition 5.2), we can always make this assumption, see also Proposition 10.8.12 in [4]. Then there exists a small ball $B_r(z)$ such that $(a_k, b_k)_{k\in\{1,\ldots n\}}$ is an (n, ε) explosion at x for all $x \in B_r(z)$ and $(a_k, b_k)_{k\in\{1,\ldots n\}}$ is in $B_s(p)\backslash B_{2\hat{r}}(z)$ where $s >> \hat{r} >> r > 0$: distance is continuous in X and comparison angles (which are measured in $M^2(-V) :=$ hyperbolic space with curvature equal $-V$) change continuously as distances change continuously and stay away from zero (see [11], equation (44)). With $s >> \hat{r} >> r$, we mean $\frac{\hat{r}}{s} << 1$ and $\frac{r}{\hat{r}} << 1$. Going back to $(B_1(q_i), h_i, q_i)$ with our Gromov–Hausdorff approximation, we see (once again dropping dependence on i for readability) that there exists a ball $B_r(\tilde{z}) \subseteq B_{1/2}(q_i)$ and an explosion $(\tilde{a}_k, \tilde{b}_k)_{k\in\{1,\ldots n\}}$ in $B_{2s}(\tilde{z})\backslash B_{\hat{r}}(\tilde{z})$ (if i is large enough) such that $(\tilde{a}_k, \tilde{b}_k)_{k\in\{1,\ldots n\}}$ is an $(n, 4\varepsilon)$ explosion at x for all $x \in B_r(\tilde{z})$: once again, this follows from the fact that angle comparisons change continuously as distances change continuously and stay away from zero, and distance changes at most by $\delta(i)$, with $\delta(i) \to 0$ as $i \to \infty$, under our Gromov–Hausdorff approximation. There are no $((n + 1), \varepsilon)$ explosions in $(B_1(q_i), h_i, q_i)$, as the Hausdorff dimension of the manifold (and hence the burst index) is n (see Theorem 5.4 in [3] or Proposition 10.8.15 in [4]). Fix $0 < \varepsilon(n) << \frac{1}{2000n}$. But then, using Theorem 5.4 in [3], see also Theorem 10.8.18 in [4], (more explicitly, using the proofs thereof) we see that there is a $\tilde{r} = \tilde{r}(n, r) > 0$ and a bi-Lipschitz homeomorphism from $f : B_{\tilde{r}}(\tilde{z}) \to f(B_{\tilde{r}}(\tilde{z})) \subseteq \mathbb{R}^n$, where the bi-Lipschitz constant may be estimated by $\frac{1}{c(n)} d_i(x, y) \leq |f(x) - f(y)| \leq c(n) d_i(x, y)$ for some $c(n) > 0$, and hence $\mathrm{vol}(B_1(q_i), h_i, q_i) \geq \varepsilon(n, r) > 0$ for i large enough, as r, n do not depend on i. This shows, that after taking a subsequence, we must have $\mathrm{vol}(B_1(q_i), h_i, q_i) \geq \varepsilon(n, r) > 0$. □

Notes on Alexandrov Spaces

The fact that $(B_1(p_i), g_i, p_i)$ and $(B_1(q_i), h_i, q_i)$ Gromov–Hausdorff converge to some metric space $(X = B_1(p), d)$ after taking a subsequence follows from Gromov's Convergence Theorem (we apply the theorem to the closed balls $\overline{B_{1-\frac{1}{i}}(p)} \subseteq B_1(p)$ with $i \in \mathbb{N}$, and then take a diagonal subsequence). See 10.7.2 in [4]. The limit space has the property that $\overline{B_s(p)}$ is complete for all $0 < s < 1$ (by construction), and $\overline{B_s(p)}$ is compact for all $0 < s < 1$, since it is also totally bounded (due to the Bishop–Gromov comparison principle: see the argument on the rough dimension of $\overline{B_a(y)}$ at the beginning of the proof above).

In order to guarantee that $(X = B_1(p), d, p)$ is an *Alexandrov space*, a local version of the *Globalisation Theorem of Alexandrov–Toponogov–Burago–Gromov–Perelman* (Theorem 3.2 in [3]) is necessary, as the spaces we are considering are not complete. Such a local version of the theorem exists, as pointed out in Remark 3.5 in [3]. Proofs of the Globalisation Theorem can be found in the book [1] and a similar proof, obtained independently, is given in the paper [10]. Examining the proofs of the Globalisation Theorem (in the case sec ≥ -1) in any of the proofs mentioned above, we see that the proofs are local. Examining any of the proofs mentioned above, we see that the following is true: if $(B_1(x_0), g)$ is compactly contained in a smooth manifold, and sec ≥ -1 on $(B_1(x_0), g)$ and $z \in B_1(x_0)$ has $d(x_0, z) = 1 - r$, then the quadruple condition (or the hinge condition , or any of the other equivalent conditions, see sect. 2 in [3] or 8.2.1 in [1], or the discussion on page 3 of [10] to see why these conditions are equivalent) hold on the ball $B_{rc}(z) \subseteq B_1(x_0)$ for some fixed constant $0 < c << 1$ independent of z or r. Note that the space $(X = B_1(p), d)$ we obtain this way is *locally intrinsic*: for all $x \in X$, for all $z, q \in B_\varepsilon(x)$ for all $B_{5\varepsilon}(x) \subseteq B_{1-\alpha}(p)$ for all $1 > \alpha, \varepsilon > 0$ there exists a length minimising geodesic between z and q which is contained in $B_{5\varepsilon}(x)$: see the proof of Theorem 2.4.16 in [4].

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